ALFRED YOUNG'S CONSTRUCTION OF THE IRREDUCIBLE REPRESENTATIONS OF $\mathbf{S_n}$

1. The representation matrices

If λ is a partition of n (in symbols $\lambda \vdash n$), a filling of the Ferrers' diagram of λ by the integers $1, 2, ..., n$ will be referred to as an "injective" tableau of shape λ. The collection of all these tableaux will be denoted by $\mathrm{INJ}(\lambda)$. If $T \in \mathrm{INJ}(\lambda)$ and the entries of T increase from left to right in the rows and from bottom to top on the columns then T is said to be standard. The collection of all these tableaux will be denoted by $\mathrm{ST}(\lambda)$.

Given an injective tableau T with n squares, a permutation

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$$

is made to act on T by replacing the entry "i" by "σ_i". The resulting tableau is denoted by σT. We say that σ is in the "*Row Group*" of T if the rows of T and σT differ only by the order of their entries. The "*Column Group*" of T is analogously defined. These two groups will be denoted by $R(T)$ and $C(T)$ respectively.

Given $T_1, T_2 \in ST(\lambda)$ we shall say that T_1 precedes T_2 in the Young "first letter order" and write

$$T_1 \ <_{YFLO} \ T_2$$

if the first entry of disagreement between T_1 and T_2 is higher in T_1 than in T_2. For instance we have

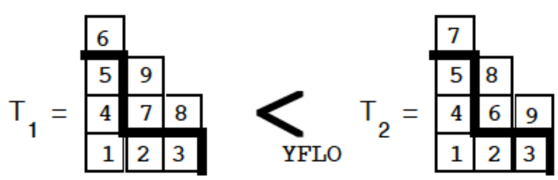
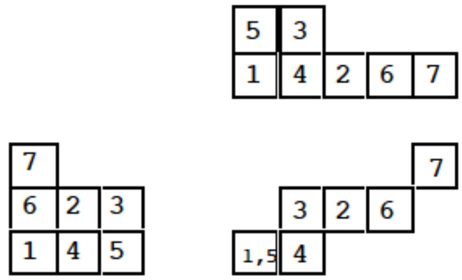


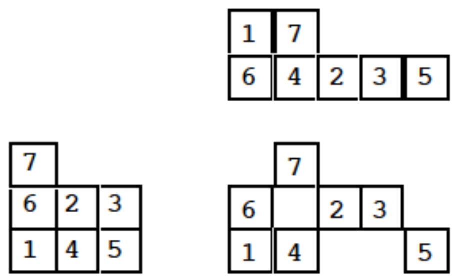
Fig 1

Here T_1 and T_2 agree in the positions of $1, 2, 3, 4, 5$. The first letter of disagreement is 6 and it is "higher" in T_1 than in T_2. The positions of the remaining letters do not matter.

Given two tableaux T_1, T_2, not necessarily of the same shape, we let $T_1 \wedge T_2$ be the diagram obtained by placing in the cell (i, j) the intersection of row i of T_1 with column j of T_2. The diagram below gives an example of this construction



Note that in this case there is a cell which contains more than one entry. When this happens we say that the pair $T_1 \wedge T_2$ is "*bad*". Note that in this next example each cell has at most one entry.



When this happens we say that $T_1 \wedge T_2$ is "*good*" . Note that, in the good case, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ is the shape of T_1 and $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ is the shape of T_2, then for each i we must have

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \;\; \leq \;\; \mu_1 + \mu_2 + \cdots + \mu_i \qquad 1.1$$

This is because the left hand side gives the number of entries in the first i rows of $T_1 \wedge T_2$ and the right hand side gives the number of entries in the tableau (necessarily of shape μ) obtained by letting the entries of $T_1 \wedge T_2$ drop down their column until they are packed tight. This means that when T_1 and T_2 have different shapes, we must have some motion of cells and a strict inequality in 1.1, for at least one i . On the other hand when T_1 and T_2 have the same shape then $\lambda = \mu$, and there can be no motion of cells. In this case $T_1 \wedge T_2$ is also a tableau of shape $\lambda = \mu$, whose rows are a rearrangement of the rows of T_1 and whose columns are a rearrangement of the columns of T_2. In summary when $\lambda = \mu$ and $T_1 \wedge T_2$ is good there are two permutations $\alpha_1 \in R(T_1)$ and $\beta_2 \in C(T_2)$ such that

$$T_1 \wedge T_2 \;\; = \;\; \alpha_1 T_1 \quad ; \quad T_2 \;\; = \;\; \beta_2 \, T_1 \wedge T_2 \qquad 1.2$$

Conversely, it is easily seen, that the existence of two such permutations giving 1.2 forces $T_1 \wedge T_2$ to be good.

This permits us to introduce an important function on pairs of tableau:

Definition 1.1

For $T_1, T_2 \in INJ(\lambda)$ we let

$$C(T_1,T_2) \;=\; \begin{cases} 0 & \text{if } T_1 \wedge T_2 \text{ is bad} \\ sign(\beta_2) & \text{if } T_1 \wedge T_2 \text{ is good} \end{cases}$$

Here and after, it will be convenient to let

$$S_1^\lambda \,,\, S_2^\lambda \,,\, \ldots \,,\, S_{f_\lambda}^\lambda \qquad 1.3$$

denote the standard tableaux of shape λ in Young's first letter order. This given, we set

$$C^\lambda(\sigma) \;=\; \|C(S_i^\lambda \,,\, \sigma S_j^\lambda)\|_{i'j=1}^{f_\lambda} \;. \qquad 1.4$$

We have the following basic property

Proposition 1.1

The matrix $C^\lambda(\epsilon)$ is upper unitriangular and is therefore invertible over the integers.

Proof

Note that if $T_1 <_{YFLO} T_2$ and a is the first letter of disagreement, then there is a letter b such that a and b are in the same column of T_1 and in the same row of T_2. Note that $b = 4$ for the example given in Fig 1. In general b is the letter which lies at the intersection of the column of the shape λ in which a lies in T_1 with the row in which a lies in T_2. This means that $T_2 \wedge T_1$ is bad. So $T_1 <_{YFLO} T_2$ implies $C(T_2, T_1) = 0$. Thus from the definition of first letter order we derive that

$$C(S_i^\lambda, S_j^\lambda) \;=\; 0 \qquad \forall \;\, i > j \;\; .$$

This proves the proposition since $C(S_i^\lambda, S_i^\lambda) = 1$ holds true trivially.

This allows us to define

$$A^\lambda(\sigma) \;=\; C^\lambda(\epsilon)^{-1}\, C^\lambda(\sigma) \qquad 1.5$$

Our goal in these notes is to show that the collection of matrix functions

$$\{A^\lambda(\sigma)\}_{\lambda\vdash n} \qquad 1.6$$

form a complete set of irreducible representations of S_n.

Remark 1.1

The simplicity of the definition in 1.5 should be compared with the pages and pages of intricate constructions that characterize the treatments of the representations of the symmetric groups given in recent and past litterature following the work of A. Young. Because of peculiarities of Young's style of writing many of his successors never bothered to read, let alone tried to understand, Young's beautiful constructions. As a result you will often see the name "Specht module" associated with representations which are in fact none other than those defined in 1.5. What is more amusing is that Young himself in QSA IV (†) , with surprising premonition, described some alternate ways of constructing representations of S_n which include as particular cases all the constructions that followed. Another justification often given to assign some significance to the work of Specht and assorted "Jonny-come-latelies" is that later developments are "*Characteristic Free*" but this is only based on a naive missunderstanding of Young's purely combinatorial arguments.

Problems

Problem 1.

Show that for any injective tableau T we have $\alpha\beta = \epsilon$ with $\alpha \in R(T)$ and $\beta \in \mathcal{C}(T)$ if and only if $\alpha = \beta = \epsilon$.

Problem 2.

Let $T \in INJ(\lambda)$ with $\lambda \vdash n$. Show that $T \wedge \sigma T$ is good if and only if $\sigma = \beta_2\alpha_1$ with $\beta_2 \in \mathcal{C}(\sigma T)$ and $\alpha_1 \in R(T)$. Show that this factorization is unique. That is show that if $\sigma = \beta_2'\alpha_1'$, with $\beta_2' \in \mathcal{C}(\sigma T)$ and $\alpha_1' \in R(T)$ then then $\beta_1' = \beta_1$ and $\alpha_1' = \alpha_1$.

Problem 3.

Construct all standard tableaux of shape $(3, 2, 1)$ in the Young first letter order. Then construct the matrix $A^{(3,2)}(\sigma)$ for $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{bmatrix}$

Problem 4.

Show that when λ is a "*hook* " that is $\lambda = (k, 1^{n-k})$ then the matrix $C^\lambda(\epsilon)$ reduces to the identity matrix.

2. Young's Tableau idempotents.

For a given tableau T here and after we set

$$P(T) = \sum_{\alpha \in R(T)} \alpha \quad , \quad N(T) = \sum_{\beta \in C(T)} sign(\beta)\beta \qquad 2.1$$

If $T_1, T_2 \in INJ(\lambda)$ by $\sigma_{T_1T_2}$ we denote the permutation that sends T_2 into T_1 that is

$$T_1 \;=\; \sigma_{T_1T_2} T_2$$

This given, it is easy to see that we have

$$\begin{aligned} &a) \quad P(\sigma T) = \sigma P(T)\, \sigma^{-1} \, ; \; N(\sigma T) = \sigma N(T) \sigma^{-1} \qquad \forall \; \sigma \in S_n \\ &b) \quad \sigma_{T_1T_2} P(T_2) \;=\; P(T_1) \sigma_{T_1T_2} \\ &c) \quad \sigma_{T_1T_2} N(T_2) \;=\; N(T_1) \sigma_{T_1T_2} \end{aligned} \qquad 2.2$$

(†) (see "The Collected Papers of A. Young, 1873-1940", Math. Expositions V. #21 Univ. of Toronto Press 1977

We shall also use the shorthand notations

$$\begin{cases} a) \quad E_{T_1T_2} = N(T_1)\,\sigma_{T_1T_2}\,P(T_2) \quad , \quad E_T = E_{TT} = N(T)P(T) \\ \\ b) \quad F_{T_1T_2} = P(T_1)\,\sigma_{T_1T_2}\,N(T_2) \quad , \quad F_T = F_{TT} = P(T)N(T) \end{cases} \qquad 2.3$$

Note that from 2.2 b) and c) we derive that

$$\begin{cases} a) \quad E_{T_1T_2} \;=\; E_{T_1}\sigma_{T_1T_2} \;=\; \sigma_{T_1T_2}E_{T_2} \\ b) \quad F_{T_1T_2} \;=\; F_{T_1}\sigma_{T_1T_2} \;=\; \sigma_{T_1T_2}F_{T_2} \end{cases} \qquad 2.4$$

Recalling that

$$S_1^\lambda \,,\, S_2^\lambda \,,\, \ldots \,,\, S_{f_\lambda}^\lambda \qquad 2.5$$

denote the standard tableaux of shape λ in Young's first letter order we set

$$\begin{cases} E_i^\lambda \;=\; E_{S_i^\lambda} \quad ; \quad E_{ij}^\lambda \;=\; E_{S_i^\lambda S_j^\lambda} \\ F_i^\lambda \;=\; F_{S_i^\lambda} \quad ; \quad F_{ij}^\lambda \;=\; F_{S_i^\lambda S_j^\lambda} \end{cases} \quad ; \quad \sigma_{ij}^\lambda \;=\; \sigma_{S_i^\lambda S_j^\lambda} \qquad 2.6$$

When dealing with a fixed shape, to lighten our formulas, sometime we shall omit the superscript λ from our symbols. We should note that many identities and results involving the elements E_{ij}^λ hold also for the $F_{ij}^\lambda\,'s$, with only minor monifications. In each case, to avoid unnecessary repetitions, we shall prove them for one set and leave it to the reader to derive the analogous results for the other set.

The group algebra elements $P(T)$, $N(T)$ have truly remarkable properties. To prove them we need some auxiliary results.

Proposition 2.1

For a pair of tableaux T_1 and T_2 of the same shape the following properties are equivalent

$$\begin{array}{ll} a) & T_1 \wedge T_2 \quad \textit{is good} \\ b) & \exists\ \alpha_1 \in R(T_1)\ \textit{and}\ \ \beta_2 \in C(T_2) \quad \textit{such that} \quad T_2 = \beta_2\alpha_1\,T_1 \\ c) & \exists\ \alpha_1 \in R(T_1)\ \textit{and}\ \ \beta_1 \in C(T_1) \quad \textit{such that} \quad T_2 = \alpha_1\beta_1\,T_1 \\ d) & \exists\ \alpha_2 \in R(T_2)\ \textit{and}\ \ \beta_2 \in C(T_2) \quad \textit{such that} \quad T_2 = \alpha_2\beta_2\,T_1 \end{array}$$

Moreover we have

$$sign(\beta_1) \;=\; sign(\beta_2) \qquad 2.7$$

Proof

We have seen that a) and b) are equivalent. From 2.2 a) we then get that b) implies

$$N(T_1) \;=\; \alpha_1^{-1}\beta_2^{-1}\,N(T_2)\,\beta_2\alpha_1 \;=\; \alpha_1^{-1}\,N(T_2)\,\alpha_1$$

Thus there is a $\beta_1 \in N(T_1)$ such that

$$\beta_1 \;=\; \alpha_1^{-1}\beta_2\alpha_1 \qquad 2.8$$

This gives that

$$T_2 \;=\; \beta_2\alpha_1\, T_1 \;=\; (\alpha_1\beta_1\alpha_1^{-1})\alpha_1\, T_1 \;=\; \alpha_1\beta_1\, T_1$$

which is c). Note that 2.7 now follows from 2.8. Conversely suppose we have c). Then we may write

$$N(T_2) \;=\; \alpha_1\beta_1 N(T_1)\beta_1^{-1}\alpha_1^{-1} \;=\; \alpha_1 N(T_1)\alpha_1^{-1}$$

and we must necessarily have a $\beta_2 \in C(T_2)$ such that

$$\beta_2 \;=\; \alpha_1\beta_1\alpha_1^{-1} \;,$$

giving

$$\sigma_{T_2T_1} \;=\; \alpha_1\beta_1 \;=\; \beta_2\alpha_1 \;.$$

This shows the equivalence of b) and c). We are left to show that d) is equivalent to b). It turns out that, in this case, the $P'\,s$ play the role the $N'\,s$ played in the previous argument. More precisely from b) and 2.2 a) we derive that

$$P(T_2) \;=\; \beta_2\alpha_1 P(T_1)\alpha_1^{-1}\beta_2^{-1} \;=\; \beta_2 P(T_1)\beta_2^{-1} \;.$$

Thus there must be an $\alpha_2 \in R(T_2)$ such that

$$\alpha_2 \;=\; \beta_2\alpha_1\beta_2^{-1}$$

and this gives

$$\sigma_{T_2T_1} \;=\; \alpha_2\beta_1 \;=\; \alpha_2\beta_2 \;.$$

Thus b) implies d). We leave it to the reader to show that d) implies b) and complete the proof of the proposition.

Remark 2.1

It is important to note that the permutations $\alpha_1, \alpha_2, \beta_1, \beta_2$ of Proposition 2.1 are uniquely determined by T_1 and T_2. In fact for a given Tableau T we cannot have two pairs $\alpha', \alpha" \in R(T)$ and $\beta', \beta" \in C(T)$ such that

$$\alpha'\beta' = \alpha"\beta"$$

Indeed this is equivalent to the existence of an $\alpha \in R(T)$ and a $\beta \in C(T)$ such that

$$\alpha \;=\; \beta$$

and this is easily seen to be impossible. Let us keep this fact in mind since we are going to make use of it several times in the future.

Before we proceed with the next result we need to make some conventions. Young used a very efficient notation to represent some elements of the group algebra of the symmetric group S_n. For a given subset

$$S = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\} \subseteq \{1, 2, \ldots, n\}$$

he let the symbol "$[S]$" represent the sum of all elements of S_n that permute the elements of S and leave fixed all the elements of the complement of S in $\{1, 2, \ldots, n\}$. He also used "$[S]'$" to denote the sum of the same permutations where each is multiplied by its sign. For instance, with this convention we see that for the tableau

$$T \;=\; \begin{array}{cccc} 2 & 3 & & \\ 5 & 7 & & \\ 6 & 9 & 10 & \\ 11 & 4 & 8 & 1 \end{array}$$

we may write

$$P(T) = [1, 4, 8, 11] \times [6, 9, 10] \times [5, 7] \times [2, 3] \quad \text{and} \quad N(T) = [2, 5, 6, 11]' \times [3, 4, 7, 9]' \times [8, 10]'$$

Remark 2.2

For a given $f \in \mathcal{A}(S_n)$, it will also be convenient to denote by "f'" the element of the group algebra obtained by replacing each permutation by the same permutation preceded by its sign. More precisely, for $f = \sum_{\sigma \in S_n} f(\sigma)\,\sigma$ we shall set

$$f' \;=\; \sum_{\sigma \in S_n} sign(\sigma) f(\sigma)\,\sigma$$

and refer to the operation $f \to f'$ as the "*priming operator*". We should note that any identity involving elements of the group algebra of S_n generates a companion identity when we apply the priming operator to both sides. In this manner identities for the $E^{\lambda}_{ij}\,'s$ may be transformed into identities for the $F^{\lambda}_{ij}\,'s$. In fact, it easy to see that

$$E_{T_1,T_2}{}' \;=\; sign(\sigma_{T_1,T_2})\, F_{T_1^{\top},T_2^{\top}} \tag{2.9}$$

where $T_1^{\top}$ and $T_2^{\top}$ denote the corresponding transposed tableaux (†). In particular we have

$$E^{\lambda}_{ij}{}' \;=\; sign(\sigma^{\lambda}_{ij})\, F^{\lambda'}_{ij} \;, \tag{2.10}$$

where λ' denotes the partition conjugate to λ.

Remark 2.3

For two partitions

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h) \;, \qquad \mu = (\mu_1, \mu_2, \ldots, \mu_k)$$

we say that μ "*dominates*" λ and write

$$\lambda \leq_D \mu$$

(†) Transposing a tableau simply means reflecting it across the 45^o diagonal emanating from its south-west corner if and only if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i \qquad \forall\, 1 \leq i \leq min(h,k)\ \ .$$

Thus our remarks at the beginning of section 1 may be restated by writing

$$T_1 \wedge T_2 \quad good \quad \longrightarrow \quad shape(T_1) \leq_D shape(T_2)\ \ . \tag{2.11}$$

Now it develops that the same implication holds true if we have

$$a) \quad P(T_1)N(T_2) \neq 0 \quad \text{ or } \quad b) \quad N(T_2)P(T_1) \neq 0 \tag{2.12}$$

To see this note that, by definition, $T_1 \wedge T_2$ " bad" implies that there are two elements a, b that are in the same row of T_1 and the same column of T_2. This means that the transposition (a, b) is at the same time in $R(T_1)$ and $C(T_2)$. We thus have

$$P(T_1)N(T_2) = \Big(P(T_1)(a,b)\Big)N(T_2) = P(T_1)\Big((a,b)N(T_2)\Big) = -P(T_1)N(T_2)\ \ .$$

this contradicts 2.12 a). Thus

$$P(T_1)N(T_2) \neq 0 \quad \longrightarrow \quad T_1 \wedge T_2 \quad good \quad \longrightarrow \quad shape(T_1) \leq_D shape(T_2)\ \ . \tag{2.13}$$

We can proceed similarly and show that

$$N(T_2)P(T_1) \neq 0 \quad \longrightarrow \quad T_1 \wedge T_2 \quad good \quad \longrightarrow \quad shape(T_1) \leq_D shape(T_2)\ \ . \tag{2.14}$$

3. Semi-simplicity of Algebras

We have seen that the Group algebra $\mathcal{A}[\Gamma]$ of a finite group Γ has two binary operations "+" (*addition*) and "$\times$" (*multiplication*), which are associative and satisfy the left and right distributivity laws. That is for all $f, h, g \in \mathcal{A}[\Gamma]$ we have

$$(left) \quad h \times (f+g) \;=\; h \times f \;+\; h \times g \qquad\qquad (right) \quad (f+g) \times h \;=\; f \times h \;+\; g \times h$$

Moreover, since its elements are formal linear combinations of group elements, such as

$$f \;=\; \sum_{\gamma \in \Gamma} f(\gamma)\gamma$$

with f a complex valued function on Γ, we also have a "*multiplication by a scalar* " operation, defined by setting for any complex number c

$$cf \;=\; \sum_{\gamma \in \Gamma} cf(\gamma)\gamma$$

this operation is clearly associative and distributive with respect to addition.

The group algebra has also an identity which we denote "ϵ" which is simply given by the identity element of the group Γ.

A structure with these properties with scalars in a field $\mathbf{K}$ is usually referred to as a "$\mathbf{K}$-Algebra". The collection of $n \times n$ matrices with entries in $\mathbf{K}$, which we denote $\mathcal{M}_n[\mathbf{K}]$, is the simplest example of a $\mathbf{K}$-Algebra. It easily seen $\mathcal{M}_n[\mathbf{K}]$ is a n^2-dimensional vector space with a natural basis consisting of the $n \times n$ matrices $E_{i,j}$ with i,j entry equal to 1 and all other entries equal to zero. With this notation, for any matrix $A = \|a_{i,j}\|_{i,j=1}^n$ we can write

$$A \;=\; \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}E_{i,j} \tag{3.1}$$

The $E_{i,j}$, which (using Young's terminology) will refer as "*matrix units* ", are easily seen to satisfy the identities

$$E_{i,j} \times E_{r,s} \;=\; \begin{cases} 0_n & \text{if } j \neq r \\ E_{i,s} & \text{if } j = r \end{cases} \tag{3.2}$$

where "0_n" here means the $n \times n$ matrix with all elements equal to 0.

The the next simplest example is obtained by taking a direct sum of a finite number of simple matrix algebras. More precisely we will denote

$$\bigoplus_{l=1}^{k} \mathcal{M}_{n_l}[\mathbf{K}]$$

the vector space of $M \times M$ block matrices, (with $M = \sum_{s=1}^k n_s$) of block diagonal form

$$A \;=\; \begin{bmatrix} A^{(1)} & 0 & \cdots & 0 \\ 0 & A^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^{(k)} \end{bmatrix} \;=\; \bigoplus_{l=1}^{k} A^{(l)} \qquad \big(\text{with } A^{(l)} \in \mathcal{M}_{n_l}[\mathbf{K}]\big) \tag{3.3}$$

It will be convenient to denote by $f_{i,j}^{(s)}$ the block matrix in 3.3 with

$$A^{(l)} \;=\; \begin{cases} 0_{n_l} & \text{if } l \neq s \\ E_{i,j}^{(n_s)} & \text{if } i = s \end{cases} \tag{3.4}$$

where here $E_{i,j}^{(n_s)}$ denotes the $n_s \times n_s$ matrix with i,j element 1 and all other elements 0. It is easily seen that the collection $\bigcup_{s=1}^k \{f_{i,j}^{(s)}\}_{i,j=1}^{n_s}$ is a basis of $\bigoplus_{i=1}^k \mathcal{M}_{n_i}[\mathbf{K}]$, since it is clearly an independent set and every element $A = \bigoplus_{l=1}^k A^{(l)}$ has an expansion of the form

$$A \;=\; \sum_{s=1}^{k}\sum_{i=1}^{n_s}\sum_{j=1}^{n_s} a_{i,j}^{(s)} f_{i,j}^{(s)}$$

In particular we see that

$$\dim \bigoplus_{i=1}^{k} \mathcal{M}_{n_i}[\mathbf{K}] \;=\; \sum_{s=1}^{k} n_s^2$$

Moreover, from 3.2 aND 3.4 it follows that

$$f_{i,j}^{l} \times f_{r,s}^{m} \;=\; \begin{cases} 0_M & \text{if } l \neq m \\ 0_M & \text{if } l = m \text{ but } j \neq r \\ f_{i,s}^{l} & l = m \text{ and } j \neq r \end{cases} \qquad \big(M = \textstyle\sum_{s=1}^{k} n_s\big) \tag{3.5}$$

It is easily shown that only multiples of the identity matrix $I \in \mathcal{M}[\mathbf{K}]$ commute with all the matrices in $\mathcal{M}[\mathbf{K}]$. In particular, it follows that the center of $\bigoplus_{i=1}^{k} \mathcal{M}_{n_i}[\mathbf{K}]$ (that is the subspace of the elements that commute with all the other elements) is spanned by the elements

$$f_s \;=\; \sum_{i=1}^{n_s} f_{i,i}^{(n_s)} \qquad (1 \leq s \leq k) \tag{3.6}$$

Which are none other than the block matrices in 3.3 with

$$A^{(l)} \;=\; \begin{cases} 0_{n_l} & \text{if } l \neq s \\ I^{(n_s)} & \text{if } i = s \end{cases}$$

where $I^{(n_s)}$ denotes the $n_s \times n_s$ identity matrix.

This gives

$$\dim Center\Big(\bigoplus_{i=1}^{k} \mathcal{M}_{n_i}[\mathbf{K}]\Big) \;=\; k \tag{3.7}$$

It is customary to call the algebras $\bigoplus_{i=1}^{k} \mathcal{M}_{n_i}[\mathbf{K}]$ "*semi-simple* ".

Frobenius' Fundamental Theorem of Representation Theory states that the group algebra of every finite group is isomorphic to a semi-simple algebra.

More precisely Frobenius result assures that given a finite group Γ of order $N = |\Gamma|$ there are integers $\{n_s\}_{s=1}^{k}$ such that

$$N \;=\; \sum_{s=1}^{k} n_s^2 \tag{3.8}$$

and a bijective map

$$\phi \;:\; \mathcal{A}(\Gamma) \quad\longleftrightarrow\quad \bigoplus_{i=1}^{k} \mathcal{M}_{n_i}[\mathbf{C}] \tag{3.9}$$

that respects addition, multiplication and multiplication by a complex number. In particular we deduce from 3.7 that the integer k must be equal to the dimension of the center $\mathcal{C}(\Gamma)$ of the group group algebra of Γ. But we have seen that $\mathcal{C}(\Gamma)$ is none other that the subspace of class functions. Thus in particular we derive that k in 3.8 and 3.9 must be none other than the number of conjugacy classes of Γ.

Of course in view of 3.5 the Frobenius result amounts to showing the existence in $\mathcal{A}(\Gamma)$ of a basis $\bigcup_{s=1}^{k} \big\{e_{i,j}^{(s)}\big\}_{i,j=1}^{n_s}$ satisfying the identities

$$e_{i,j}^{l} \times e_{r,s}^{m} \;=\; \begin{cases} 0 & \text{if } l \neq m \\ 0 & \text{if } l = m \text{ but } j \neq r \\ e_{i,s}^{l} & l = m \text{ and } j = r \end{cases} \tag{3.10}$$

This given, the desired map ϕ is simply obtained by setting

$$\phi\, e^l_{i,j} \;=\; f^l_{i,j}$$

and extending it by linearity to the rest of $\mathcal{A}(\Gamma)$.

Remarkably, Alfred Young, initially unaware of Frobenius work, set himself the task, in a series of papers written in a span of 17 years, to produce a purely combinatorial construction of such matrix units for several reflection groups including all the symmetric groups. While doing so he completely upstaged Frobenius who only succeeded in constructing the k elements f_s in 3.6 for S_n.

In the next few section we will present Young's construction for the symmetric groups and derive some of its most significant properties.

4. Young's matrix units for $\mathcal{A}(\mathbf{S_n})$.

This section is dedicated to the proof of the following remarkable result

Theorem 4.1(A.Young)

With $S_1^\lambda, S_2^\lambda, \ldots, S_{n_\lambda}^\lambda$ the standard tableaux of shape $\lambda \vdash n$ in Young's first letter order, let $h_\lambda = n!/n_\lambda$ and set

$$\gamma_i^\lambda \;=\; \frac{1}{h_\lambda}\, N(S_i^\lambda)P(S_i^\lambda) \qquad 4.1$$

This given, for $\sigma_{S_i^\lambda, S_j^\lambda} = \sigma_{ij}^\lambda$, the group algebra elements

$$e_{ij}^\lambda \;=\; \gamma_i^\lambda\, \sigma_{ij}^\lambda\, (1-\gamma_{j+1}^\lambda)(1-\gamma_{j+2}^\lambda)\cdots(1-\gamma_{f_\lambda}^\lambda) \qquad 4.2$$

do not vanish and satisfy the identities

$$e_{ij}^\lambda\; e_{rs}^\mu \;=\; \begin{cases} e_{is}^\lambda & \text{if} \quad \lambda=\mu \quad \text{and} \quad j=r \\ 0 & \text{otherwise} \end{cases} \qquad 4.3$$

We will obtain our proof by combining a few, very simple. purely combinatorial properties of the group algebra elements γ_i^λ which we will state as separate Propositions.

We will start with the following very beautiful fact which considerably simplifies Young's original argument.

Proposition 4.1 (Von Neuman Sandwich Lemma)

For any element $f \in \mathcal{A}(S_n)$ and any $T \in INJ(\lambda)$ we have

$$N(T)\, f\, P(T) \;=\; c_T(f)\quad N(T)P(T)\;, \qquad 4.4$$

with

$$c_T(f) \;=\; f\, P(T)N(T)\,|_\epsilon\;. \qquad 4.5$$

Proof

Expanding f in the left hand side of 4.4 we get

$$N(T)\, f\, P(T) \;=\; \sum_{\sigma\in S_n} f(\sigma)\, N(T)\, \sigma\, P(T)\ . \qquad 4.6$$

Now note that using 2.2 a) we can write

$$N(T)\, \sigma\, P(T) \;=\; N(T)\, P(\sigma T)\, \sigma\ .$$

That means that the only terms that survive in 4.6 are those for which $N(T)\, P(\sigma T) \neq 0$. However, as we have seen in Remark 2.3, this can happen only if $\sigma T \wedge T$ is good. But part d) of Proposition 2.1 gives that we must have

$$T \;=\; \alpha\, \beta\, \sigma\, T$$

with $\alpha \in R(T)$ and $\beta \in C(T)$. In other words the only terms that survive in 4.6 are those coming from permutations σ of the form

$$\sigma \;=\; \beta^{-1}\alpha^{-1} \qquad\qquad (\text{with}\quad \alpha \in R(T)\quad,\quad \beta \in C(T)\ )$$

Since any permutation that can be so expressed has a unique expression of this form we can rewrite 4.6 as

$$N(T)\, f\, P(T) \;=\; \sum_{\alpha\in R(T)} \sum_{\beta\in C(T)} f(\beta^{-1}\alpha^{-1})\, N(T)\, \beta^{-1}\alpha^{-1}\, P(T)\ . \qquad 4.7$$

But then the simple identity

$$N(T)\, \beta^{-1}\alpha^{-1}\, P(T) \;=\; sign(\beta)\, N(T)P(T)$$

reduces 4.7 to

$$N(T)\, f\, P(T) \;=\; \Big(\sum_{\alpha\in R(T)} \sum_{\beta\in C(T)} sign(\beta) f(\beta^{-1}\alpha^{-1}) \Big)\, N(T)P(T)\ ,$$

which is 4.4 with

$$c_T(f) \;=\; \sum_{\alpha\in R(T)} \sum_{\beta\in C(T)} sign(\beta) f(\beta^{-1}\alpha^{-1})$$

However the latter is but another way of writing 4.5. **Q.E.D.**

Proposition 4.1 yields us the following fundamental fact

Proposition 4.2

For each $\lambda \vdash n$ we have a non vanishing constant h_λ depending only on λ such that for all tableaux $T \in INJ(\lambda)$ we have, $E_T = N(T)P(T)$

$$E_T^2 \;=\; h_\lambda\ E_T \qquad 4.8$$

Proof

We simply use 4.4 with $f = P(T)N(T)$ and get

$$N(T)\, P(T)N(T)\, P(T) \;=\; h_\lambda \;\; N(T)P(T) \;,$$

with

$$h_\lambda \;=\; P(T)N(T)\, P(T)N(T)\, |_\epsilon \;.$$

Next we need to show that h_λ is the same for all injective tableaux of shape λ. However this is a simple consequence of the fact that if $T_1 = \sigma T$ then $P(T_1) = \sigma P(T)\sigma^{-1}$ and $N(T_1) = \sigma N(T)\sigma^{-1}$, thus

$$P(T_1)N(T_1)\, P(T_1)N(T_1)\, |_\epsilon \;=\; \sigma P(T)N(T)\, P(T)N(T)\sigma^{-1}\, |_\epsilon \;=\; P(T)N(T)\, P(T)N(T)\, |_\epsilon$$

Finally we must show that $h_\lambda \neq 0$. But if $h_\lambda = 0$ the element E_T would be nilpotent. This in turn would imply that its image $L(E_T)$ by the left regular representation would be a nilpotent matrix. Since nilpotent matrices have trace zero and for any group algebra element $f = \sum_{\sigma \in S_n} f(\sigma)\sigma$ we have

$$trace L(f) \;=\; \sum_{\sigma \in S_n} f(\sigma)\, trace L(\sigma) \;=\; f(\epsilon) n!$$

the vanishing of h_λ would imply that $E_T\, |_\epsilon = 0$. Now this is absurd since, from Remark 2.1 we immediately derive that $E_T\, |_\epsilon = 1$. This completes our proof.

Remark 4.1

We should note that one of the simplifications to Young's arguments due to the Von Neuman Lemma is to avoid a direct identification of the constant h_λ as $n!/n_\lambda$. This is obtained in Young's work as the final result of gruesome brute force proof of the idempotency of the group algebra elements E_T/h_λ. In the present development the identification of h_λ will be carried out at the very end. This given, until that time we will assume that h_λ is the mysterious constant appearing in Von Neuman's lemma. In particular, this assumption yields us that all the elements γ_i^λ in are idempotent.

Proposition 4.3

$$\gamma_j^\lambda \gamma_i^\mu \;=\; \begin{cases} 0 & \text{if } \lambda \neq \mu \\ 0 & \text{if } \lambda = \mu \text{ but } j > i \\ \gamma_i & \text{if } \lambda = \mu \text{ and } j = i \end{cases} \tag{4.9}$$

Proof

We have seen in the proof of Proposition 1.1 that if $T_1 <_{YFLO} T_2$ then there is a pair of entries r, s that are in the same column of T_1 and same row of T_2 . This gives

$$P(T_2)N(T_1) \;=\; P(T_2)(r,s)N(T_1) \;=\; -P(T_2)N(T_1)$$

and thus $E_{T_2}E_{T_1}$ must necessarily vanish. Since $S_1^\lambda, S_2^\lambda, \ldots, S_{n_\lambda}^\lambda$ are in Young's first letter order it follows that

$$E_{S_j^\lambda} E_{S_i^\lambda} \;=\; 0 \quad \text{when} \quad j > i$$

in view of 4.1 this proves the second case of 4.9.

To prove the first case of 4.9 we need only show that if T_1 and T_2 are injective tableaux of shapes λ and μ respectively then

$$\lambda \neq \mu \qquad \longrightarrow \qquad E_{T_1}E_{T_2} \;=\; 0$$

To this end note that 2.13 immediately gives that

$$E_{T_1}E_{T_2} \;=\; N(T_1)P(T_1)N(T_2)P(T_2) \neq 0 \qquad \longrightarrow \qquad \lambda \leq \mu \tag{4.10}$$

On the other hand we expand the mid element $P(T_1)N(T_2)$ in the product $E_{T_1}E_{T_2}$ we will obtain a sum of terms of the form

$$N(T_1)\sigma P(T_2) \;=\; N(T_1)P(\sigma T_2)\sigma$$

If $E_{T_1}E_{T_2}$ does not vanish then at least one of these terms must not vanish. But then from 2.14 we derive that $\mu \leq \lambda$ which together with 4.10 contradicts $\lambda \neq \mu$. This completes our proof of the first case of 4.9 and we are done since the last case, under our assumption that h_λ is the Von Neuman constant, is as we have seen an immediate consequence of Proposition 4.2.

We are now finally in a position to prove Theorem 4.1, except for the identification of the constant h_λ. We will start with a

Proof of the identities in 4.3

Note first that if $\lambda \neq \mu$ then the first case of 4.9 gives

$$(1-\gamma_i^\lambda)\gamma_j^\mu \;=\; \gamma_j^\mu \qquad\qquad \text{(for all } i,j) \tag{4.11}$$

Thus

$$\begin{aligned} e_{ij}^\lambda\, e_{rs}^\mu \;&=\; \sigma_{ij}^\lambda\gamma_j^\lambda(1-\gamma_{j+1}^\lambda)\cdots(1-\gamma_{n_\lambda}^\lambda)\gamma_r^\mu\sigma_{rs}^\mu(1-\gamma_{s+1}^\mu)\cdots(1-\gamma_{n_\lambda}^\mu)\\ &=\; \sigma_{ij}^\lambda\gamma_j^\lambda\gamma_r^\mu\sigma_{rs}^\mu(1-\gamma_{s+1}^\mu)\cdots(1-\gamma_{n_\lambda}^\mu)\\ &=\; 0 \qquad \text{(again by the first case of 4.9)} \end{aligned}$$

We are thus reduced to proving the identities in 4.3 when $\lambda=\mu$ in all our elements. Note then that the second case of 4.9 gives

$$a)\quad \gamma_j^\lambda\gamma_i^\lambda = 0 \qquad \text{as well as} \qquad b)\quad (1-\gamma_j^\lambda)\gamma_i^\lambda = \gamma_i^\lambda \qquad \text{when } j>i \tag{4.12}$$

Now note that we can write, for $r<j$

$$\begin{aligned} e_{ij}^\lambda\, e_{rs}^\lambda \;&=\; \sigma_{ij}^\lambda\gamma_j^\lambda(1-\gamma_{j+1}^\lambda)\cdots(1-\gamma_{n_\lambda}^\lambda)\gamma_r^\lambda\sigma_{rs}^\lambda(1-\gamma_{s+1}^\lambda)\cdots(1-\gamma_{n_\lambda}^\lambda)\\ (\text{ by 4.12 b) })\;&=\; \sigma_{ij}^\lambda\gamma_j^\lambda\gamma_r^\mu\sigma_{rs}^\mu(1-\gamma_{s+1}^\mu)\cdots(1-\gamma_{n_\lambda}^\mu)\\ (\text{ by 4.12 a) })\;&=\; 0 \end{aligned}$$

and for $r=j$

$$\begin{aligned} e_{ij}^\lambda\, e_{rs}^\lambda \;&=\; \sigma_{ij}^\lambda\gamma_j^\lambda(1-\gamma_{j+1}^\lambda)\cdots(1-\gamma_{n_\lambda}^\lambda)\gamma_j^\lambda\sigma_{js}^\lambda(1-\gamma_{s+1}^\lambda)\cdots(1-\gamma_{n_\lambda}^\lambda)\\ (\text{ by 4.12 a})\;&=\; \sigma_{ij}^\lambda\gamma_j^\lambda\gamma_j^\lambda\sigma_{js}^\lambda(1-\gamma_{s+1}^\lambda)\cdots(1-\gamma_{n_\lambda}^\lambda)\\ (\text{ by the last case of 4.9) })\;&=\; \sigma_{ij}^\lambda\gamma_j^\lambda\sigma_{js}^\lambda(1-\gamma_{s+1}^\lambda)\cdots(1-\gamma_{n_\lambda}^\lambda)\\ &=\; \gamma_i^\lambda\sigma_{ij}^\lambda\sigma_{js}^\lambda(1-\gamma_{s+1}^\lambda)\cdots(1-\gamma_{n_\lambda}^\lambda) \;=\; e_{is}^\lambda \end{aligned}$$

Finally for $r > j$, after successive uses of 4.12 will necessarily get to a point where the resulting expression for $e^\lambda_{ij}\, e^\lambda_{rs}$ contains the factor $(1-\gamma^\lambda_r)\gamma^\lambda_r$ which of course vanishes by the last case of 4.9. This completes our argument.

From these relations we can now establish that

Proposition 4.4

The group algebra elements e^λ_{ij} cannot vanish

Proof

Note first that the identity

$$e^\lambda_{i,j}e^\lambda_{j,i} \;=\; e^\lambda_{i,i}$$

reduces us to showing that the $e^\lambda_{i,i}$ themselves cannot vanish. Now this result is an immediate consequence of the very useful fact (as we shall see) that for any two group algebra elements f, g we have

$$fg\Big|_\epsilon \;=\; gf\Big|_\epsilon \tag{4.13}$$

the reader is urged to work out a proof of this identity in full generality. This given, note first that from the definition in 4.2 and 4.13 it follows that

$$\begin{aligned} e^\lambda_{i,i}\Big|_\epsilon &= \gamma^\lambda_i(1-\gamma^\lambda_{i+1})\cdots(1-\gamma^\lambda_{n_\lambda})\Big|_\epsilon \\ &= (1-\gamma^\lambda_{i+1})\cdots(1-\gamma^\lambda_{n_\lambda})\gamma^\lambda_i\Big|_\epsilon \\ \text{(by 4.12 b))} &= \gamma^\lambda_i\Big|_\epsilon \;=\; 1/h_\lambda \neq 0 \end{aligned}$$

As a first step in proving that Young's matrix units are a basis we can now establish that

Proposition 4.5

The group algebra elements $\bigcup_{\lambda\vdash n}\{e^\lambda_{ij}\}^{n_\lambda}_{i,j=1}$ are independent

Proof

Assume if possible that for some constants c^λ_{ij} we have

$$\sum_{\lambda\vdash n}\sum_{i,j=1}^{n_\lambda} c^\lambda_{ij}\;e^\lambda_{ij} \;=\; 0$$

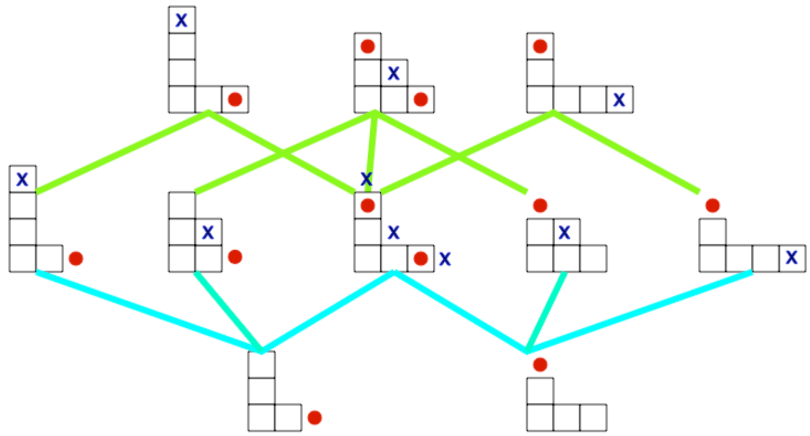
then for all possible choices of μ, r and s we derive from the identities in 4.3 that

$$c^\mu_{rs}\;e^\mu_{rs} \;=\; e^\mu_{rr}\Big(\sum_{\lambda\vdash n}\sum_{i,j=1}^{n_\nu} c^\lambda_{ij}\;e^\lambda_{ij}\Big)e^\mu_{ss} \;=\; 0$$

and since, as we have seen, the $e^\mu_{rs}\,'s$ do not vanish we must necessarily have $c^\mu_{rs}=0$ as desired.

To prove that an independent set in a vector space V is a basis we need only show that it spans V or that its cardinality equals the dimension of V. Alfred Young proved that his matrix units are a basis, both ways. We will follow the latter approach first since it it based on two beautiful combinatorial identities.

But before we can state them we need a few preliminary observations. In the figure below we have depicted the Ferrers diagram of the partition $(3,1,1)$ and its immediate neighbors in the Young lattice. (the lattice of Ferrers diagrams ordered by inclusion).



We have at the center the Ferrers diagram of $(3,1,1)$ with the circles indicating its "*removable*" (inner) corner cells and the $\mathbf{x}'s$ showing its "*addable*" (outer) corner cells. In the row below we have the partitions obtained by removing one of its inner corner cells and in the row above we have the partitions obtained by adding one of its addable outer corner cells. We can clearly see that every Ferrers diagram has one more addable cells than removable ones. In the same row as $(3,1,1)$ we have depicted the partitions that are obtained by going up a row by adding an addable cell then down a row by removing a removable cell. The fundamental property that yields our desired "basis" result is that the same collection of partitions can be obtained by first removing a removable cell then adding one of the addable cells. Using the symbol "$\nu\to\mu$" to express that μ is obtained from ν by adding an addable cell of ν, or equivalently, ν is obtained from μ by removing one of the removable cells of μ, we can derive the following basic recursions for the number of standard tableaux.

Proposition 4.5

Denoting by n_λ the number of standard tableaux of shape λ we have

$$a)\quad \sum_{\nu\vdash n-1} n_\nu \chi(\nu\to\mu) \;=\; n_\mu \quad , \qquad b)\quad \sum_{\lambda\vdash n+1} n_\lambda \chi(\mu\to\lambda) \;=\; (n+1)n_\mu \qquad 4.14$$

Proof

The identity in 4.14 a) is immediate. In fact, if μ is a partition of n we obtain a bijection between the standard tableaux enumerated by the right hand side and the collection of standard tableau enumerated by the left hand side by the removal of n and the cell containing it. For 4.14 b) we will proceed by induction on n. We can take as base case $\mu=(1)$ since we trivially see there is only one standard tableau of shape (1) and exactly one each of shapes $(1,1)$ and (2). So let us assume that 4.14 b) is true for $n-1$ and note that multiple applicationc of 4.14 a) give

$$\sum_{\lambda\vdash n+1} n_\lambda \chi(\mu\to\lambda) \;=\; \sum_{\lambda\vdash n+1}\sum_{\gamma\vdash n} n_\gamma \chi(\mu\to\lambda\leftarrow\gamma) \qquad 4.15$$

Since in the sums on the right hand side we are allowed to remove the cell we are adding, we can see that, if μ has r inner corner cells (and therefore $r+1$ outer corner sells), then by separating out the terms with $\gamma = \mu$ we can rewrite 4.15 in the form

$$\sum_{\lambda \vdash n+1} n_\lambda \chi(\mu \to \lambda) \;=\; \sum_{\lambda \vdash n+1} \sum_{\gamma \neq \mu} n_\gamma \chi(\mu \to \lambda \leftarrow \gamma) \;+\; (r+1) n_\mu \tag{4.16}$$

But the observation, that for any given μ, adding and addable and then removing a removable yields the same collection we obtain by the reverse process of removing a removable and then adding an addable, can be simply translated into the following beautiful identity

$$\sum_{\lambda \vdash n+1} \sum_{\gamma \neq \mu} n_\gamma \chi(\mu \to \lambda \leftarrow \gamma) \;=\; \sum_{\nu \vdash n-1} \sum_{\gamma \neq \mu} n_\gamma \chi(\mu \leftarrow \nu \to \gamma) \tag{4.17}$$

Thus, using 4.17 in 4.16 we derive that

$$\sum_{\lambda \vdash n+1} n_\lambda \chi(\mu \to \lambda) \;=\; \sum_{\nu \vdash n-1} \sum_{\gamma \neq \mu} n_\gamma \chi(\mu \leftarrow \nu \to \gamma) \;+\; (r+1) n_\mu \tag{4.18}$$

But now, since μ has r removable corners we see that the sum $\sum_{\gamma \vdash n} n_\gamma \chi(\mu \leftarrow \nu \to \gamma)$ will contain exactly r terms where $\gamma = \mu$. This gives

$$\sum_{\nu \vdash n-1} \sum_{\gamma \vdash n} n_\gamma \chi(\mu \leftarrow \nu \to \gamma) \;=\; \sum_{\nu \vdash n-1} \sum_{\gamma \neq \mu} n_\gamma \chi(\mu \leftarrow \nu \to \gamma) \;+\; r n_\mu$$

Using this 4.18 becomes

$$\begin{aligned} \sum_{\lambda \vdash n+1} n_\lambda \chi(\mu \to \lambda) \;&=\; \sum_{\nu \vdash n-1} \sum_{\gamma \vdash n} n_\gamma \chi(\mu \leftarrow \nu \to \gamma) \;+\; n_\mu \\ &=\; \sum_{\nu \vdash n-1} \chi(\nu \to \mu) \sum_{\gamma \vdash n} n_\gamma \chi(\nu \to \gamma) \;+\; n_\mu \\ \text{(by 4.14 b) for } n-1) \;&=\; \sum_{\nu \vdash n-1} \chi(\nu \to \mu)\, n\, n_\nu \;+\; n_\mu \\ \text{(by 4.14 a))} \;&=\; n\, n_\mu \;+\; n_\mu \;=\; (n+1) n_\mu \end{aligned}$$

completing the induction and the proof.

These two recursions combined have the following remarkable corollary.

Proposition 4.6

$$\sum_{\lambda \vdash n} n_\lambda^2 \;=\; n! \tag{4.19}$$

In particular it follows that the group algebra elements $\bigcup_{\lambda \vdash n} \{e_{ij}^\lambda\}_{i,j=1}^{n_\lambda}$ *yield a basis for* $\mathcal{A}(S_n)$.

Proof

Note that for the base case $n=2$ we have $n^2_{(1,1)}+n^2_{(2)}=1+1=2!$. Thus we can again proceed by induction on n. We will suppose then that 4.19 is true for n. Now, using 14.4 a), the left hand side of 4.19 for $n+1$ becomes

$$
\begin{aligned}
\sum_{\lambda\vdash n+1} n_\lambda^2 &= \sum_{\lambda\vdash n+1} n_\lambda \sum_{\mu\vdash n} n_\mu \chi(\mu\to\lambda)\\
&= \sum_{\mu\vdash n} n_\mu \sum_{\lambda\vdash n+1} n_\lambda \chi(\mu\to\lambda)\\
\big(\text{by 4.14 b) }\big) &= \sum_{\mu\vdash n} n_\mu\,(n+1)\,n_\mu\\
\big(\text{by induction }\big) &= (n+1)\,n! \;=\; (n+1)!
\end{aligned}
$$

Completing the induction and the proof.

It is good to keep in mind that Young's matrix units satisfy the following useful identities

Proposition 4.7

For all λ we have

$$
e^\lambda_{i,j}\Big|_\epsilon \;=\; \begin{cases} 0 & \text{if } i\neq j\\ \gamma_i^\lambda\Big|_\epsilon = \frac{1}{h_\lambda} & \text{if } i=j \end{cases} \tag{4.19}
$$

Proof

These identities are immediate consequences of 4.13. In fact, to begin with we obtain

$$
e^\mu_{i,j}\Big|_\epsilon \;=\; e^\mu_{i,1}e^\mu_{1,j}\Big|_\epsilon \;=\; e^\mu_{1,j}e^\mu_{i,1}\Big|_\epsilon \;=\; \begin{cases} 0 & \text{if } i\neq j\\ e^\lambda_{11}\Big|_\epsilon & \text{if } i=j\,\text{¸} \end{cases}
$$

But on the other hand we have

$$
e^\lambda_{11}\Big|_\epsilon \;=\; \gamma_1^\lambda(1-\gamma_2^\lambda)\cdots(1-\gamma^\lambda_{n_\lambda})\Big|_\epsilon \;=\; (1-\gamma_2^\lambda)\cdots(1-\gamma^\lambda_{n_\lambda})\gamma_1^\lambda\Big|_\epsilon \;=\; \gamma_1^\lambda\Big|_\epsilon \;=\; \frac{1}{h_\lambda}E_{S_1^\lambda}\Big|_\epsilon \;=\; \frac{1}{h_\lambda}
$$

as desired.

This given, it develops that there is a very simple explicit formula yielding the expansion of any element of the group algebra of S_n in terms of Young's matrix units.

Theorem 4.1

For any $f\in\mathcal{A}(S_n)$ we have

$$
f \;=\; \sum_{\lambda\vdash n} h_\lambda \sum_{i,j=1}^{n_\lambda} f\times e^\lambda_{j,i}\Big|_\epsilon\, e^\lambda_{i,j} \tag{4.20}
$$

Proof

Since the collection $\bigcup_{\lambda\vdash n}\{e^\lambda_{ij}\}_{i,j=1}^{n_\lambda}$ is a basis there will be coefficients $c^\lambda_{ij}(f)$ yielding

$$
f \;=\; \sum_{\lambda\vdash n}\sum_{i,j=1}^{n_\lambda} c^\lambda_{i,j}(f)\, e^\lambda_{i,j}
$$

Thus

$$f\times e^{\mu}_{rs}\Big|_{\epsilon} \;=\; \sum_{i,j=1}^{n_\mu} c^{\mu}_{i,j}(f)\, e^{\mu}_{i,j}e^{\mu}_{rs}\Big|_{\epsilon} \;=\; \sum_{i=1}^{n_\mu} c^{\mu}_{i,r}(f)\, e^{\mu}_{i,s}\Big|_{\epsilon}$$

and 4.19 gives

$$f\times e^{\mu}_{rs}\Big|_{\epsilon} \;=\; c^{\mu}_{s,r}(f)\frac{1}{h_\lambda}$$

This proves 4.20.

Note that, if we interpret every permutation $\sigma\in S_n$ as the element of $\mathcal{A}(S_n)$ which is 1 on σ and 0 elsewhere, we will have coefficients $a^{\lambda}_{i,j}(\sigma)$ yielding the expansion

$$\sigma \;=\; \sum_{\lambda\vdash n}\sum_{i,j=1}^{n_\lambda} a^{\lambda}_{i,j}(\sigma)\, e^{\lambda}_{i,j} \tag{4.21}$$

Now we have the following remarkable fact

Theorem 4.2

For each $\lambda\vdash n$ the matrices $\{A^{\lambda}(\sigma)=\|a^{\lambda}_{i,j}(\sigma)\|_{i,j=1}^{n_\lambda}\}_{\sigma\in S_n}$ yield an irreducible representation of S_n

Proof

For any $\alpha,\beta\in S_n$ we have, using 4.21 and the identities in 4.2

$$\begin{aligned}\alpha\beta \;&=\; \sum_{\lambda\vdash n}\sum_{i,j=1}^{n_\lambda}\sum_{r,s=1}^{n_\lambda} a^{\lambda}_{i,j}(\alpha)a^{\lambda}_{r,s}(\beta)\, e^{\lambda}_{i,j}e^{\lambda}_{r,s}\\ &=\; \sum_{\lambda\vdash n}\sum_{i,j=1}^{n_\lambda}\sum_{s=1}^{n_\lambda} a^{\lambda}_{i,j}(\alpha)a^{\lambda}_{j,r}(\beta)\, e^{\lambda}_{i,s} \;=\; \sum_{\lambda\vdash n}\sum_{i=1}^{n_\lambda}\sum_{s=1}^{n_\lambda}\Big(\sum_{j=1}^{n_\lambda} a^{\lambda}_{i,j}(\alpha)a^{\lambda}_{j,r}(\beta)\Big)\, e^{\lambda}_{i,s}\end{aligned} \tag{4.22}$$

But on the other hand a direct use of 4.21 for $\alpha\beta$ gives

$$\alpha\beta \;=\; \sum_{\lambda\vdash n}\sum_{i,j=1}^{n_\lambda} a^{\lambda}_{i,j}(\alpha\beta)\, e^{\lambda}_{i,j}$$

Comparing with 4.22 yields the desired product identity

$$A^{\lambda}(\alpha\beta) \;=\; A^{\lambda}(\alpha)A^{\lambda}(\beta). \tag{4.23}$$

Note further that an application of 4.20 yields (by 4.19)

$$a^{\lambda}_{i,j}(\epsilon) \;=\; h_\lambda \epsilon\, e^{\lambda}_{j,i}\Big|_{\epsilon} \;=\; \begin{cases}0 & \text{if } i\neq j\\ 1 & \text{if } i=j\end{cases}$$

In other words we also have

$$A^{\lambda}(\epsilon) \;=\; I_{n_\lambda}$$

(the $n_\lambda\times n_\lambda$ identity matrix!).

Next note that using 4.21 we derive that

$$\sigma e^{\mu}_{r,s} \;=\; \sum_{i,j=1}^{n_\mu} a^{\mu}_{i,j}(\sigma)\, e^{\mu}_{i,j} e^{\mu}_{r,s} \;=\; \sum_{i=1}^{n_\mu} a^{\mu}_{i,r}(\sigma)\, e^{\mu}_{i,s} \tag{4.24}$$

In matrix form, this can be written as

$$\sigma\big\langle e^{\mu}_{1,s}, e^{\mu}_{2,s}, \ldots, e^{\mu}_{n_\mu,s}\big\rangle \;=\; \big\langle e^{\mu}_{1,s}, e^{\mu}_{2,s}, \ldots, e^{\mu}_{n_\mu,s}\big\rangle A^{\mu}(\sigma) \tag{4.25}$$

In other words the have shown that the subspace $\mathcal{L}[e^{\mu}_{1,s}, e^{\mu}_{2,s}, \ldots, e^{\mu}_{n_\mu,s}]$ spanned by the independent set $\{e^{\mu}_{1,s}, e^{\mu}_{2,s}, \ldots, e^{\mu}_{n_\mu,s}\}$ is invariant under the action of S_n and the matrix corresponding to the action of σ on the basis $\{e^{\mu}_{1,s}, e^{\mu}_{2,s}, \ldots, e^{\mu}_{n_\mu,s}\}$ is precisely $A^{\mu}(\sigma)$.

To show that $A^{\mu}(\sigma)$ is an irreducible representation we need only show that there is no non-trivial invariant subspace or, equivalently that any non vanishing element $g \in \mathcal{L}[e^{\mu}_{1,s}, e^{\mu}_{2,s}, \ldots, e^{\mu}_{n_\mu,s}]$, has any one of the basis elements $\{e^{\mu}_{1,s}, e^{\mu}_{2,s}, \ldots, e^{\mu}_{n_\mu,s}\}$ as one of its images. To this end suppose that

$$g \;=\; \sum_{i=1}^{n_\mu} c_i e^{\mu}_{i,s}$$

with $c_u \neq 0$, (there clearly must be at least one such u). Then note that we must have

$$\tfrac{1}{c_u} e^{\mu}_{r,u}\, g \;=\; \sum_{i=1}^{n_\mu} \tfrac{c_i}{c_u}\, e^{\mu}_{ru} e^{\mu}_{i,s} \;=\; e^{\mu}_{r,s}$$

and the arbitrariness of r proves the irreducibility, completing our proof.

5. Properties of the representations $\{A^{\lambda}\}_{\lambda}$ and their characters

To identify Young's matrices $A^{\lambda}(\sigma)$ and their traces we will need an auxiliary fact which reveals a beautiful property of the tableaux function $C(T_1, T_2)$.

Proposition 5.1

For any $T_1, T_2 \in INJ(\lambda)$ and $\tau \in S_n$

$$E_{T_2T_1}\,|_{\tau} \;=\; C(\tau T_1, T_2) \tag{5.1}$$

Proof

$$E_{T_2T_1}\,|_{\tau} = \sum_{\beta_2 \in C(T_2)} \sum_{\alpha_2 \in R(T_2)} sign(\beta_2)\; \beta_2\alpha_2\, \sigma_{T_2T_1}\,|_{\tau} = \sum_{\beta_2 \in C(T_2)} \sum_{\alpha_2 \in R(T_2)} sign(\beta_2)\;\; \chi\big(\beta_2\alpha_2\, \sigma_{T_2,T_1} = \tau\big)$$

now there may not be a single pair α_2, β_2 for which $\beta_2\alpha_2\, \sigma_{T_2,T_1} = \tau$, in which case

$$E_{T_2T_1}\,|_{\tau} = 0$$

But if there is such a pair, there will be one and only one (*) , and then we must have

$$\beta_2\alpha_2\,\sigma_{T_2T_1} = \tau \quad \to \quad \beta_2\alpha_2 = \tau\,\sigma_{T_1T_2} = \sigma_{\tau T_1,T_2}$$

or

$$\sigma_{T_2,\tau T_1} \;=\; \alpha_2^{-1}\beta_2^{-1}$$

which combined with part d) of Proposition 2.1 gives that $\tau T_1 \wedge T_2$ is good. Thus from 2.7 we derive that

$$E_{T_2T_1}\,|_\tau \;=\; \begin{cases} 0 & \text{if } \;\tau T_1 \wedge T_2 \text{ is bad} \\ \\ sign(\beta_2) & \text{if } \;\tau T_1 \wedge T_2 \text{ is good} \end{cases}$$

or

$$E_{T_2T_1}\,|_\tau \;=\; C(\tau T_1, T_2)$$

as desired.

Q.E.D.

We should note that in particular we must have

$$E_{T_2T_1}\,|_\epsilon \;=\; sign(\beta_2) \;=\; C(T_1,T_2) \tag{5.2}$$

This brings us in a position to prove

Theorem 5.1

For any injective tableau T of shape $\mu \vdash n$ we have coefficients $a_i(T)$ giving

$$E_T \;=\; \sum_{i=1}^{n_\mu} a_i(T) E_{S_i^\mu,T} \tag{5.3}$$

In fact these coefficients may be directly obtained from the vector identity

$$\begin{pmatrix} a_1(T) \\ a_2(T) \\ \vdots \\ a_{n_\mu}(T) \end{pmatrix} \;=\; C^\mu(\epsilon)^{-1} \begin{pmatrix} C(S_1^\mu,T) \\ C(S_2^\mu,T) \\ \vdots \\ C(S_{n_\mu}^\mu,T) \end{pmatrix} \tag{5.4}$$

Proof.

Assume first that these coefficients do exist. Multiplying 5.3 on the right by $\sigma_{TS_j^\mu}$ gives

$$E_{TS_j^\mu} \;=\; \sum_{i=1}^{n_\mu} a_i(T)\,E_{S_i^\mu S_j^\mu} \;\; .$$

Equating coefficients of the identity, and using 5.2, we get

$$C(S_j^\mu,T) \;=\; \sum_{i=1}^{n_\mu} a_i(T)\,C(S_j^\mu,S_i^\mu)$$

(*) Recall Remark 2.1

which, in matrix notation, may be expressed as

$$\begin{pmatrix} C(S_1^\mu, T) \\ C(S_2^\mu, T) \\ \vdots \\ C(S_{n_\mu}^\mu, T) \end{pmatrix} \;=\; C^\mu(\epsilon) \begin{pmatrix} a_1(T) \\ a_2(T) \\ \vdots \\ a_{n_\mu}(T) \end{pmatrix} \;,$$

and formula 5.4 follows upon left multiplication of both sides of this identity by $C^\mu(\epsilon)^{-1}$.

It turns out that existence is an immediate consequence of 4.21, namely the general expansion identity

$$\sigma \;=\; \sum_{\lambda \vdash n} \sum_{i,j=1}^{n_\lambda} a_{i,j}^\lambda(\sigma)\, e_{i,j}^\lambda \qquad 5.5$$

In fact, for any given μ and $1 \le r \le n_\mu$ the identities in 4.3 give (using 5.5)

$$\begin{aligned} \sigma\, \gamma_r^\mu \;&=\; \sum_{i,j=1}^{n_\mu} a_{i,j}^\mu(\sigma)\, e_{i,j}^\mu \gamma_r^\mu \;=\; \sum_{i,j=1}^{n_\mu} a_{i,j}^\mu(\sigma)\, \gamma_i^\mu \sigma_{i,j}^\mu (1-\gamma_j^\mu)\cdots(1-\gamma_{n_\mu}^\mu)\gamma_r^\mu \\ &=\; \sum_{i=1}^{n_\mu} a_{i,r}^\mu(\sigma)\, \gamma_i^\mu \sigma_{i,r}^\mu \gamma_r^\mu \;=\; \sum_{i=1}^{n_\mu} a_{i,r}^\mu(\sigma)\, \gamma_i^\mu \sigma_{i,r}^\mu \end{aligned}$$

Recalling the definition in 4.1, we derive from this that

$$\sigma E_{S_r^\mu} \;=\; \sum_{i=1}^{n_\mu} a_{i,r}^\mu(\sigma)\, E_{S_i^\mu, S_r^\mu}$$

Now this can be rewritten as

$$E_{\sigma S_r^\mu} \;=\; \sum_{i=1}^{n_\mu} a_{i,r}^\mu(\sigma)\, E_{S_i^\mu, S_r^\mu}\, \sigma^{-1}$$

and for $\sigma = \sigma_{T, S_r^\mu}$ we get

$$E_T \;=\; \sum_{i=1}^{n_\mu} a_{i,r}^\mu(\sigma_{T,S_r^\mu})\, E_{S_i^\mu, S_r^\mu}\, \sigma_{S_r^\mu, T} \;=\; \sum_{i=1}^{n_\mu} a_{i,r}^\mu(\sigma_{T,S_r^\mu})\, E_{S_i^\mu, T}$$

This proves 5.3 with

$$a_i(T) \;=\; a_{i,r}^\mu(\sigma_{T,S_r^\mu}) \qquad 5.6$$

and completes our argument.

This proof has a most desirable by-product.

Theorem 5.2

For all $\mu \vdash n$ and all $\sigma \in S_n$ we have

$$A^\mu(\sigma) \;=\; C^\mu(\epsilon)^{-1} C^\mu(\sigma) \qquad 5.7$$

Proof

From 5.6 and 5.4 it follows for $\sigma S_r^\mu = T$ that

$$\begin{pmatrix} a^\mu_{1,r}(\sigma) \\ a^\mu_{2,r}(\sigma) \\ \vdots \\ a^\mu_{n_\mu,r}(\sigma) \end{pmatrix} = C^\mu(\epsilon)^{-1} \begin{pmatrix} C(S_1^\mu, \sigma S_r^\mu) \\ C(S_2^\mu, \sigma S_r^\mu) \\ \vdots \\ C(S_{n_\mu}^\mu, \sigma S_r^\mu) \end{pmatrix}$$

But this simply says that the r^{th} column of the matrix on the left of 5.7 equals the r^{th} of the matrix on the right. Finally yielding the result identity which we have anticipated since the beginning of these notes.

We are now in a position to obtain Young's formula giving the character's of his irreducible representations. This result may be stated as follows.

Theorem 5.3

For a given $\mu \vdash n$ set

$$\chi^\mu(\sigma) = trace\, A^\mu(\sigma) \qquad and \qquad \chi^\mu = \sum_{\sigma \in S_n} \chi^\mu(\sigma)\, \sigma \tag{5.8}$$

then

$$\chi^\mu = \sum_{T \in INJ(\mu)} \frac{P(T)(N(T)}{n!/n_\mu} \tag{5.9}$$

Proof

Using 5.7 and dropping some of the μ superscripts we can write

$$\chi^\mu = \sum_{\sigma \in S_n} \sigma\, trace\, C(\epsilon)^{-1} C(\sigma) = trace\, C(\epsilon)^{-1}\, F \tag{5.10}$$

where $F = \|F_{ij}\|_{i=1}^{n_\mu}$ is a matrix with group algebra entries

$$F_{ij} = \sum_{\sigma \in S_n} \sigma\, C(S_i, \sigma S_j) \ .$$

Now Proposition 5.1 gives that

$$C(S_i, \sigma S_j) = C(\sigma^{-1} S_i, S_j) = E_{S_j S_i}\, |_{\sigma^{-1}}$$

Thus

$$F_{ij} = \sum_{\sigma \in S_n} E_{S_j S_i}\, |_{\sigma^{-1}}\, \sigma = \sum_{\sigma \in S_n} E_{S_j S_i}\, |_\sigma\, \sigma^{-1} = P(S_i) \sigma_{ij} N(S_j) \ .$$

Note further that since χ^μ is a class function it follows that we also have

$$\chi^\mu = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \chi^\mu \sigma^{-1}$$

and we can thus write, using 5.10

$$\chi^\mu \;=\; \frac{1}{n!}\sum_{\sigma\in S_n} \sigma\left(trace\, C(\epsilon)^{-1}F\right)\sigma^{-1} \;=\; \frac{1}{n!}\, trace\, C(\epsilon)^{-1}\sum_{\sigma\in S_n} \sigma\, F\sigma^{-1}\ .$$

But we see that

$$\sum_{\sigma\in S_n}\sigma\, F_{ij}\sigma^{-1} \;=\; \sum_{\sigma\in S_n}\sigma\, P(S_i)\sigma_{ij}N(S_j)\sigma^{-1} \;=\; \sum_{\sigma\in S_n}\sigma\, N(S_j)P(S_i)\sigma_{ij}\sigma^{-1} \;=\; 0$$

at least when $i>j$. This means that the matrix

$$\sum_{\sigma\in S_n}\sigma\, F\sigma^{-1}$$

is upper triangular and since $C(\epsilon)^{-1}$ is upper triangular with unit diagonal elements we must conclude that

$$\begin{aligned}\chi^\mu \;&=\; \frac{1}{n!}\, trace \sum_{\sigma\in S_n}\sigma\, F\sigma^{-1} \;=\; \frac{1}{n!}\sum_{\sigma\in S_n}\sigma\left(\sum_{i=1}^{n_\mu}F_{i,i}\right)\sigma^{-1}\\ &=\; \frac{1}{n!}\sum_{i=1}^{n_\mu}\sum_{\sigma\in S_n}\sigma\, P(S_i)N(S_i)\sigma^{-1} \;=\; \frac{n_\mu}{n!}\sum_{T\in INJ(\lambda)}P(T)N(T)\end{aligned}$$

which is 5.9 precisely as asserted.

The character χ^μ is in fact very closely related to the matrix units. More precisely

Proposition 5.2

Setting

$$e^\mu \;=\; \sum_{i=1}^{n_\mu} e^\mu_{i,i} \qquad 5.11$$

we have

$$\chi^\mu \;=\; h_\mu\, e^\mu \qquad 5.12$$

in particular it follows from the identities in 4.3 that

$$\chi_\mu\times\chi_\mu \;=\; h_\mu\,\chi_\mu \qquad 5.13$$

Proof

Note that using 4.21 we get

$$\begin{aligned} e^\mu_{i,i}\Big|_{\sigma^{-1}} = \sigma\, e^\mu_{i,i}\Big|_\epsilon \;&=\; \sum_{\lambda\vdash n}\sum_{r,s=1}^{n_\lambda} a^\lambda_{r,s}(\sigma)\, e^\lambda_{r,s}\, e^\mu_{i,i}\Big|_\epsilon \;=\; \sum_{r,s=1}^{n_\mu} a^\mu_{r,s}(\sigma)\, e^\mu_{r,s}\, e^\mu_{i,i}\Big|_\epsilon\\ &=\; \sum_{r=1}^{n_\mu} a^\mu_{r,i}(\sigma)\, e^\mu_{r,i}\Big|_\epsilon \;=\; a^\mu_{i,i}(\sigma)\tfrac{1}{h_\mu}\ \ \text{(by 4.19)}\end{aligned}$$

and summing for i

$$e^{\mu}\Big|_{\sigma^{-1}} \;=\; \sum_{i=1}^{n_\mu} e^{\mu}_{i,i}\Big|_{\sigma^{-1}} \;=\; \tfrac{1}{h_\mu}\chi^{\mu}(\sigma) \tag{5.14}$$

Now note that since e^{μ} is a central element of $\mathcal{A}(S_n)$ it follows that

$$e^{\mu}\Big|_{\sigma^{-1}} \;=\; e^{\mu}\Big|_{\sigma}.$$

Thus multiplying both sides of 5.14 by σ and summing over S_n gives 5.12 and completes our proof.

Now a result parallel to 5.13 can be obtained by direct manipulations of Young idempotents.

Proposition 5.3

$$\Big(\sum_{T\in INJ(\mu)} N(T)P(T)\Big)^2 \;=\; h_\mu^2 \sum_{T\in INJ(\mu)} N(T)P(T) \tag{5.15}$$

Proof

Note we may rewrite the left hand aside of 5.15 as

$$LHS \;=\; \sum_{T\in INJ(\mu)}\sum_{\sigma\in S_n} N(T)P(T)N(\sigma T)P(\sigma T) \tag{5.16}$$

But $P(T)N(\sigma T)\neq 0$ implies $T\wedge \sigma T$ is good. Thus from Proposition 2.1 we derive that the only non vanishing summands in are obtained for $\sigma=\alpha\beta$ with $\alpha\in R(T)$ and $\beta\in C(T)$. Since we have

$$\begin{aligned} N(T)P(T)N(\alpha\beta T)P(\alpha\beta T) &= N(T)P(T)\alpha\beta N(T)P(T)\beta^{-1}\alpha^{-1} \\ &= N(T)P(T)N(T)P(T)\, sign(\beta)\beta^{-1}\alpha^{-1} \\ \text{(by 4.8)} &= h_\mu N(T)P(T)\, sign(\beta)\beta^{-1}\alpha^{-1} \end{aligned}$$

the identity in 5.16 becomes

$$\begin{aligned} LHS &= h_\mu \sum_{T\in INJ(\mu)} N(T)P(T) \sum_{\substack{\alpha\in R(T)\\ \beta\in C(T)}} sign(\beta)\beta^{-1}\alpha^{-1} \\ &= h_\mu \sum_{T\in INJ(\mu)} N(T)P(T)N(T)P(T) \end{aligned}$$

and a further use of 4.8 gives 5.15 as desired.

The result we have anticipated since Von Neuman Lemma is now finally within reach.

Theorem 5.4

$$h_\mu \;=\; \frac{n!}{n_\mu} \tag{5.17}$$

Proof

Squaring both sides of 5.9 gives, using 5.13 and 5.15

$$h_\mu\,\chi^{\mu} \;=\; \frac{h_\mu^2}{(n!/n_\mu)^2}\sum_{T\in INJ(\mu)} P(T)N(T)$$

canceling the common factor h_μ and using 5.9 again gives

$$\chi^{\mu} \;=\; \frac{h_\mu}{n!/n_\mu}\chi^{\mu}$$

and 5.17 follows from the non-vanishing of χ^{μ}.

6. The Frobenius map

The group algebra $\mathcal{A}(S_n)$ has a natural scalar product obtained by setting for any $f, g \in \mathcal{A}(S_n)$

$$< f, g > = \frac{1}{n!} \sum_{\gamma \in S_n} f(\gamma)\overline{g(\gamma)}. \qquad 6.1$$

Note that since for all $\alpha \in S_n$

$$\alpha f = \sum_{\gamma \in S_n} f(\gamma)\alpha\gamma = \sum_{\beta \in S_n} f(\alpha^{-1}\beta)\beta.$$

we have

$$\begin{aligned} < \alpha f, g > &= \frac{1}{n!} \sum_{\beta \in S_n} f(\alpha^{-1}\beta)\overline{g(\beta)} \\ &= \frac{1}{n!} \sum_{\gamma \in S_n} f(\gamma)\overline{g(\alpha\sigma)} = < f, \alpha^{-1} g > . \end{aligned} \qquad 6.2$$

Similarly we show that

$$< f\alpha, g > = < f, g\alpha^{-1} > . \qquad 6.3$$

Let us denote by C_μ the element of the group algebra of S_n that is obtained by summing all the permutatiions with cycle structure μ. Now we have shown that the cardinality of this collection of permutations is given by the ratio

$$|C_\mu| \;=\; \frac{n!}{z_\mu} \qquad 6.4$$

where for a μ with α_i parts equal to i it is customary to set

$$z_\mu \;=\; 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n} \alpha_1! \alpha_2! \cdots \alpha_n! \qquad 6.5$$

Using this and the fact that conjugacy classes are disjoint it follows that

$$\big\langle C_\mu \,,\, C_\lambda \big\rangle \;=\; \begin{cases} \frac{1}{n!}\frac{n!}{z_\mu} = \frac{1}{z_\mu} & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases} \qquad 6.6$$

There is also a natural scalar product in the space $\Lambda^{=n}$ of symmetric polynomial homogeneous of degree n which is obtained by setting for two power basis elements p_λ, p_μ

$$\big\langle p_\lambda \,,\, p_\mu \big\rangle \;=\; \begin{cases} z_\mu & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases} \qquad 6.7$$

This given note that if we set

$$F\, C_\mu \;=\; p_\mu / z_\mu \qquad 6.8$$

and use the fact that $\Lambda^{=n}$ and the center $\mathcal{C}(S_n)$ have the same dimension, we can define an invertible linear map

$$F : \quad \mathcal{C}(S_n) \;\longleftrightarrow\; \Lambda^{=n}$$

which is also an isometry. In fact, note that from 6.6 and 6.7 it follows that

$$\langle FC_\mu \, , \, FC_\lambda \rangle \;=\; \tfrac{1}{z_\mu^2}\langle p_\mu \, , \, p_\lambda \rangle \;=\; \tfrac{1}{z_\mu}\chi(\lambda=\mu)] \;=\; \langle C_\mu \, , \, C_\lambda \rangle$$

Now Frobenius introduced this map precisely for the purpose of constructing all the irreducible characters of S_n. Frobenius result can be stated as follows

Theorem 6.1

The class functions

$$\xi^\lambda \;=\; F^{-1}\, s_\lambda \qquad 6.9$$

form a complete set of the irreducible characters of S_n.

The surprising fact is that Young's and Frobenius' partition indexing of the irreducible characters turn out to be identical. This result, established by Young himself, may be simply be stated as follows

Theorem 6.2 (A. Young)

$$F^{-1} s_\lambda \;=\; \frac{n_\lambda}{n!} \sum_{T\in INJ(\lambda)} N(T)P(T) \qquad 6.10$$

We should mention that to the best our knowledge there is at the moment no direct proof of this fact. Moreover, even Young's "proof" is based on a very dubious use of his quite heuristic (hair) "raising operators". Our proof as we will later see is very indirect. This given, it is worthwhile taking a close look at what needs to be proved. To begin we need the following very convenient form of the Frobenius map

Proposition 6.1

For each $\sigma \in S_n$ *set*

$$\psi(\sigma) \;=\; p_{\lambda(\sigma)} \qquad 6.11$$

where $\lambda(\sigma)$ *is the partition giving the cycle structure of* σ. *This given, for any class function* $g \in \mathcal{C}(S_n)$ *we have*

$$F\, g \;=\; \langle g \, , \, \psi \rangle \qquad 6.12$$

Proof

The proof is immediate. We need only verify 6.12 for the conjugacy class elements C_μ with $\mu \vdash n$. In this case the definition in 6.1 gives

$$F\, C_\mu \;=\; \langle C_\mu \, , \, \psi \rangle \;=\; \frac{1}{n!}\sum_{\sigma\in S_n} C_\mu\Big|_\sigma p_{\lambda(\sigma)} \;=\; \frac{1}{n!}\, p_\mu \frac{n!}{z_\mu} \;=\; \frac{p_\mu}{z_\mu}$$

which is precisely the definition in 6.8.

As a corollary we obtain

Proposition 6.2

The equality in 6.10 amounts to showing that for any injective tableau T of shape λ we have

$$s_\lambda \;=\; \frac{1}{h_\lambda}\sum_{\alpha\in R(T)}\sum_{\beta\in C(T)} sign(\beta)\; p_{\lambda(\alpha\beta)} \qquad 6.13$$

Proof

Note that 6.10 may also be written in the form

$$s_\lambda \;=\; \frac{n_\lambda}{n!}\,F\sum_{\sigma\in S_n} N(\sigma T)P(\sigma T) \;=\; \frac{n_\lambda}{n!}\,F\sum_{\sigma\in S_n} \sigma N(T)P(T)\sigma^{-1}$$

Now, using 6.12 we get

$$\begin{aligned}
s_\lambda &= \frac{n_\lambda}{n!}\sum_{\sigma\in S_n}\big\langle \sigma N(T)P(TY)\sigma^{-1}\,,\,\psi\big\rangle\\
\text{(by 6.2 and 6.3)} &= \frac{n_\lambda}{n!}\sum_{\sigma\in S_n}\big\langle N(T)P(TY)\,,\,\sigma^{-1}\psi\sigma\big\rangle\\
\text{(since } \psi \text{ is central)} &= \frac{n_\lambda}{n!}\sum_{\sigma\in S_n}\big\langle N(T)P(T)\,,\,\psi\big\rangle \;=\; n_\lambda\big\langle N(T)P(T)\,,\,\psi\big\rangle\\
&= \frac{n_\lambda}{n!}\sum_{\alpha\in R(T)}\sum_{\beta\in C(T)} sign(\beta)\; p_{\lambda(\alpha\beta)}
\end{aligned}$$

This proves 6.13.

Theorem 3.6

For any given tableau T_o of shape $\mu \vdash n$, the characters of the action of S_n on the left ideals

$$a) \quad \mathcal{A}(S_n)P(T_o) \quad \text{and} \quad b) \quad \mathcal{A}(S_n)N(T_o)$$

are respectively given by the group algebra elements

$$a) \quad P^\mu = \sum_{T\in INJ(\mu)} \frac{P(T)}{\mu!} \quad \text{and} \quad b) \quad N^\mu = \sum_{T\in INJ(\mu)} \frac{N(T)}{\mu'!} \tag{3.19}$$

Proof

Since both

$$\frac{P(T_o)}{\mu!} \quad \text{and} \quad \frac{N(T_o)}{\mu'!}$$

are idempotents, formula 8.6 (**R**) yields that these characters are

$$P^\mu \;=\; \sum_{\sigma\in S_n} \sigma \frac{P(T_o)}{\mu!}\sigma^{-1} \quad \text{and} \quad N^\mu \;=\; \sum_{\sigma\in S_n} \sigma \frac{N(T_o)}{\mu'!}\sigma^{-1}$$

Now these exressions can be rewritten in the form

$$P^\mu = \sum_{\sigma\in S_n} \frac{P(\sigma T_o)}{\mu!} \quad \text{and} \quad N^\mu = \sum_{\sigma\in S_n} \frac{N(\sigma T_o)}{\mu'!}$$

and the identities in 3.19 follow since, as σ varies in S_n, the tableau σT_o describes all the injective tableaux of shape μ.

It develops that we have the tools to derive the following basic facts about these these two characters.

Theorem 3.7

For $\mu \vdash n$ and $T_o \in INJ(\mu)$, let $K_{\lambda\mu}$ denote the multiplicity of Young's representation A^λ in the action of S_n on $\mathcal{A}(S_n)P(T_o)$ then

$$a) \quad K_{\lambda\mu} > 0 \quad \Longrightarrow \quad \lambda \geq_D \mu \quad \text{and} \quad b) \quad K_{\mu\mu} = 1 \ . \tag{3.20}$$

It then follows that we have the expansions

$$a) \quad P^\mu \;=\; \chi^\mu + \sum_{\lambda >_D \mu} \chi^\lambda K_{\lambda\mu} \quad \text{and} \quad b) \quad N^\mu \;=\; \chi^\mu + \sum_{\lambda <_D \mu} \chi^\lambda K_{\lambda'\mu'} \tag{3.21}$$

In particular we derive that

$$P^\mu \times N^\mu \;=\; h_\mu \, \chi^\mu \tag{3.22}$$

Proof

The orthonormality of the irreducible character basis immediately gives the expansions

$$a) \quad P^\mu \;=\; \sum_{\lambda\vdash n} \chi^\lambda \left\langle P^\mu \, , \, \chi^\lambda \right\rangle \quad \text{and} \quad b) \quad N^\mu \;=\; \sum_{\lambda\vdash n} \chi^\lambda \left\langle N^\mu \, , \, \chi^\lambda \right\rangle \tag{3.23}$$

Now note that our definition of priming introduced in section 2 applied to formula 3.9 gives that

$$(\chi^\lambda)' \;=\; \chi^{\lambda'} \tag{3.24}$$

Thus priming 3.23 a), written for μ replaced by μ', gives

$$N^\mu \;=\; \sum_{\lambda\vdash n} \chi^{\lambda'} \big\langle P^{\mu'}\,,\,\chi^\lambda\big\rangle\;,$$

and this may be rewritten as

$$N^\mu \;=\; \sum_{\lambda\vdash n} \chi^{\lambda} \big\langle P^{\mu'}\,,\,\chi^{\lambda'}\big\rangle\,.$$

Thus we see that 3.21 b) follows from 3.21 a). To prove 3.20 note that we may write

$$K_{\lambda\mu} = \big\langle P^\mu\,,\,\chi^\lambda\big\rangle = \frac{1}{\mu! h_\lambda} \sum_{T_1\in INJ(\mu)}\;\sum_{T_2\in INJ(\lambda)} \big\langle P(T_1)\,,\,N(T_2)P(T_2)\,\big\rangle\;. \tag{3.25}$$

But since

$$\big\langle P(T_1)\,,\,N(T_2)P(T_2)\,\big\rangle \;=\; \big\langle N(T_2)P(T_1)\,,\,P(T_2)\,\big\rangle\,,$$

we see from 2.28 that, if a single summand in 3.25 fails to vanish, we must necessarily have

$$\lambda \geq_D \mu\,.$$

This proves 3.20 a). To prove 3.21 a) assume next that $\lambda=\mu$. We then simultaneously have

$$P^\mu = \frac{1}{\mu!}\sum_{\sigma\in S_n} \sigma\,P(\sigma T_o)\sigma^{-1} \quad\text{and}\quad \chi^\mu = \frac{1}{h_\mu}\sum_{\sigma\in S_n} N(\sigma T_o)P(\sigma T_o)$$

Thus

$$\begin{aligned} K_{\mu\mu} \;=\; \big\langle P^\mu\,,\,\chi^\mu\big\rangle \;&=\; \frac{1}{\mu!}\sum_{\sigma\in S_n} \big\langle \sigma\,P(T_o)\sigma^{-1},\,\chi^\mu\big\rangle \\ &=\; \frac{1}{\mu!}\sum_{\sigma\in S_n} \big\langle\, P(T_o),\,\sigma^{-1}\chi^\mu\,\sigma\,\big\rangle \\ &=\; \frac{n!}{\mu!}\big\langle\, P(T_o),\,\chi^\mu\;\big\rangle \\ &=\; \frac{n!}{\mu! h_\mu}\sum_{\sigma\in S_n} \big\langle P(T_o)\,,\,N(\sigma T_o)P(\sigma T_o)\big\rangle\;. \end{aligned} \tag{3.26}$$

But we see that we have

$$\big\langle P(T_o)\,,\,N(\sigma T_o)P(\sigma T_o)\big\rangle \;=\; \big\langle N(\sigma T_o)P(T_o)\,,\,P(\sigma T_o)\big\rangle$$

and so the only case in which this summand fails to vanish is when

$$\sigma T_o \;=\; \alpha\beta T_o \qquad\text{with}\quad \alpha\in R(T_o)\quad\text{and}\quad \beta\in C(T_o)$$

This reduces 3.26 to

$$
\begin{aligned}
K_{\mu\mu} &= \frac{n!}{\mu! h_\mu} \sum_{\alpha \in R(T_o)} \sum_{\beta \in C(T_o)} \big\langle P(T_o)\,,\ \alpha\beta N(T_o) P(T_o) \beta^{-1}\alpha^{-1} \big\rangle \\
&= \frac{n!}{\mu! h_\mu} \sum_{\alpha \in R(T_o)} \sum_{\beta \in C(T_o)} sign(\beta) \big\langle \alpha^{-1} P(T_o)\, \alpha\,,\ N(T_o) P(T_o) \beta^{-1} \big\rangle \\
&= \frac{n!}{h_\mu} \sum_{\beta \in C(T_o)} sign(\beta) \big\langle P(T_o)\,,\ N(T_o) P(T_o) \beta^{-1} \big\rangle = \frac{n!}{h_\mu} \ \big\langle P(T_o)\,,\ N(T_o) P(T_o) N(T_o)\, \big\rangle \\
&= \frac{n!}{h_\mu} \ \big\langle id\,,\ N(T_o) P(T_o) N(T_o) P(T_o)\, \big\rangle = n! \ \big\langle id\,,\ N(T_o) P(T_o)\, \big\rangle \ = \ 1
\end{aligned}
$$

as desired. This completes the proof of our theorem since 3.22 follows immediately from 3.21 because of the relations in 5.10 (**R**)