A BIG EXAMPLE - MATH 6161

MAY 22, 2003

I want to work through a complete example of the theorems that we have been talking about for the last two weeks. This exam is a perfect opportunity to do this as a group. The following questions tell a story about how to apply the theory that we have been learning. Each question represents a step in a fairly large problem so it is important that all steps are correct before we proceed to the next question. In addition, I expect that this example will take a long time so SPEED is a factor. This is tag team math. Go as fast as possible.

Here are the rules of the game:

(1) You will be assigned some of the questions below by draw of a number.
(2) You will be asked to write the answer on the blackboard for the problem(s) that you draw.
(3) In addition, you are to actively help the person at the blackboard for the problem before and the problem after if they should get stuck and/or need a hint.
(4) Try to be consistent with notation from problem to problem.
(5) I will record the answer and distribute it next time.
(6) I hope to have my computer with me and you may ask me to do calculations that may speed up the process.

Let D_n be the group generated by two elements x, y satisfying the relations $x^n = e$, $y^2 = e$, $yx = x^{-1}y$. We will take $\{e, x, x^2, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y\}$ as a complete set of representatives of this group.

(1) What are the conjugacy classes of D_n?
(2) Find all of the 1-dimensional representations for D_n. (Hint: 1-dimensional representations will be maps from $D_n \to \mathbb{C}$ so you not only have that the characters are orthonormal but that they are group homomorphisms as well)
(3) What are the dimensions of the irreducible characters of D_n?
(4) Give a character table for D_4 and D_5.

Consider now the D_4 module $\mathcal{L}\{a_0, a_1, b_0, b_1\}$ with the action $xa_i = a_{1-i}$ and $ya_i = b_i$.

(5) Prove that this does in fact define a module.
(6) Find the action of the group elements on the basis and compute the character of this module.
(7) Find the multiplicities of each of the irreducible modules in this module by computing the scalar products of the character of this module with the character table.

(8) Find the decomposition of this module into irreducible components (find a decomposition into irreducible submodules).

Now consider the space of polynomials in the variables a_0, a_1, b_0, b_1. Let $P = \mathbb{C}[a_0, a_1, b_0, b_1]$ and let P_k be the polynomials of degree k so that $P = \bigoplus_{k \geq 0} P_k$.

(9) What is the dimension of the polynomials of degree k?
(10) Define an action on P so that P_k is a submodule for any $k \geq 0$.
(11) For $k = 2$ find the character of this module
(12) For $k = 2$ find the multiplicities of each of the irreducibles.

I thought about asking you to find the decomposition of the module into irreducible submodules for any $k > 0$. We should be able to do this but it will take a while. Instead I will ask you some questions about induction and restriction because these are important subjects that didn't come up in the example above.

(13) Write down the character table for $C_4 = \{e, x, x^2, x^4\}$ and show that the restricted irreducible characters of D_4 can be written as sums of characters in C_4.
(14) Considering P_1 as a C_4 module only (with the same action as before on the basis elements) decompose the space into irreducible submodules.

Consider the C_4-module $\mathcal{L}\{a_0, a_1, a_2, a_3\}$ with the action $xa_i = a_{i+1 mod 4}$.

(15) Write down the matrix representation associated to this basis (give all four $4x4$ matricies).
(16) Write down the character and determine how it decomposes into irreducibles.
(17) Find the induced character from C_4 to D_4.
(18) Determine the multiplicities of the irreducible D_4-characters in this character.
(19) Write down a few matricies of the induced representation and try to define a module that is represented by the induced character.