MATH 6121 lecture notes

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11.1 Maschke's theorem and Schur's lemma

Maschke's theorem: Letting V be a G-module over $\mathbb{C}$, then if W is an invariant subspace of V, there exists an invariant subspace $U \subseteq V$ such that $V = W \oplus U$.

Schur's lemma: All mappings T which are G-homomorphisms of irreducible modules are either trivial or are isomorphisms. If the irreducible modules are equal then $T = c \cdot \mathrm{Id}$ for some scalar c.

If the characteristic of the scalar ring of a G-module divides $|G|$, things can go wrong with respect to Maschke's theorem.

For example, let $G = \{e, a\} = \{e_G, a\}$ be the multiplicative cyclic group of order 2, and let V denote the vector space over the field $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ whereby $V = \mathscr{L}_{\mathbb{Z}_2}\{e_1, e_2\}$.

$(\mathbb{Z}_2, +)$	0	1
0	0	1
1	1	0

$(\mathbb{Z}_2, \cdot)$	0	1
0	0	0
1	0	1

$(G, \cdot)$	e	a
e	e	a
a	a	e

By constructing an appropriate G-action on the module V, we may thus construct an invariant submodule $W = \mathscr{L}\{e_1\}$ of V which does not have an orthogonal complement as a submodule. We need to be able to divide by $|G|$ in order to construct an appropriate inner product according to Maschke's theorem, but $|G| \equiv 0 (\mathrm{mod}\ 2)$ in this case.

Let G act on the additive abelian group $V = \{0, e_1, e_2, e_1+e_2\}$ so that $g \bullet 0 = 0$, and $g \bullet (x+y) = g \bullet x + g \bullet y$ for $x, y \in V$, with:

$$\begin{aligned} e_G \bullet e_1 &= e_1, \\ e_G \bullet e_2 &= e_2, \\ a \bullet e_1 &= e_1, \\ a \bullet e_2 &= e_1 + e_2. \end{aligned}$$

From the above equalities, we have that the mapping

$$\bullet \colon G \times V \to V$$

satisfies the group action axiom whereby $e_G \bullet x = x$ for $x \in V$. We proceed to observe that the mapping $\bullet$ as given above is indeed a group action, since the group action axiom whereby

$$g_1 \bullet (g_2 \bullet x) = (g_1 \cdot_G g_2) \bullet x$$

for $g_1, g_2 \in G$ and $x \in V$ is easily seen to hold:

$$(a \cdot_G a) \bullet e_2 = a^2 \bullet e_2$$

$$
\begin{aligned}
&= e_G \bullet e_2 \\
&= e_2 \\
&= e_1 + (e_1 + e_2) \\
&= a \bullet e_1 + a \bullet e_2 \\
&= a \bullet (e_1 + \bullet e_2) \\
&= a \bullet (a \bullet e_2)\,.
\end{aligned}
$$

We thus find that the module $V = \{0, e_1, e_2, e_1 + e_2\}$ has a very natural structure as a G-module.

Although we have a G-invariant submodule

$$W = \mathscr{L}_{\mathbb{Z}_2}\{e_1\} = \{0, e_1\} \leq_G V$$

of V, V is not equal to W plus another G-invariant submodule.

In this case, something goes wrong with Maschke's theorem. We're not working over $\mathbb{C}$ in this case.

Working over $\mathbb{C}$ with respect to Maschke's theorem is intuitively very useful.

Recall that a **Hermitian inner product**[1] is such that if you reverse the order, you get the complex conjugate.

$[\cdot, \cdot]$ is a G-invariant Hermitian inner product on V. So, we can construct an orthonomal basis $\mathcal{B}$ of V.

$A_{\mathcal{B}}(g)$ will be a unitary matrix: $A^{-1} = \overline{A^T}$.

$A_{\mathcal{B}}$ is called a **unitary representation**[2].

Let V be finite dimensional. Write

$$V \cong V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

where V_i is irreducible for all i.

Now, let V be decomposed in the following manner.

$$
\begin{aligned}
V \cong\ & \underbrace{V_1 \oplus V_1 \oplus \cdots \oplus V_1}_{m_1} \oplus \\
& \underbrace{V_2 \oplus V_2 \oplus \cdots \oplus V_2}_{m_2} \oplus \\
& \cdots \\
& \underbrace{V_k \oplus V_k \oplus \cdots \oplus V_k}_{m_k}\,.
\end{aligned}
$$

We want all bases to be essentially the same.

Every time we have a representation, it will "break down" in the manner suggested above.

Observe that given a morphism from V_1 to V_2, by Schur's lemma, a morphism of this form must either be trivial, or an isomorphism.

[1]See `http://mathworld.wolfram.com/HermitianInnerProduct.html`.
[2]See `https://en.wikipedia.org/wiki/Unitary_representation`.

It is very important to choose essentially the same basis for each subspace of the form V_i.

For example, consider the isomorphic equivalence whereby

$$\mathscr{L}\{e_1, e_2\} \cong \mathscr{L}\{e_1 + e_2, e_1 - e_2\},$$

letting $\mathcal{B}$ denote the basis $\{e_1, e_2\}$, and letting $\mathcal{B}' = \{e_1 + e_2, e_1 - e_2\}$. Compute the transition matrices with respect to these bases:

$$\begin{aligned} {}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{B}'} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ {}_{\mathcal{B}'}[\mathrm{id}]_{\mathcal{B}} &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}. \end{aligned}$$

V as given above is a module, but a representation is a map from the group to the corresponding group of matrices.

Choose a basis carefully so that each copy of V_i has an isomorphic copy of that basis.

$$\begin{matrix} \boxed{A^{\binom{1}{1}}_{(g)}} & & & & & & \\ & \boxed{A^{\binom{1}{2}}_{(g)}} & & & & & \\ & & \ddots & & & & \\ & & & \boxed{A^{\binom{1}{m_1}}_{(g)}} & & & \\ & & & & \boxed{A^{\binom{2}{1}}_{(g)}} & & \\ & & & & & \ddots & \\ & & & & & & \boxed{A^{\binom{2}{m_2}}_{(g)}} \end{matrix}$$

Smaller example: $V = X^{(1)} \oplus X^{(1)} \oplus X^{(2)}$.

In this case, let $\dim(X^{(1)}) = 3$ and $\dim(X^{(2)}) = 4$.

Also, let $A^{\binom{1}{1}}(g) = A^{\binom{1}{2}}(g)$.

$$A_{\mathcal{B}}(g) = \begin{array}{c} \\ 3 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 3 & 3 & 4 \end{array} \\ \left[\begin{array}{c|c|c} A^{\binom{1}{1}}_{(g)} & & 0 \\ \hline & A^{\binom{1}{2}}_{(g)} & \\ \hline 0 & & A^{\binom{2}{1}}_{(g)} \end{array}\right] \end{array}$$

$$\begin{array}{c} 3 \\ 3 \\ 4 \end{array} \left[\begin{array}{c|c|c} T^{\binom{1}{1}\binom{1}{1}} & T^{\binom{1}{1}\binom{1}{2}} & T^{\binom{1}{1}\binom{2}{1}} \\ \hline T^{\binom{1}{2}\binom{1}{1}} & T^{\binom{1}{2}\binom{1}{2}} & T^{\binom{1}{2}\binom{2}{1}} \\ \hline T^{\binom{2}{1}\binom{1}{1}} & T^{\binom{2}{1}\binom{1}{2}} & T^{\binom{2}{1}\binom{2}{1}} \end{array}\right]$$

Question 11.1. What is $\mathrm{Com}(A) = \{T \in \mathrm{Mat}_{n\times n}(\mathbb{C}) : TA_{\mathcal{B}}(g) = A_{\mathcal{B}}(g)T\}$?

If $i \neq k$, then $T^{\binom{i}{j}\binom{k}{\ell}}$ is a G-homomorphism from V_i to V_k. So, in this case, we have that $T^{\binom{i}{j}\binom{k}{\ell}} \equiv 0$, by Schur's lemma.

If $i = k$, then $T^{\binom{i}{j}\binom{k}{\ell}}$ is a G-homomorphism from V_i to V_i. So, in this case, we have that $T^{\binom{i}{j}\binom{k}{\ell}} = c \cdot \mathrm{Id}_{d_i \times d_i}$, with $d_1 = \dim(X^{(1)}) = 3$ and $d_2 = \dim(X^{(2)}) = 4$.

If T is in the commutator of A, what does this tell us? It shows that T has a very specific form.

In particular, if T is in $\mathrm{Com}(A)$, then T must be of the form

$$\begin{bmatrix} a & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & b & 0 & 0 & 0 & 0 \\ c & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f \end{bmatrix}$$

for $a, b, c, d, f \in \mathbb{C}$.

Now observe that:

$$\deg(A_{\mathcal{B}}(g)) = \sum_{i=1}^{k} m_i d_i.$$

Recall that m_i denotes the number of times the irreducible component V_i appears in the decomposition of V, and $\dim(V_i) = d_i$.

Recall that

$$V \cong V_1^{\oplus m_1} \oplus V_2^{\oplus m_2} \oplus \cdots V_k^{\oplus m_k},$$

with $\dim(V_i) = d_i$. We thus have that:

(i) $\deg(A_{\mathcal{B}}(g)) = \sum_{i=1}^{k} m_i d_i$;

(ii) $\mathrm{Com}(A) \cong \oplus_{i=1}^{k} \mathrm{Mat}_{m_i \times m_i}(\mathbb{C}) \otimes \mathrm{Id}_{d_i \times d_i}$; and

(iii) $\dim(\mathrm{Com}(A)) = \sum_{i=1}^{k} m_i^2$.

It is natural to ask: What is the center of $\mathrm{Com}(A)$?

If $\mathcal{U} \in Z(\mathrm{Com}(A))$, then $\mathcal{U}$ must be of the form:

$$\begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b \end{bmatrix}$$

for $a, b \in \mathbb{C}$, because $Z(\mathrm{Mat}_{r \times r}(\mathbb{C})) = \mathbb{C}\mathrm{Id}_{r \times r}$.

We thus find that:

(iv) $Z(\mathrm{Com}(A)) = \oplus_{i=1}^{k} \mathbb{C}\mathrm{Id}_{d_i m_i \times d_i m_i}$; and

(v) $\dim(Z(\mathrm{Com}(A))) = k$.

Property (i) is clear because we chose the notation that way.

Property (ii) is less clear. It follows because

$$A^{\binom{i}{j}} T^{\binom{i}{j}\binom{k}{\ell}} = T^{\binom{i}{j}\binom{k}{\ell}} A^{\binom{k}{\ell}}.$$

Kronecker product of the matrices $\longrightarrow$ tensor product. Recall that:

$$A \otimes B \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1\ell}B \\ a_{21}B & a_{22}B & \cdots & a_{2\ell}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{n\ell}B \end{bmatrix}.$$

It is clear that (ii) $\implies$ (iii), if we choose an appropriate basis for $\mathrm{Mat}_{m_i \times m_i}(\mathbb{C})$.

It is clear that (ii) $\implies$ (iv), because of the previous exercise which required proving that $Z(\mathrm{Mat}_{n\times n}(\mathbb{C})) \cong \mathbb{C}\mathrm{Id}_{n\times n}$.

It is clear that (iv) $\implies$ (v).

11.2 Elementary character theory

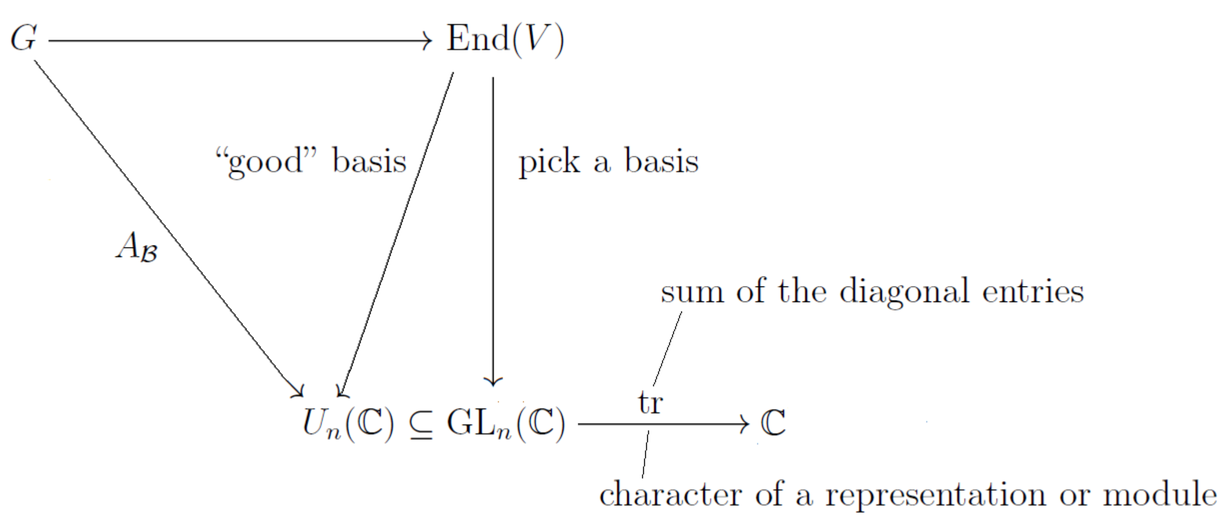
Recall that we have discussed a correspondence between G-modules and G-representations:

$$G\text{-module} \iff G\text{-representation}$$

G-module: group acting on a vector space.

(G, V)	choose a basis	$A_{\mathcal{B}}\colon G \to \mathrm{GL}_n(\mathcal{C})$
G-module	$\implies$	G-representation
G-set	$\iff$	embedding in symmetric group on set

Remark 11.2. Intuitively, the invertible matrices are a much richer class of groups compared to S_n.



$\mathrm{SL}_n(\mathbb{C}) \longrightarrow$ condition for embedding?

Recall that $U_n(\mathbb{C})$ denotes the group of $n \times n$ unitary matrices. Recall that a complex square matrix U is **unitary** if its conjugate transpose U^* is also its inverse.

The character for a module V will be denoted χ^V, with

$$\chi^V : G \to \mathbb{C},$$

and

$$\chi^V(g) = \mathrm{tr}(A_{\mathcal{B}}(g)),$$

for an arbitrary element $g \in G$. The above definition is independent of one's choice of a basis $\mathcal{B}$ of V.

The right-hand side of the above equation seems to depend on a basis, but the left-hand side does not depend on any basis.

To show that the above definition does not depend upon a choice of a basis $\mathcal{B}$, we make use of the following identity:

$$\operatorname{tr}(ABA^{-1}) = \operatorname{tr}(B).$$

This is a consequence of the formula whereby:

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

The matrix ${}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{B}'}$ is nonsingular. Now, observe that:

$$\begin{aligned} {}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{B}'}^{-1} &= {}_{\mathcal{B}'}[\mathrm{id}]_{\mathcal{B}} \\ A_{\mathcal{B}}(g) &= {}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{B}'} \\ A_{\mathcal{B}'}(g) &= {}_{\mathcal{B}'}[\mathrm{id}]_{\mathcal{B}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \operatorname{tr}(A_{\mathcal{B}(g)}) &= \operatorname{tr}\left({}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{B}'} A_{\mathcal{B}'}(g) {}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{B}'}^{-1}\right) \\ &= \operatorname{tr}(A_{\mathcal{B}'}(g)). \end{aligned}$$

This implies that χ^V is independent of the basis.

Choose a "good" basis so that $A_{\mathcal{B}} \colon G \to U_{n\times n}(\mathbb{C})$. In this case, we have that:

$$\begin{array}{ccc} \chi^V(g) & = & \chi^V(hgh^{-1}) \\ \| & & \| \\ \operatorname{tr}(A_{\mathcal{B}}(g)) & = & \operatorname{tr}(A_{\mathcal{B}}(h)A_{\mathcal{B}}(g)A_{\mathcal{B}}(h^{-1})) \end{array}$$

Theorem 11.3. *The character χ^V is constant on conjugacy classes of the corresponding group* $\operatorname{dom}(\chi^V)$.

Remark 11.4. Observe that $A_{\mathcal{B}}(h^{-1}) = A_{\mathcal{B}}(h)^{-1}$.

Proposition 11.5. *If $M \cong N$, then $\chi^M(g) = \chi^N(g)$.*

Amazing fact:

$$\boxed{\forall g\ \chi^M(g) = \chi^N(g) \implies M \cong N.}$$

Remark 11.6. The above result shows that we basically "*just need the sum of the diagonals*" when dealing with representations. This gives us one number for each conjugacy class. Then you can tell if these G-modules are isomorphic or not. This character-theoretic approach solves so many problems in algebraic combinatorics.

Remark 11.7. One may be tempted to use the word "representation" casually in mathematics. Do not do this. The word "representation" has a specific meaning in mathematics.

Remark 11.8. Representation theory has been an active research area since around the turn of the 20th century since Schur's thesis was in 1905. That may not be the first reference on the subject, but historically it does seem to solidify the place of representation theory as a well defined subject.