Summary of Vector Spaces

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Given V n-dimensional vector space over F with basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ and W an m dimensional vector space with basis $\mathcal{E} = \{w_1, w_2, \ldots, w_m\}$. If $\phi(v_j) = \sum_{i=1}^{n} a_{ij} w_i$, then $M_{\mathcal{B}}^{\mathcal{E}} : Hom(V, W) \to M_{m \times n}(F)$ where

$$M_{\mathcal{B}}^{\mathcal{E}}(\phi) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}$$

Using this map, $Hom(V, W) \cong M_{m \times n}(F)$ as vector spaces over F.

Exercise 1 *If $\mathcal{D} = \{u_1, u_2, \ldots u_k\}$ is a basis for a k dimensional space U over F and $\psi(u_j) = \sum_{i=1}^{n} b_{ij} v_i$, calculate $\phi \circ \psi(u_i)$ and explain how this shows $M_{\mathcal{E}}^{\mathcal{D}}(\phi \circ \psi) = M_{\mathcal{B}}^{\mathcal{E}}(\phi) M_{\mathcal{D}}^{\mathcal{B}}(\psi)$.*

We can conclude from this that $Hom(V, V) \cong M_{n \times n}(F)$ as F-algebras (recall the product on $Hom(V, V)$ is $\circ$ and the product on $M_{n \times n}(F)$ is matrix multiplication) since they are isomorphic as vector spaces and now we know that $M_{\mathcal{B}}^{\mathcal{B}}$ is a homomorphism with respect to the $\circ$ operation on $Hom(V, V)$ and the matrix product on $M_{n \times n}(F)$.

Define $V^* = Hom(V, F)$ and $\mathcal{B}^* = \{v_1^*, v_2^*, \ldots, v_n^*\} \subseteq Hom(V, F)$ is called the dual space and dual basis of V where $v_i^*(v_j) = \delta_{ij}$. Elements of V^* are called linear functionals. V^{**} (the dual of V^*) is called the double dual of V.

If V is finite dimensional then the dimension of V^* is equal to the dimension of V and $V^{**} \cong V$ (in a natural way).

If V is infinite dimensional then V^* is larger than V and V^{**} is not isomorphic to V.

Exercise 2 *If V is infinite dimensional with basis $\mathcal{A}$, prove that $\mathcal{A}^* = \{v^* | v \in \mathcal{A}\}$ does not span V^*.*

Exercise 3 *Let $\mathcal{A}$ be a basis for the infinite dimensional space V. Prove that V is isomorphic to the direct sum of copies of the field F indexed by the set $\mathcal{A}$. Prove that the direct product of copies of F indexed by $\mathcal{A}$ is a vector space over F and it has strictly larger dimension than the dimension of V (see exercises in section 10.3).*

Exercise 4 *If V is infinite dimensional with basis $\mathcal{A}$, prove that V^* is isomorphic to the direct product of copies of F indexed by $\mathcal{A}$. Deduce that dim $V^* >$ dim V.*

With $\phi \in Hom(V, W)$, define ϕ^* to be an element of $Hom(W^*, V^*)$ given for all $f \in W^*$, $\phi^*(f) = f \circ \phi \in V^*$.

Exercise 5 *Compute $\phi^*(w_i^*)$ and use this calculation to determine $M_{\mathcal{E}^*}^{\mathcal{B}^*}(\phi^*)$.*