MATH 6121 lecture notes

Transcribed and formatted by John M. Campbell
jmaxwellcampbell@gmail.com

1 September 08 lecture

1.1 Contact information

http://garsia.math.yorku.ca/~zabrocki/math6121f16/
Mike Zabrocki - Ross N518.
Office hours: email zabrocki@mathstat.yorku.ca.

1.2 Related seminars

The Algebraic Combinatorics Seminar at the Fields Institute will take place on Fridays. The "theme" of this seminar during the Fall term is based on geometric combinatorics.

The York University Applied Algebra Seminar will be taking place on Monday afternoons at 3 p.m.

1.3 Topics in MATH 6121

The syllabus of this course is based on the previous MATH 6121 syllabus (see http://garsia.math.yorku.ca/~nantel/teaching/).

There are three main topics to be covered in MATH 6121:

(i) Linear algebra;

(ii) Group theory and representation theory; and

(iii) Rings and modules.

1.4 Linear algebra: basic definitions and theorems

Definition 1.1. A **vector space** over $\mathbb{C}$ is a set endowed with two operations,

$$+\colon V \times V \to V$$

and

$$\cdot\colon \mathbb{C} \times V \to V,$$

such that the following axioms hold:

(i) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ for all $\vec{u}, \vec{v} \in V$ (i.e. the binary operation $+$ is commutative);

(ii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ for all $\vec{u}, \vec{v}, \vec{w} \in V$ (i.e. the binary operation $+$ is associative);

(iii) There exists an element $\vec{0} \in V$ such that $\vec{w} + \vec{0} = \vec{w}$;

(iv) For all $\vec{w} \in V$, there exists an element $-\vec{w} \in V$ such that $\vec{w} + (-\vec{w}) = \vec{0}$;

(v) $r(s\vec{v}) = (rs)\vec{v}$ for all $r, s \in \mathbb{C}$ and all $\vec{v} \in V$;

(vi) $(r + s)\vec{v} = r\vec{v} + s\vec{v}$ for all $r, s \in \mathbb{C}$ and all $\vec{v} \in V$;

(vii) $r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}$ for all $r \in \mathbb{C}$ and all $\vec{v}, \vec{w} \in V$; and

(viii) $1\vec{v} = \vec{v}$ for all $\vec{v} \in V$.

Definition 1.2. A **linear transformation** from a vector space V to a vector space W is a map $T\colon V \to W$ such that:

(i) $T(\vec{v} +_V \vec{w}) = T(\vec{v}) +_W T(\vec{w})$ for all $\vec{v}, \vec{w} \in V$; and

(ii) $T(r \cdot_V \vec{v}) = r \cdot_W T(\vec{v})$.

Examples of a vector space:

(i) $\mathbb{C}^n$: ordered n-tuples of elements from $\mathbb{C}$;

(ii) Continuous functions on $\mathbb{C}$;

(iii) All functions on a set S to $\mathbb{C}$, with addition is defined so that $f(s) + g(s) = (f + g)(s)$; and

(iv) Polynomials in $\mathbb{C}[x]$ for a variable x, and polynomials in $\mathbb{C}[x_1, x_2, \ldots, x_n]$ for multiple variables.

Definition 1.3. The **span** of a set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq V$ is equal to the set

$$\{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n : c_1, c_2, \ldots, c_n \in \mathbb{C}\}.$$

Definition 1.4. We say that V is **finitely-generated** if there is a finite set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ of vectors such that $\mathrm{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = V$.

Definition 1.5. A set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ of vectors is said to be **linearly independent** if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$$

implies $c_1 = c_2 = \cdots = c_n = 0$.

Definition 1.6. A **basis** of a vector space V is a linearly independent spanning set of V.

Theorem 1.7. *Given a set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ that spans V, if this set has the property that no subset of these vectors also span V, then this set is a basis.*

Proof. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be as given above, and suppose that this set is such that no (proper) subset of this set also spans V. By way of contradiction, suppose that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is not linearly independent. So there exist scalars $c_1, c_2, \ldots, c_n$ such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0},$$

but it is not the case that all of the scalars $c_1, c_2, \ldots, c_n$ are equal to 0. We may assume without loss of generality that $c_1 \neq 0$. But then

$$\vec{v}_1 = -\frac{1}{c_1}\left(c_2\vec{v}_2 + \cdots + c_n\vec{v}_n\right),$$

so the set $\{\vec{v}_2, \vec{v}_3, \ldots, \vec{v}_n\}$ spans V, contradicting our initial assumption that the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is such that no proper subset of this set also spans V. $\square$

Intuition concerning the above theorem: we can "keep throwing away vectors until we are left with a basis".

Lemma 1.8. *If $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_m\}$ is a set of linearly independent elements and $\mathcal{A} = \{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$ is a basis, then $\mathcal{A}$ can be reordered in such a way so that*

$$\left\{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_k, \vec{a}_{k+1}, \dots, \vec{a}_n\right\}$$

is also a basis, where $1 \leq k \leq n$.

Theorem 1.9. *Letting $\mathcal{B}$ and $\mathcal{A}$ be as given in the above lemma, we have that*

$$\#\text{ of elements in }\mathcal{B} \leq \#\text{ of elements in }\mathcal{A}.$$

Corollary 1.10. *Every basis for a finitely-generated vector space V has the same number of elements.*

Definition 1.11. We call the cardinality of a basis of a vector space V the **dimension** of V.

There is a natural isomorphism between V and $\mathbb{C}^n$ where $n = \dim(V)$.

Intuition: "Every time I pick an ordered basis, there will be an isomorphism."

Let $\mathcal{B} = (\vec{b}_1, \vec{b}_2, \dots, \vec{b}_n)$ be an ordered bases. It will be important for our purposes that this basis is ordered, since the isomorphism based on this basis which we will construct will depend on the order of the elements in this basis.

Define the mapping $L_{\mathcal{B}} \colon V \to \mathbb{C}^n$ as follows:

$$L_{\mathcal{B}}\left(\vec{v} = \sum_{i=1}^{n} c_i \vec{b}_i\right) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Take $T \colon V \to W$ and fix a basis $\mathcal{B}$ of V and $\mathcal{C}$ of W. Then there is an isomorphism which sends T to a matrix M. If $\dim(V) = n$ and $\dim(W) = m$, then this matrix will be an $m \times n$ matrix.

In other words, every linear transformation may be regarded as a matrix transformation.

Let V denote the vector space of all degree-2 homogeneous polynomials in x_1 and x_2.

Then $\{x_1^2, x_2^2, x_1x_2\}$ is a basis for this vector space.

Now let $T = \partial x_1 + \partial x_2$.

That is, $T = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$.

Now let W denote the vector space consisting of degree-1 homogeneous polynomials in x_1 and x_2. Then $\mathcal{C} = \{x_1, x_2\}$ is a basis of this vector space.

Consider the mapping $L_{\{x_1^2, x_2^2, x_1x_2\}}$ from V to $\mathbb{C}^{\dim(V)} = \mathbb{C}^3$:

$$L_{\{x_1^2,x_2^2,x_1x_2\}}\left(ax_1^2 + bx_2^2 + cx_1x_2\right) = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{C}^3.$$

Now consider the matrix M corresponding to the linear transformation T.

$$\begin{aligned} M &= \begin{bmatrix} L_{\mathcal{C}}(T(x_1^2)) & L_{\mathcal{C}}(T(x_2^2)) & L_{\mathcal{C}}(T(x_1x_2)) \end{bmatrix} \\ &= \begin{bmatrix} L_{\mathcal{C}}(2x_1) & L_{\mathcal{C}}(2x_2) & L_{\mathcal{C}}(x_1 + x_2) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}. \end{aligned}$$

Now consider the inverse mapping $L_{\mathcal{C}}^{-1} \colon \mathbb{C}^2 \to W$. The function $L_{\mathcal{C}}^{-1}$ evaluated at

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a + c \\ 2b + c \end{bmatrix}$$

is equal to:

$$\begin{aligned} L_{\mathcal{C}}^{-1}\left(\begin{bmatrix} 2a + c \\ 2b + c \end{bmatrix}\right) &= (2a + c)x_1 + (2b + c)x_2 \\ &= \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)\left(ax_1^2 + bx_2^2 + cx_1x_2\right) \\ &= T\left(ax_1^2 + bx_2^2 + cx_1x_2\right). \end{aligned}$$

Intuition: It's easier to "work with" matrices compared to, say, linear transformations on polynomial expressions. For example, it's easier to compute the null space of a matrix given in a standardized format, compared to the null space of a function on polynomials.

Notation: write $L_{\mathcal{B}}(\vec{v}) = [\vec{v}]_{\mathcal{B}}$.

Let $T\colon V \to W$, and let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_m\}$ be bases of V and W respectively. Then T "corresponds to" the following $m \times n$ matrix, which is denoted by ${}_{\mathcal{C}}[T]_{\mathcal{B}}$:

$$\left[L_{\mathcal{C}} \left(T \left(\vec{b}_1 \right) \right) \quad \cdots \quad L_{\mathcal{C}} \left(T \left(\vec{b}_n \right) \right) \right]_{m \times n}.$$

Problem 1.12. Say that $M = {}_{\mathcal{C}}[T]_{\mathcal{B}}$ for some bases $\mathcal{B}$ and $\mathcal{C}$ of V and W, with $T\colon V \to W$. Show that if $M\vec{u} = \vec{0}_{\mathbb{C}^m}$ for $\vec{u} \in \mathbb{C}^n$, then $T(L_{\mathcal{B}}^{-1}(\vec{u})) = \vec{0}_V$.

Remark 1.13. This correspondence has the property that

$$L_{\mathcal{C}} \circ T = M \circ L_{\mathcal{B}}$$

and the property that

$$L_{\mathcal{C}} \circ T = {}_{\mathcal{C}}[T]_{\mathcal{B}} \circ L_{\mathcal{B}},$$

where the latter expression is given by matrix multiplication by the $m \times n$ matrix ${}_{\mathcal{C}}[T]_{\mathcal{B}}$.

Next lecture: direct sums of linear transformation, how the direct sums of two linear transformations corresponds to the direct sum of the corresponding matrices, tensor products of linear transformations, how the tensor product of two linear transformations corresponds to the tensor product of the corresponding matrices.