# FINAL EXAM - MATH 6161

JUNE 12, 2003

PART I: written and computational. Instructions: Do any 4 of the following 6 problems.

There are several ways of approaching each of these problems. I will give one or two solutions to each.

(1) Find the value of $\langle p_1^n, h_k h_{n-k}\rangle$. Use this to compute the dimension of the irreducible character $\chi^\mu$ for $\mu$ a two row partition.

$h_k = p_1^k/k! +$ other terms and $h_{n-k} = p_1^{n-k}/(n-k)! +$ other terms so $h_k h_{n-k} = p_1^n/(k!(n-k)!) +$ other terms. $\langle p_1^n, h_k h_{n-k}\rangle = \langle p_1^n, p_1^n/(k!(n-k)!)\rangle = \frac{n!}{k!(n-k)!}$.

$\langle p_1^n, h_k h_{n-k}\rangle = p_{1^n}^\perp(h_k h_{n-k}) = p_{1^{n-1}}^\perp(p_1^\perp(h_k)h_{n-k} + h_k p_1^\perp(h_{n-k})) = \sum_{i=0}^n \binom{n}{i} p_{1^i}^\perp(h_k) p_{1^{n-i}}^\perp(h_{n-k})$ the only term in this sum which is non-zero is for $i = k$ and $p_{(1^k)}^\perp(h_k) = 1$ therefore it is equal to $\binom{n}{k}$.

Now to consider the dimension of the irreducible character indexed by the partition $\mu$ when $\mu$ has only two rows we note that

$$dim\ \chi^{(\mu_1,\mu_2)} = \left\langle p_{1^n}, s_{(\mu_1,\mu_2)}\right\rangle = \langle p_{1^n}, h_{\mu_1}h_{\mu_2} - h_{\mu_1+1}h_{\mu_2-1}\rangle = \binom{n}{\mu_1} - \binom{n}{\mu_1+1}$$

(2) Expand $h_{(3,2,1)}$ in the

(a) $p$-basis

$h_3 = p_3/3 + p_{21}/2 + p_{1^3}/6$, $h_2 = p_2/2 + p_{1^2}/2$ and $h_1 = p_1$. Therefore take the product and find

$$h_{(3,2,1)} = 1/12\, p_{(1^6)} + 1/3\, p_{(21^4)} + 1/6\, p_{(31^3)} + 1/4\, p_{(2^21^2)} + 1/6\, p_{(321)}$$

(b) $e$-basis

$$h_{(3,2,1)} = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & e_1 & e_2 \\ 0 & 1 & e_1 \end{vmatrix} \cdot \begin{vmatrix} e_1 & e_2 \\ 1 & e_1 \end{vmatrix} \cdot e_1$$

Also using the recurrence $h_k = \sum_{i=1}^k (-1)^{i-1} h_{k-i} e_i$, we have $h_1 = e_1$, $h_2 = h_1 e_1 - e_2 = e_{1^2} - e_2$, $h_3 = h_2 e_1 - h_1 e_2 + e_3 = e_{1^3} - 2e_{21} + e_3$.

$$h_{(3,2,1)} = e_{1^6} - 3\, e_{21^4} + e_{31^3} + 2\, e_{2^21^2} - e_{321}$$

(c) $s$-basis

Use the Pieri rule or the combinatorial interpretation in the notes. These are the easiest ways of solving this. $h_3 = s_{(3)}$, $h_{(3,2)} = s_{(3)}h_2 = s_{(3,2)} + s_{(4,1)} + s_{(5)}$, and finally:

$$\begin{aligned} h_{(3,2,1)} &= h_{(3,2)}h_1 = (s_{(3,2)} + s_{(4,1)} + s_{(5)})h_1 \\ &= (s_{(3,2,1)} + s_{(3,3)} + s_{(4,2)}) + (s_{(4,1,1)} + s_{(4,2)} + s_{(5,1)}) + (s_{(5,1)} + s_{(6)}) \\ &= s_{(3,2,1)} + s_{(3,3)} + 2\, s_{(4,2)} + s_{(4,1,1)} + 2\, s_{(5,1)} + s_{(6)} \end{aligned}$$

There is also one term for each column strict tableau with content $(3, 2, 1)$.

$$\begin{array}{lll} 3 & & \\ 2 & 2 & \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{lll} 2 & 2 & 3 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{llll} 3 & & & \\ 2 & & & \\ 1 & 1 & 1 & 2 \end{array} \qquad \begin{array}{llll} 2 & 3 & & \\ 1 & 1 & 1 & 2 \end{array}$$

$$\begin{array}{llll} 2 & 2 & & \\ 1 & 1 & 1 & 3 \end{array} \qquad \begin{array}{lllll} 2 & & & & \\ 1 & 1 & 1 & 2 & 3 \end{array} \qquad \begin{array}{lllll} 3 & & & & \\ 1 & 1 & 1 & 2 & 2 \end{array} \qquad \begin{array}{llllll} 1 & 1 & 1 & 2 & 2 & 3 \end{array}$$

(3) Use the following identity,

$$\Delta(p_\mu) = \sum_\lambda \frac{p_\lambda}{z_\lambda} \otimes \left(p_\lambda^\perp p_\mu\right)$$

to show in general that for any dual bases $\{a_\lambda\}_\lambda$ and $\{b_\lambda\}_\lambda$, and for any $f \in \Lambda$,

$$\Delta(f) = \sum_\lambda a_\lambda \otimes \left(b_\lambda^\perp f\right).$$

$$\begin{aligned} \Delta(p_\mu) &= \sum_\lambda \sum_{\nu \vdash |\lambda|} \left\langle \frac{p_\lambda}{z_\lambda}, b_\nu \right\rangle a_\nu \otimes \left(p_\lambda^\perp p_\mu\right) \\ &= \sum_\nu \sum_{\lambda \vdash |\nu|} a_\nu \otimes \left(\left\langle \frac{p_\lambda}{z_\lambda}, b_\nu \right\rangle p_\lambda^\perp p_\mu\right) \\ &= \sum_\nu a_\nu \otimes \left(b_\nu^\perp p_\mu\right) \end{aligned}$$

Now for any $f \in \Lambda$, $f = \sum_\gamma c_\gamma p_\gamma$ and we have

$$\Delta(f) = \sum_\gamma c_\gamma \Delta(p_\gamma) = \sum_\gamma c_\gamma \sum_\mu a_\mu \otimes \left(b_\mu^\perp p_\gamma\right) = \sum_\mu a_\mu \otimes \left(\sum_\gamma c_\gamma b_\mu^\perp p_\gamma\right) = \sum_\mu a_\mu \otimes \left(b_\mu^\perp f\right)$$

(4) Calculate $\left\langle h_{(3,3)}, h_{(3,2,1)} \right\rangle$, or equivalently, find the coefficient of $m_{(3,3)}$ in $h_{(3,2,1)}$.

Method 1 would be to expand these functions in the $p$-basis and compute the scalar product. The problem is there is a lot of room for error. Fine if you are a computer, but it is easy to make a mistake if you are not. The expansion we did in the first problem should help.

Alternatively we compute the expansion of $h_3 h_3$ in variables to get a value and find the coefficient of the monomial $x_1^3 x_2^2 x_3$.

$$h_3[x_1 + x_2 + x_3] = {x_1}^3 + {x_2}^3 + {x_3}^3 + x_1 {x_2}^2 + x_1 {x_3}^2 + x_2 {x_1}^2 + x_2 {x_3}^2 + x_3 {x_1}^2 + x_3 {x_2}^2 + x_1 x_2 x_3$$

How can we get a monomial $x_1^3x_2^2x_3$? This is the number of ways of filling a Young diagram for the shape $(3,2,1)$ with three 1s and three 2s such that the values are increasing in the rows. Like the following 6 diagrams

$$\begin{array}{|c|c|c|}\hline 2 \\ \cline{1-1} 2 & 2 \\ \cline{1-2} 1 & 1 & 1 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 2 \\ \cline{1-1} 2 & 1 \\ \cline{1-2} 1 & 1 & 2 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1 \\ \cline{1-1} 2 & 2 \\ \cline{1-2} 1 & 1 & 2 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 2 \\ \cline{1-1} 1 & 1 \\ \cline{1-2} 1 & 2 & 2 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1 \\ \cline{1-1} 1 & 2 \\ \cline{1-2} 1 & 2 & 2 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1 \\ \cline{1-1} 1 & 1 \\ \cline{1-2} 2 & 2 & 2 \\ \hline\end{array}$$

This combinatorial description is symmetric and we can also compute the number of Young diagrams with three 1s, two 2s and one 3 of shape $(3,3)$ such that the entries are increasing in the rows.

$$\begin{array}{|c|c|c|}\hline 2 & 2 & 3 \\ \hline 1 & 1 & 1 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 1 & 1 & 2 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 1 & 1 & 3 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 1 & 2 & 2 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 1 & 2 & 3 \\ \hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & 3 \\ \hline\end{array}$$

(5) Determine the coefficient of $z^0, z^1, z^2, z^3$ and $z^4$ in the expression $m_{(3,2,1)}[X+z]$.

This can be done by definition, first expand the symmetric function in the $p$-basis, replace each $p_k$ by a $\sum_i x_i^k + z^k$ and then take a coefficient, but we don't want to do that because it would take forever.

Method 2 would be to look in the notes at the example where we have $f[X+z]\big|_{z^k} = h_k^\perp f[X]$. Therefore the coefficient of $z^0$ in $m_{(3,2,1)}[X+z]$ is $m_{(3,2,1)}[X]$, the coefficient of $z^k$ is $h_k^\perp m_{(3,2,1)}[X]$. If you want to reduce $h_k^\perp m_{(3,2,1)}[X]$ we can compute the coefficient of $m_\lambda[X]$ by taking the scalar product with $h_\lambda[X]$. That is, $\sum_\lambda \left\langle h_k^\perp m_{(3,2,1)}, h_\lambda \right\rangle m_\lambda[X] = \sum_\lambda \left\langle m_{(3,2,1)}, h_k h_\lambda \right\rangle m_\lambda[X]$ is the coefficient of $z^k$ in $m_{(3,2,1)}[X+z]$ and since the $h_\mu$ and $m_\lambda$ bases are dual we have that the coefficient of $z^1$ is $m_{(3,2)}[X]$, $z^2$ is $m_{(3,1)}[X]$, $z^3$ is $m_{(2,1)}[X]$ and $z^4$ will be 0.

But there is even an easier way to look at this problem. $m_\lambda[X+z]$ is a monomial symmetric function in the $x_i$ variables and $z$ variable. This means that it is equal to

$$m_{(3,2,1)}[X+z] = \sum_{\alpha \sim (3,2,1)} z^{\alpha_1} x_1^{\alpha_2} x_2^{\alpha_3} \cdots$$

In general, if $\alpha_1 = 0$ then $(\alpha_2, \alpha_3, \ldots) \sim (3,2,1)$ and the coefficient of $z^0$ is $m_{(3,2,1)}[X]$, if $\alpha_1 = 1$ then $(\alpha_2, \alpha_3, \ldots) \sim (3,2)$ and the coefficient of $z^1$ is $m_{(3,2)}[X]$, if $\alpha_1 = 2$ then $(\alpha_2, \alpha_3, \ldots) \sim (3,1)$ and the coefficient of $z^2$ is $m_{(3,1)}[X]$, if $\alpha_1 = 3$ then $(\alpha_2, \alpha_3, \ldots) \sim (2,1)$ and the coefficient of $z^3$ is $m_{(2,1)}[X]$, if $\alpha_1 = 4$ then $\alpha$ does not sort to $(3,2,1)$ and so the coefficient is 0.

(6) You are given below a table of coefficients of $p_\lambda/z_\lambda$ in $h_\mu$ ($\mu$ indexes the left side of the table and $\lambda$ the row across the top). Use this to calculate the first 6 rows of the character table for $S_6$. Explain in a few words how you can easily find the last 5 rows from the first 5.

| | $(1^6)$ | $(2,1^4)$ | $(2^2,1^2)$ | $(3,1^3)$ | $(2^3)$ | $(3,2,1)$ | $(4,1^2)$ | $(3^2)$ | $(4,2)$ | $(5,1)$ | $(6)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $(6)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $(5,1)$ | 6 | 4 | 2 | 3 | 0 | 1 | 2 | 0 | 0 | 1 | 0 |
| $(4,2)$ | 15 | 7 | 3 | 3 | 3 | 1 | 1 | 0 | 1 | 0 | 0 |
| $(4,1,1)$ | 30 | 12 | 2 | 6 | 0 | 0 | 2 | 0 | 0 | 0 | 0 |
| $(3,3)$ | 20 | 8 | 4 | 2 | 0 | 2 | 0 | 2 | 0 | 0 | 0 |
| $(3,2,1)$ | 60 | 16 | 4 | 3 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $(3,1,1,1)$ | 120 | 24 | 0 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(2,2,2)$ | 90 | 18 | 6 | 0 | 6 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(2,2,1,1)$ | 180 | 24 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(2,1,1,1,1)$ | 360 | 24 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $(1,1,1,1,1,1)$ | 720 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Since $s_{(6)} = h_6$ the first row is done. Since $s_{(5,1)} = h_5h_1 - h_6$ we only need to subtract the first row from the second to obtain the character corresponding to $(5,1)$. Similarly, we have $s_{(4,2)} = h_4h_2 - h_5h_1$, $s_{(3,3)} = h_3h_3 - h_4h_2$ and so the third row of the character table will be $3^{rd}$ row above minus the $2^{nd}$ and the fifth row of the character table will be the $5^{th}$ row of the table above minus the $3^{rd}$. $h_{(4,1,1)} = s_{(4,1,1)} + s_{(4,2)} + 2s_{(5,1)} + s_{(6)}$. This means that to compute the $4^{th}$ row of the character table, take the $4^{th}$ row of the table above and subtract the first row, third row and 2 times the second row of the character table. Finally to compute $s_{(3,2,1)}$ we know the expansion of $h_{(3,2,1)}$ in the Schur basis from the second problem. But notice from that expansion that $s_{(3,2,1)} = h_{(3,2,1)} - h_{(4,1,1)} - s_{(3,3)} - s_{(4,2)}$ which says that the $6^{th}$ row of the character table will be the $6^{th}$ row of the table above minus the $4^{th}$ row minus the $5^{th}$ and $3^{rd}$ rows of the character table.

| | $(1^6)$ | $(2,1^4)$ | $(2^2,1^2)$ | $(3,1^3)$ | $(2^3)$ | $(3,2,1)$ | $(4,1^2)$ | $(3^2)$ | $(4,2)$ | $(5,1)$ | $(6)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $(6)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $(5,1)$ | 5 | 3 | 1 | 2 | $-1$ | 0 | 1 | $-1$ | $-1$ | 0 | $-1$ |
| $(4,2)$ | 9 | 3 | 1 | 0 | 3 | 0 | $-1$ | 0 | 1 | $-1$ | 0 |
| $(4,1,1)$ | 10 | 2 | $-2$ | 1 | $-2$ | $-1$ | 0 | 1 | 0 | 0 | 1 |
| $(3,3)$ | 5 | 1 | 1 | $-1$ | $-3$ | 1 | $-1$ | 2 | $-1$ | 0 | 0 |
| $(3,2,1)$ | 16 | 0 | 0 | $-2$ | 0 | 0 | 0 | $-2$ | 0 | 1 | 0 |

Finally the bottom half of this table can be obtained by multiplying the top half by the sign character which is given as

| | $(1^6)$ | $(2,1^4)$ | $(2^2,1^2)$ | $(3,1^3)$ | $(2^3)$ | $(3,2,1)$ | $(4,1^2)$ | $(3^2)$ | $(4,2)$ | $(5,1)$ | $(6)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $(1^6)$ | 1 | $-1$ | 1 | 1 | $-1$ | $-1$ | $-1$ | 1 | 1 | 1 | $-1$ |