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Modules Over A Ring

For the purpose of this presentation we will be considering a commutative ring A with unity (1_A)

Modules

Definition: M is an A-**module** is an abelian group w.r.t. addition and with a mapping
⍰: $A \times M \longrightarrow M$ such that ⍰ (a,x) is ax that satisfies the following conditions $\forall$ a,b $\in$ A and $\forall$ x,y $\in$ M :

(1) $a(x + y) = ax + ay$

(2) $(a + b)x = ax + bx$

(3) $(ab)x = a(bx)$

(4) $1_A x = x$

Definition: M, N modules, $f: M \longrightarrow N$ is an A-**module homomorphism** if it satisfies $\forall$ a $\in$ A and $\forall$ x,y $\in$M:

(1) $f(x+y) = f(x) + f(y)$

(2) $f(ax) = af(x)$

Definition: M, N modules. $Hom_A(M,N)$ is the set of A-module homomorphisms from M to N.

Submodules

Definition: M'$\subseteq$M is a submodule if it is a subgroup of M and it is closed under multiplication. That is, $\forall$ a $\in$ A, and $\forall$ x $\in$M then ax $\in$ M.

Prop . (The Submodule Criterion) Let A be a ring and let M be an A-module.
A subset M' of M is a submodule of M if and only if
(1) $M' \neq \emptyset$, and
(2) $x + ay \in M'$ for all $a \in A$ and for all $x , y \in M'$.

Proof

$\Longrightarrow$ M' is a submodule, then $0 \in M'$ so $N \neq \emptyset$. Also M' is closed under addition and is sent to itself under the action of elements of A.

$\Longleftarrow$ Assume (1) and (2). Let $a = -1$ and apply the subgroup criterion (in additive form) to see that M' is a subgroup of M. In particular, $0 \in M'$. Now let $x = 0$ and apply hypothesis (2) to see that M' is sent to itself under the action of R.

Quotient Modules

Definition: M a module and $M' \subseteq M$ a submodule then M/M' is an A-module if we define an action, $a(x + M') = ax + M'$ for every $a \in A$ and $(x + M') \in M/M'$.

Prop. There is a mapping $\Phi: M \longrightarrow M/M'$ such that $\Phi(x) = x + M'$. It is an A-module homomorphism and $\ker(\Phi)=M'$.

Proof.

- M abelian additive group $\Longrightarrow$ M/M' is an additive abelian group
- Is the action of $a \in A$ on $x + M'$ well defined? Let $x + M' = y + M' \Longrightarrow x - y \in M'$ and M' submodule $\Longrightarrow a(x - y) \in M' \Longrightarrow ax - ay \in M' \Longrightarrow ax + M' = ay + M' \Longrightarrow$ action is well defined.
- Check the axioms of the module. For example, $\forall a1, a2 \in A$ and $x + M' \in M/M'$ then $(a_1a_2)(x + M') = (a_1a_2x) + M' = a_1 (a_2x + M') = a_1 (a_2(x + M'))$. This proves the 3rd condition in the definition of an A-module
- $\Phi: M \longrightarrow M/M'$ is the projection of a an abelian group onto an abelian group. Every subgroup of an abelian group is normal $\Longrightarrow \ker(\Phi) = M'$ by exercise 11 that we did in the beginning of the term.
- $\Phi(x + y) = x + y + M' = x + M' + y + M' = \Phi(x) + \Phi(y)$ and $\Phi(ax) = ax + M' = a(x + M') = a\,\Phi(x) \Longrightarrow \Phi$ is an A- module homomorphism

More Definitions

For $f: M \longrightarrow N$ an A-module homomorphism then the **kernel** of f is the set

$$\mathrm{Ker}(f) = \{x \in M \mid f(x) = 0\}$$

The kernel of f is a submodule of M. The **image** of f is the set

$$\mathrm{Im}(f)=f(M)$$

The image of f is a submodule of N. The **cokernel** of f is the set

$$\mathrm{Coker}(f) = N/\mathrm{Im}(f)$$

Exact Sequences of Modules

A sequence of modules and module homomorphisms

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \longrightarrow \cdots$$

is called **exact** at M_i if $\mathrm{Im}(f_i) = \mathrm{Ker}(f_{i+1})$.

Equivalently (i) $g \circ f = 0$

(ii) $\forall\ x_i \in M_i$, if $f_{i+1}(x_i) = 0$ then $\exists x_{i-1} \in M_{i-1}$ s.t. $x_i = f_i(x_{i-1})$

If the above sequence has an infinite amount of modules and module homomorphisms then it is a **long exact sequence.**

Simple examples of exact sequences

$0 \longrightarrow M' \xrightarrow{f} M$ is exact at M' iff $\mathrm{Ker}(f)=\mathrm{Im}(0 \longrightarrow M')= 0$. This implies that f is a monomorphism(injective).

$M \xrightarrow{g} M'' \longrightarrow 0$ is exact at M'' iff $\mathrm{Im}(g)=\mathrm{Ker}(M'' \longrightarrow 0)= M''$. This implies that g is an epimorphism(surjective).

A **short exact sequence** is of the form

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

An s.e.s is exact at M', M, M'', f is injective, and g is surjective.

Exact Functors

A **functor** is a mapping between categories (in this case A-modules)

An **exact functor** is a functor that preserves exact sequences. For example, F is an exact functor if a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \Longrightarrow 0 \longrightarrow F(M') \longrightarrow F(M) \longrightarrow F(M'') \longrightarrow 0$$

is exact.

Some functors are left or right exact. Left exact means that

(i)$0 \to M' \to M \to M'' $ exact implies $0 \to F(M') \to F(M) \to F(M'')$ exact.

Or

(ii) $M' \to M \to M'' \to 0$ exact implies $0 \to F(M'') \to F(M) \to F(M')$ exact

A good example of this is Hom_A which is left exact. In particular $Hom_A(-,N)$ satisfies (ii) such

that (*) $M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$ exact implies

(**) $0 \longrightarrow Hom_A(M'', N) \xrightarrow{v^*} Hom_A(M, N) \xrightarrow{u^*} Hom_A(M', N)$ is exact for every N.

What is v* and u*?

$$\begin{tikzcd} & N \\ M \arrow[ur, "v^*"] \arrow[r, "v"'] & M'' \arrow[u, "a''"'] \end{tikzcd} \qquad \begin{tikzcd} & N \\ M' \arrow[ur, "u^*"] \arrow[r, "u"'] & M \arrow[u, "a"'] \end{tikzcd}$$

Proof 1) Exact at $Hom_A(M'', N)$ iff v* is injective. Take a in $Hom_A(M'', N)$ and assume $v^*(a) = 0$ iff $a \circ v = 0$ iff $a(v(m)) = 0$, for every m, v is surjective. Since (*) is exact at M'' this implies that every element of M'' is of the form v(m) for some m in M. Thus $a(m'') = 0$ for every m'' in M'' implies that a= 0. Then $\ker(v^*) = 0$ implies v* injective.
2) Exact at $Hom_A(M,N)$. i) $u^* \circ v^*$ iff $(v \circ u)^* = 0$. True because $v \circ u$=in (*) since (*) exact at M. ii) take any b in $Hom_A(M,N)$, assume $u^*(b) = 0$ iff $b \circ u = 0$ which implies that b vanishes on ker(v) since ker(v) = im(u) which implies b factors through $M \longrightarrow M/\ker(v)$ which is isomorphic to M'' which implies there exists $b'': M'' \longrightarrow N$ such that $b'' \circ v = b$. Conclusion, the sequence is exact at $Hom_A(M,N)$.

Note: $Hom_A(N,-)$ satisfies (i) for every N

The Snake Lemma

$$
\begin{tikzcd}
 & 0 \arrow[r] & \ker(f') \arrow[r, "u^*"] \arrow[d, "i_{f'}"] & \ker(f) \arrow[r, "v^*"] \arrow[d, "i_f"] & \ker(f'') \arrow[d, "i_{f''}"] \arrow[dddll, "d"] & \\
 & 0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, "f'"] & M \arrow[r, "v"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
 & 0 \arrow[r] & N' \arrow[r, "u'"] \arrow[d, "p'"] & N \arrow[r, "v'"] \arrow[d, "p"] & N'' \arrow[r] \arrow[d, "p''"] & 0 \\
 & & \mathrm{coker}(f') \arrow[r, "u^{*\prime}"] & \mathrm{coker}(f) \arrow[r, "v^{*\prime}"] & \mathrm{coker}(f'') \arrow[r] & 0
\end{tikzcd}
$$

Assumptions

(i)The rows are short exact sequences

(ii) all the squares commute

Then the sequence

$$0 \longrightarrow \ker(f') \xrightarrow{u^*} \ker(f) \xrightarrow{v^*} \ker(f'') \xrightarrow{d} \mathrm{coker}(f') \xrightarrow{u^{*\prime}} \mathrm{coker}(f) \xrightarrow{v^{*\prime}} \mathrm{coker}(f'') \longrightarrow 0$$

is exact at all 6 of the modules and a connecting homomorphism $d: \ker(f'') \longrightarrow \mathrm{coker}(f')$ exists.

How do we define $d: \ker(f'') \longrightarrow \mathrm{coker}(f')$?

Let $m'' \in \ker(f'')$ which is contained in M''. The top row is exact thus v is surjective which implies there exists an m in M s.t. $v(m) = m''$ (not necessarily unique). Let $n := f(m)$ implies $v'(n) = v'(f(m)) = f''(m'') = 0$ because m'' is an element of $\ker(f'')$ and therefore $f(m)$is in the kernel of v'. The bottom row is exact which implies that there exists a unique n' in N' by injectivity of $u'(n') = f(m) = n$. Then define

$$d(m''):= p'(n')$$

Exactness at ker(f)

Since the square with edges $v \circ i_f$ commutes, then $v \circ i_f \circ u^*(a) = i_{f''} \circ v^* \circ u^*(a) = 0$.

Proof: (i)to be verified: $v^*\circ u^*=0$. Take a in $\ker(f')$. Then, by commutativity of squares $i_f\circ u^*(a) = u\circ i_{f'}(a)$ then $v\circ i_f \circ u^*(a) = 0$ by exactness. $I_{f''}$ is injective therefore $v^*\circ u^*=0$

(ii)to be verified: $\ker(v^*)\subseteq \mathrm{im}(u^*)$. Take b in $\ker(v^*)$. Then $v^*(b)= 0 \in\ker(f'')$. $i_{f''}$ is injective thus $0\in M''$. Then $v\circ i_f(b) = 0$ by commutativity of the squares. $i_f(b) \in\ker(v) = \mathrm{im}(u)$ by exactness. Then $i_f(b) = u(c)$ for some $c \in M'$. $i_{f'}$ is injective therefore there exists $c'\in\ker(f')$ where $u^*(c') = b$