PRACTICE FINAL

(1) Show that a group of order $2001 = 3 \cdot 23 \cdot 29$ must contain a normal cyclic subgroup of index 3.

(2) For $n \geq 3$, the dihedral group D_n of order $2n$ is given by $D_n = \langle a, b | a^n = b^2 = 1, ba = a^{-1}b \rangle$.
   (a) What are the conjugacy classes of D_n?
   (b) When n is odd, how many Sylow 2-subgroup does D_n have?
   (c) What are the composition factors of D_n?
   (d) How many irreducibles does D_n have? What are the dimensions?

(3) For R a U.F.D., show that a non-zero element is prime if and only if it is irreducible.

(4) For a ring R we say that $a \in R$ is right quasi-regular (r.q.r.) if $a + x = ax$ has a solution $x = b \in R$. The unity is never r.q.r. and 0 is always r.q.r. Show the following:
   (a) If a^2 is r.q.r., then a is r.q.r.
   (b) If R is a division ring, then the unity is the only non-r.q.r elements of R.
   (c) R is a division ring if and only if all elements of R but one are r.q.r.

(5) Let $\phi : G \to Aut(M)$ be a representation of a finite group G and let χ be its associated character.
   (a) Show that $\frac{1}{|G|} \sum_{g \in G} \chi(g)$ is equal to the number of time the trivial representation appears in decomposition of ϕ into irreducibles.
   (b) Show that if $\frac{1}{|G|} \sum_{g \in G} ||\chi(g)||^2 = 3$, then ϕ is the direct sum of three distinct irreducible representations.

(6) Let R be a commutative ring with identity 1 and M a finitely generated left R-module.
   (a) Show that for any $m \in M$, the set $Ann(m) = \{a \in R : am = 0\} \subseteq R$ is an ideal.
   (b) Let $P = \{Ann(m) : 0 \neq m \in M\}$. Show that a maximal element of P must be a prime ideal.