Definition Normal Subgroup

A subgroup H of a group G is called a *normal subgroup* of G if $aH = Ha$ for all a in G.

Theorem Normal Subgroup Test

A subgroup H of G is normal in G if and only if $ghg^{-1} \subseteq H$ for all $g \in G$.

Prove that $\boldsymbol{ker(\phi) \trianglelefteq G}$, where $\boldsymbol{ker(\phi) = \{g \in G | \phi(g) = e_H\}}$.

Suppose that $\phi: G \rightarrow H$ is a homomorphism.

We first need to prove that ker (ϕ) is a subgroup of G by showing that ker (ϕ) is a nonempty subset of G, closed under the operation and closed under taking inverses.

$$\phi(e_G) = e_H$$

This shows that $e_G \in$ ker (ϕ), so ker (ϕ) is a nonempty.

Let $a, b \in$ ker (ϕ). Suppose $\phi(a) = \phi(b) = e$, then

$$\phi(ab) = \phi(a)\phi(b) = ee = e$$

This shows $ab \in$ ker (ϕ), so ker (ϕ) is closed under the operation.

And for all $a \in G$, assume $\phi(a) = e$. Then

$$\phi(aa^{-1}) = \phi(a)\phi(a^{-1}) = \phi(a)\phi(a)^{-1} = ee^{-1} = e$$

This shows that if $a \in$ ker (ϕ) then $a^{-1} \in$ ker (ϕ), so ker (ϕ) closed under inverses.

Thus, ker (ϕ) is a subgroup of G.

Next, we need to show that $ghg^{-1} \in H$, which means, we need to show that $\phi(ghg^{-1}) = e_H$.

Let $g \in G$ and $h \in$ ker (ϕ), then

$$\phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g^{-1}) = \phi(g)e_H\phi(g^{-1}) = \phi(g)\phi(g^{-1}) = \phi(gg^{-1}) = \phi(e_G) = e_H$$

as desired.