HOMEWORK PROBLEMS - MATH 6161

A RUNNING LIST OF HOMEWORK PROBLEMS

(1) If a group G acts on a set S and s is in S, then the *stablizer* of s is $G_s = \{g \in G | gs = s\}$. The *orbit* of s is the set $O_s = \{gs | g \in G\}$.
   (a) Prove that G_s is a subgroup of G.
   (b) Find a bijection between cosets of G/G_s and elements of O_s.
   (c) Show that $|O_s| = |G| / |G_s|$ and use this to show that the number of elements conjugate to an element g is equal to $|G| / |Z_s|$ where $Z_s = \{h \in G | hgh^{-1} = g\}$

(2) If X is a matrix representation of a group G, then its kernel is the set $N = \{g \in G | X(g) = I\}$. A representation is faithful if it is one-to-one.
   (a) Show that N is a normal subgroup of G and find a condition of on N equivalent to the representation being faithful.

(3) (a) If $X(g) = [b_{ij}(g)]_{1 \leq i,j \leq d}$ is a matrix representation of G and $Y(g) = [a(g)]$ is a one dimensional matrix representation of G then show that $X'(g) = [a(g)b_{ij}(g)]_{1 \leq i,j \leq d}$ is a matrix representation of G
   (b) Show that if X is irreducible then X' is irreducible.

(4) Consider the group S_n and the subgroup A_n consisting of the even permutations. Give a description of the action of the symmetric group elements on the cosets. Give the matrix representation corresponding to the action on the formal linear span of the cosets (the coset representation). Find a basis for this representation broken into irreducible submodules.

(5) The *hyperoctahedral group* B_n consists of all $n \times n$ signed permutation matrices (matrices with exactly one non-zero entry in each row and column), and in which these non-zero entries are in ± 1. B_n has order $2^n n!$. The representation of B_n by these matrices is called the defining representation.
   (a) Prove that the defining representations is irreducible.
   (b) Find four distinct one-dimensional representations of B_n. (Hint: they all take on the values ± 1).

(6) Let V be a G module and $W \subseteq V$ be a submodule. Prove that the quotient space V/W is also a G module.

(7) Let V and W be G modules. Prove that $Hom(V, W)$ is a G-module with the action for $\phi \in Hom(V, W)$, $g \cdot \phi = g_W \phi g_V^{-1}$ where g_V (resp. g_W) is the action of g on V (resp. W).

(8) Show that if S_3 acts on the set $\mathcal{L}\{x_1, x_2, x_3\}$ by $\sigma(x_i) = x_{\sigma(i)}$ then $\mathcal{L}\{x_1 - x_2, x_2 - x_3\}$ is an irreducible submodule.
   Bonus: More generally show that $\mathcal{L}\{x_1 - x_n, x_2 - x_n, \ldots, x_{n-1} - x_n\}$ is an irreducible module of S_n.

(9) If kX is a permutation representation of G derived from an action of G on a finite set X, show that its character $\chi(g)$ is given by $\#\{x \in X : gx = x\}$, the number of fixed points of g. Show that the multiplicity of the trivial representation in kX is the number of G-orbits in X and deduce

$$\# \text{ orbits of } X \text{ in } G = \frac{1}{|G|} \sum_{g \in G} \#\{x \in X : gx = x\}$$

Note: Those of you who took Math 4160 with me this term should recognize this as Polya's Theorem.

(10) Show that if χ is an irreducible character of G and ρ is a 1-dimensional character then $\chi \cdot \rho(g) := \chi(g)\rho(g)$ is an irreducible character.

(11) Work out the irreducible characters of S_4 by finding two one-dimensional characters and two 3-dimensional characters (consider the decomposition of the defining representation). Use character orthogonality to find the last one.

MAPLE exercises

(1) Write a function which given an n, returns a Maple representation of the dihedral group of order 2n. The elements can be represented as pairs `[a,b]` with $0 \leq$`a`$< n$ and $b = 0$ or 1. Multiplication will be defined as
`[a,0] * [b,c] = [a+b mod n, c]` and
`[a,1] * [b,c] = [a-b mod n, 1-c]`
Make sure that your dihedral groups of order 2 through 12 all satisfy the conditions so that `isgroup( Dn );` is true.

(2) Consider the permutation representation of $C_5 = \{0, 1, 2, 3, 4\}$ where

$$X : g \to \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^g$$

for all g in C_5. Find a matrix T such that $TX(g)T^{-1}$ is diagonal for all $g \in C_5$. Hint: use the built in Maple functions to find eigenvectors.

(3) The section on matrix representations has a function '`ismatrep`' which tests if a map into the set of matrices is a group homomorphism. Find such a map which is not a group homomorphism which makes this function return the value `true` anyway. Correct the function so that it will return true if and only if the map is a homomorphism onto the set of invertible matrices.

(4) For $n = 1, 3, 5$ find two 1-dimensional representations of the dihedral group of order $2n$. For $n = 2, 4, 6$ find four 1-dimensional representations. Find one 2 dimensional representation for the dihedral group of order 6 and 8 which is irreducible.

(5) Using the defining representation, write a class function for the symmetric group which represents an $n - 1$-dimensional irreducible character (you should be able to produce 2 for $n \geq 3$ by using the result of problem #10). Compute the multiplicity of this character in the character of the action of S_4 on the representation given in the examples (the one corresponding to the S_4 action on monomials of the form $x_i x_j$ with $i \neq j$) and the regular representation of S_4 and S_5.

(6) The function `reynoldsop( char, module, vect )` is a function which accepts an irreducible character function, a module, and an element of the module. It returns the action of the Reynolds operator acting on the vector. Consider the group S_3 and the group algebra (which is given as an example function). What happens to the action of `reynoldsop` on each of the basis elements? Find a maximal collection of linearly independent elements which are the image of the function `reynoldsop` for each of the irreducible characters of S_3. What is the dimension of these images? Are they irreducible?