Georgios Katsimpas September 21, 2016

Applied Algebra Assignment

Let $(G, \circ_G)$ be a group and H be a normal subgroup of G. On the set G/H of all left cosets of H (that is, $G/H = \{g \cdot H \,/\, g \in G\}$ where for $g \in G$, we have that $g \cdot H = \{g \cdot h \,/\, h \in H\}$) we define the following operation:

$$\circ_{G/H} : G/H \times G/H \to G/H$$

such that for $g_1, g_2 \in G$ we have that

$$(g_1 \cdot H) \circ_{G/H} (g_2 \cdot H) = (g_1 \cdot g_2) \cdot H$$

Then the pair $(G/H, \circ_{G/H})$ is a group.
We observe that $e_{G/H} = e_G \cdot H$. Indeed, if $g \in G$, then:

$$(g \cdot H) \circ_{G/H} (e_G \cdot H) = (g \cdot e_G) \cdot H = g \cdot H$$
and $$(e_G \cdot H) \circ_{G/H} (g \cdot H) = (e_G \cdot g) \cdot H = g \cdot H$$

Also, for $g \in H$, the inverse of $g \cdot H \in G/H$ is $g^{-1} \cdot H$. Indeed:
$$(g \cdot H) \circ_{G/H} (g^{-1} H) = (g^{-1} H) \circ_{G/H} (g \cdot H) = (g^{-1} g) \cdot H = e_G \cdot H$$

We define a map $\phi : G \to G/H$ such that $\phi(g) = g \cdot H$.
Show that:
(i) ϕ is a group homomorphism
(ii) $\operatorname{Ker}\phi = H$

$\boxed{\text{Proof}}$ (i) Let $g_1, g_2 \in G$. Then:
$$\phi(g_1) \circ_{G/H} \phi(g_2) = (g_1 \cdot H) \circ_{G/H} (g_2 \cdot H) = (g_1 \cdot g_2) \cdot H = \phi(g_1 \cdot g_2)$$

Hence, ϕ is a group homomorphism.

(ii) We will show that $H = \ker\phi = \{g \in G \mid \phi(g) = e_{G/H}\} = \{g \in G \mid \phi(g) = e_G \cdot H\}$

Let $g \in \ker\phi \Rightarrow \phi(g) = e_G \cdot H \Leftrightarrow g \cdot H = e_G \cdot H = H$

$\Rightarrow g \cdot H = H$

Since $(g \cdot e_H) \in gH \Rightarrow (g \cdot e_G) \in gH = H \Leftrightarrow g \in g \cdot H = H \Rightarrow g \in H$

Thus $g \in H$ and we have proved that $\ker\phi \subseteq H$.

Let $h \in H$. Then $\phi(h) = h \cdot H = \{h \cdot h' \mid h' \in H\} = H$ (since H as a subgroup is closed under multiplication)

Thus $\phi(h) = H = e_G \cdot H = e_{G/H}$ and we have proved that $H \subseteq \ker\phi$.

This completes the proof. $\blacksquare$