Proof of the 2-part Compositional Shuffle Conjecture

by

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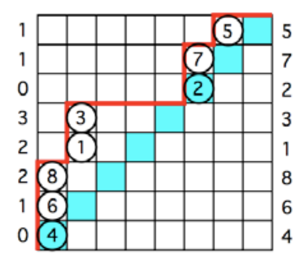
ABSTRACT

In a recent paper [9] J. Haglund, J. Morse and M. Zabrocki advanced a refinement of the Shuffle Conjecture of Haglund et all [8]. They introduce the notion of "*touch composition* " of a Dyck path, whose parts yield the positions where the path touches the diagonal. They conjectured that the polynomial $\langle \nabla \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}1\, ,\, h_{\mu_1}h_{\mu_2}\cdots h_{\mu_l}\rangle$, where $\mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}1$ is essentially a rescaled Hall-Littlewood polynomial and ∇ is the Macdonald eigen-operator introduced in [1], enumerates by $t^{area}q^{dinv}$ the parking functions whose Dyck paths hit the diagonal by $(p_1, p_2, \ldots, p_k)$ and whose diagonal word is a shuffle of l increasing words of lengths $\mu_1, \mu_2, \ldots, \mu_k$. In this paper we prove the case $l = 2$ of this conjecture.

I. Introduction

Parking functions are endowed by a colorful history and jargon (see for instance [7]) that is very helpful in dealing with them combinatorially as well as analytically. Here we will represent them interchangeably as two line arrays or as tableaux. A single example of this correspondence should be sufficient for our purposes. In the figure below we have on the left the two line array, with the list of cars $V = (v_1, v_2, \ldots, v_n)$ on top and their diagonal numbers $U = (u_1, u_2, \ldots, u_n)$ on the bottom. In the corresponding $n \times n$ tableau of lattice cells we have shaded the "*main diagonal* " (or 0-diagonal) and drawn the "*supporting*" Dyck path. The component u_i gives the number of lattice cells EAST of the i^{th} NORTH step and WEST of the main diagonal. The cells adjacent to the NORTH steps of the path are filled with the corresponding cars from bottom to top.

$$PF = \begin{bmatrix} 4 & 6 & 8 & 1 & 3 & 2 & 7 & 5 \\ 0 & 1 & 2 & 2 & 3 & 0 & 1 & 1 \end{bmatrix} \quad \Longleftrightarrow \tag{I.1}$$



The resulting tableau uniquely represents a parking function if and only if the cars increase up the columns.

A necessary and sufficient condition for the vector U to give a Dyck path is that

$$u_1 = 0 \quad \text{and} \quad 0 \le u_i \le u_{i-1} + 1 \tag{I.2}$$

This given, the column increasing property of the corresponding tableau is assured by the requirement that $V = (v_1, v_2, \ldots, v_n)$ is a permutation in S_n satisfying

$$u_i = u_{i-1} + 1 \implies v_i > v_{i-1} \tag{I.3}$$

We should mention that the component u_i may also be viewed as the order of the diagonal supporting car v_i. In the example above, car 3 is in the third diagonal, 1 and 8 are in the second diagonal, $5, 7$ and 6 are in the first diagonal and 2 and 4 are in the main diagonal. We have purposely listed the cars by diagonals from right to left starting with the highest diagonal. This gives the "*diagonal word*" of PF which we will denote $\sigma(PF)$. It is easily seen that $\sigma(PF)$ can also be obtained directly from the 2-line array by successive right to left readings of the components of the vector $V = (v_1, v_2, \ldots, v_n)$ according to decreasing values of $u_1, u_2, \ldots, u_n$. In previous work, each parking function is assigned a "*weight*"

$$w(PF) \;=\; t^{area(PF)} q^{dinv(PF)} \qquad \text{I.4}$$

where

$$area(PF) = u_1 + u_2 + \cdots + u_n \qquad \text{I.5}$$

and

$$dinv(PF) = \sum_{1 \le i < j \le n} \Big(\chi(u_i = u_j \,\&\, v_i < v_j) \;+\; \chi(u_i = u_j + 1 \,\&\, v_i > v_j) \Big) \qquad \text{I.6}$$

It is clear from this imagery, that the sum in I.5 gives the total number of cells between the supporting Dyck path and the main diagonal. We also see that two cars in the same diagonal with the car on the left smaller than the car on the right will contribute a unit to $dinv(PF)$, we call this a "*primary diagonal inversion*". The same holds true when a car on the left is bigger than a car on the right with the latter in the adjacent lower diagonal, we call this a "*secondary diagonal inversion*". For instance in I.1 we see $(6, 7)$ as the only primary diagonal inversion and $(6, 2)$, $(8, 7)$, $(8, 5)$ as the secondary ones. Thus in the the present example we have

$$area(PF) = 10, \;\; dinv(PF) = 4, \;\; \sigma(PF) = 31857624,$$

yielding

$$w(PF) \;=\; t^{10} q^4$$

Here and after, the vectors U and V in the two line representation will be also referred to as $U(PF)$ and $V(PF)$. It will also be convenient to denote by $\mathcal{PF}_n$ the collection of parking functions in the $n \times n$ lattice square.

In [9] J. Haglund, M. Morse and M. Zabrocki introduced a new parking function statistic they call "*touch composition*". This is the composition $p(PF)$ whose parts give the sizes of the intervals between successive 0's of the vector $U(PF)$. Geometrically the parts of $p(PF)$ yield the places where the supporting Dyck path hits the main diagonal. For instance for the PF in I.1 we have $p(PF) = (5, 3)$.

The "*Compositional Shuffle conjecture*" [9] states that for any composition $(p_1, p_2, \ldots, p_k) \models n$ and any partition $\mu = (\mu_1, \mu_2, \ldots, \mu_l) \vdash n$ we have the identity

$$\big\langle \nabla \mathbf{C}_{p_1} \mathbf{C}_{p_2} \cdots \mathbf{C}_{p_k} 1 \,,\, h_{\mu_1} h_{\mu_2} \cdots h_{\mu_l} \big\rangle = \sum_{\substack{PF \in \mathcal{PF}_n \\ p(PF) = (p_1, p_2, \ldots, p_k)}} t^{area(PF)} q^{dinv(PF)} \chi(\sigma(PF) \in E_1 \sqcup\!\sqcup E_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup E_l) \qquad \text{I.7}$$

where ∇ is the Macdonald eigen-operator introduced in [1], $h_{\mu_1} h_{\mu_2} \cdots h_{\mu_l}$ is the "homogeneous" symmetric function basis indexed by μ, $E_1, E_2, \ldots, E_l$ are successive segments of the word $1234 \cdots n$ of respective lengths $\mu_1, \mu_2, \ldots, \mu_l$ and the symbol "$\chi(\sigma(PF) \in E_1 \sqcup\!\sqcup E_2 \sqcup\!\sqcup \cdots \sqcup\!\sqcup E_l)$" is to indicate that the sum is to be carried out over parking functions in $\mathcal{PF}_n$ whose diagonal word is a shuffle of the words $E_1, E_2, \ldots, E_l$. Last but not least the operator $\mathbf{C}_a$ acts on a a symmetric polynomial $F[X]$ according to the plethystic formula

$$\mathbf{C}_a F[X] \;=\; (-\tfrac{1}{q})^{a-1} F\big[X - \tfrac{1-1/q}{z}\big] \sum_{m \ge 0} z^m h_m[X] \Big|_{z^a}, \qquad \text{I.8}$$

In this paper we show that the symmetric function methods developed in [5] can be used to prove the $l = 2$ case of I.7, that is the identity

$$\big\langle \nabla \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k} 1\ ,\ h_r h_{n-r}\big\rangle = \sum_{\substack{PF\in\mathcal{PF}_n \\ p(PF)=(p_1,p_2,\ldots,p_k)}} t^{area(PF)} q^{dinv(PF)} \chi\big(\sigma(PF)\in 12\cdots r \sqcup r+1\cdots n\big) \qquad \text{I.9}$$

Since in [9] it is shown that

$$\sum_{p\models n} \mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots \mathbf{C}_{p_k} 1 \;=\; e_n, \qquad \text{I.10}$$

summing I.9 over all compositions of n we obtain that

$$\big\langle \nabla e_n\ ,\ h_r h_{n-r}\big\rangle \;=\; \sum_{PF\in\mathcal{PF}_n} t^{area(PF)} q^{dinv(PF)} \chi\big(\sigma(PF)\in 12\cdots r \sqcup r+1\cdots n\big) \qquad \text{I.11}$$

which is the 2-part case of the original Shuffle Conjecture. The identity in I.11 was, in fact, established, in a 2004 paper [6], by Haglund as the ultimate bi-product of an intricate variety of new identities of Macdonald Polynomial Theory. Our proof of I.9 turns out to be much simpler and uses even less machinery than the simplified version of Haglund's original proof given in [4] . Basically, as was done in [5], we only use a small collection of Macdonald polynomial identities established much earlier in [2] and [3] to prove a recursion satisfied by the left hand side of I.9 . Then show that the right hand side satisfies the same recursion, with equality in the base cases.

This recursion, which is the crucial result of this paper, may be stated as follows

Theorem I.1

For all compositions $p = (p_1, p_2, \ldots, p_k)$ and $0 < r < n$ we have

$$\begin{aligned}\big\langle \nabla C_{p_1}C_{p_2}\cdots C_{p_k}1\ ,h_r h_{n-r}\big\rangle \;&=\; t^{p_1-1}\big\langle \nabla B_{p_1-2}C_{p_2}\cdots C_{p_k}1\ ,\ h_{r-1}h_{n-1-r}\big\rangle + \\ &+\ \chi(p_1=1)\Big(\big\langle \nabla C_{p_2}\cdots C_{p_k}1\ ,\ h_r h_{n-1-r}\big\rangle + \big\langle \nabla C_{p_2}\cdots C_{p_k}1\ ,\ h_{r-1}h_{n-r}\big\rangle\Big)\end{aligned} \qquad \text{I.9}$$

with $\mathbf{B}_a = \omega\widetilde{\mathbf{B}}_a\omega$ and for any symmetric function $F[X]$

$$\widetilde{\mathbf{B}}_a F[X] \;=\; F\big[X - \tfrac{1-q}{z}\big]\sum_{m\geq 0} z^m h_m[X]\Big|_{z^a} \qquad \text{I.10}$$

What is remarkably different in this case in contrast with the developments in [5], is that here the symmetric function side guided us on what had to be done in the combinatorial side. In fact we shall see that I.9 unravels in a totally unexpected manner some surprising inclusion-exclusions of Parking Functions.

The reader is advised to have at hand a copy of [5] in reading the present work not only for specific references to the identities we use here but also for the notation and definitions of the various symmetric function constructs we deal with in this writing. We already gave in section 2 of [5] titled a " A Macdonald Polynomial kit"' a detailed list of the Macdonald Polynomial Theory identities that play an essential role in this Branch of Algebraic combinatorics, and thus we will not repeat it here .

This paper is divided into three sections, in the first section we prove some auxiliary symmetric function identities we use here that are not in [5], in the second section we prove Theorem I.1 and in the third section we derive all its combinatorial consequences.

Acknowledgment. The Authors are indebted to Angela Hicks for helpful guidance in the combinatorial part of this work.

1. Auxiliary symmetric function identities

As we mentioned in the Introduction, this section makes heavy use of the notation, definitions and identities listed in section 2 of [5]. We will present this auxiliary material as a sequence of propositions.

The first obstacle that is encountered in dealing with the Shuffle Conjecture is to obtain a useable expression for the scalar product of a Macdonald polynomial with a homogeneous basis element, in the two-part case this obstacle can be overcome by means of the following identity proved in [3]

Proposition 1.1

For all $f \in \Lambda^{=r}$ and $\mu \vdash n$ we have

$$\langle f h_{n-r}\, ,\, \tilde{H}_\mu \rangle \;=\; \nabla^{-1}\big(\omega f[\tfrac{X-\epsilon}{M}]\big)\big|_{X\to MB_\mu -1} \tag{1.1}$$

This given, we obtain

Proposition 1.2

For $\mu \vdash n$ and $0<r<n$

$$\langle h_r h_{n-r}\, ,\, \tilde{H}_\mu \rangle \;=\; F_r[MB_\mu - 1] \tag{1.2}$$

with

$$F_r[X] \;=\; \sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}]\nabla^{-1} e_k[\tfrac{X}{M}] \tag{1.3}$$

Proof

From 1.1 with $f = h_r$ we derive that

$$\langle h_r h_{n-r}\, ,\, \tilde{H}_\mu \rangle \;=\; \nabla^{-1}\big(e_r[\tfrac{X-\epsilon}{M}]\big) \;\;=\;\; \sum_{k=0}^{r} e_{r-k}[\tfrac{-\epsilon}{M}]\nabla^{-1}e_k[\tfrac{X}{M}] \;\;=\;\; \sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}]\nabla^{-1}e_k[\tfrac{X}{M}]$$

and 1.2 is thus a consequence of Proposition 1.1.

Proposition 1.3

With n factors $\mathbf{C}_1$ we have

$$\mathbf{C}_1\mathbf{C}_1\cdots\mathbf{C}_1 1 \;\;=\;\; q^{-\binom{n}{2}}(q,q)_n h_n\big[\tfrac{X}{1-q}\big] \;\;=\;\; q^{-\binom{n}{2}}\tilde{H}_n[X;q] \tag{1.4}$$

In particular it follows that

$$\nabla\mathbf{C}_1\mathbf{C}_1\cdots\mathbf{C}_1 1 \;\;=\;\; (q,q)_n h_n\big[\tfrac{X}{1-q}\big] \tag{1.5}$$

Proof

From the definition in 1.8 it follows that

$$C_1 1 \;\;=\;\; e_1[X] \;\;=\;\; (1-q)e_1[\tfrac{X}{1-q}]$$

which the case $n=1$ of 1.4. So we will proceed by induction and assume that we have

$$C_1^{n-1}1 \;\;=\;\; q^{-\binom{n-1}{2}}(q,q)_{n-1}h_{n-1}\big[\tfrac{X}{1-q}\big].$$

This given, applying $\mathbf{C}_1$ to both sides and using I.8 again we get

$$\begin{aligned}
\tfrac{q^{\binom{n-1}{2}}}{(q,q)_{n-1}}\, C_1^n 1 \;&=\; h_{n-1}\big[\tfrac{(X-\frac{1-1/q}{z})}{1-q}\big] \sum_{m\geq 0} z^m h_m[X]\Big|_z\Big|_z \\
&=\; h_{n-1}\big[\tfrac{X}{1-q} \,+\, (X - \tfrac{1-1/q}{z(1-q)})\big] \sum_{m\geq 0} z^m h_m[X]\Big|_z \;=\; h_{n-1}\big[\tfrac{X}{1-q} + \tfrac{1}{qz}\big] \sum_{m\geq 0} z^m h_m[X]\Big|_z\Big|_z \\
&=\; \sum_{k=0}^{n-1} h_{n-1-k}[\tfrac{X}{1-q}]\tfrac{1}{q^k} h_{k+1}[X] \;=\; q\sum_{k=0}^{n-1} h_{n-1-k}[\tfrac{X}{1-q}] h_{k+1}[\tfrac{X}{q}] \\
&=\; q\sum_{k=0}^{n} h_{n-k}[\tfrac{X}{1-q}] h_k[\tfrac{X}{q}] \,-\, q h_n[\tfrac{X}{1-q}] \\
&=\; q h_n\big[\tfrac{X}{1-q} + \tfrac{X}{q}\big] \,-\, q h_n[\tfrac{X}{1-q}] \;=\; q h_n\big[\tfrac{X(q+(1-q)}{q(1-q)}\big] \,-\, q h_n[\tfrac{X}{1-q}] \\
&=\; \tfrac{1}{q^{n-1}} h_n\big[\tfrac{X}{(1-q)}\big] \,-\, q h_n[\tfrac{X}{1-q}] \;=\; \tfrac{1-q^n}{q^{n-1}} h_n[\tfrac{X}{1-q}]
\end{aligned}$$

This completes the induction proves the first equality in 1.4. The second equality i results from a well known formula for the Macdonald polynomial $\tilde{H}_\mu$ when $\mu = (n)$ The equality in 1.5 follows then from the definition of the operator ∇.

Our next auxiliary result shows how the $\mathbf{C}$ and $\mathbf{B}$ operators commute, but to prove it we need some notation. For $E_1, E_2, \ldots, E_k$ given expressions and $P[X]$ a symmetric polynomial we set

$$P^{(r_1,r_2,\ldots,r_k)}[X] \;=\; P[X + E_1u_1 + E_2u_2 + \cdots + E_ku_k]\Big|_{u_1^{r_1}u_2^{r_2}\cdots u_k^{r_k}}$$

The important property is that if

$$Q^{(r_1)}[X] \;=\; P[X + E_1u_1]\Big|_{u_1^{r_1}}$$

then

$$Q^{(r_1)}[X + E_2u_2]\Big|_{u_2^{r_2}} \;=\; P[X + E_1u_1 + E_2u_2]\Big|_{u_1^{r_1}u_2^{r_2}} \;=\; P^{(r_1,r_2)}[X]$$

Proposition 1.4

$$\big(q\,\mathbf{C}_b\mathbf{B}_a P[X] \,-\, \mathbf{B}_a\mathbf{C}_b\big)P[X] \;=\; (q-1)(-1)^{a+b-1}/q^{b-1} \times \begin{cases} 0 & \text{if } a+b>0 \\ P[X] & \text{if } a+b=0 \\ \sum_{r_1+r_2=-(a+b)} P^{r_1,r_2}[X] & \text{if } a+b<0 \end{cases} \qquad 1.6$$

Proof

Using I.8 we get (with $E_1 = \epsilon(1-q)$)

$$(-q)^{b-1}\mathbf{C}_bP[X] \;=\; \sum_{r_1=0}^{d} P^{(r_1)}[X]\tfrac{1}{z^{r_1}} \sum_{m\geq 0} z^m h_m[X]\Big|_{z^b} \;=\; \sum_{r_1=0}^{d} P^{(r_1)}[X] h_{b+r_1}[X]$$

and 1.10 gives (with $E_2=\epsilon(1-q)$)

$$\begin{aligned}(-q)^{b-1}\mathbf{B}_a\mathbf{C}_bP[X] &= \sum_{r_1=0}^{d} P^{(r_1)}\big[X+\epsilon\tfrac{1-q}{z_2}\big]h_{b+r_1}\big[X+\epsilon\tfrac{1-q}{z_2}\big]\Omega[-\epsilon z_2X]\Big|_{z_2^a}\\ &= \sum_{r_1,r_2=0}^{d} P^{(r_1,r_2)}(\tfrac{1}{z_2})^{r_2}\sum_{s=0}^{b+r_1} h_{b+r_1-s}[X]h_s\big[\epsilon\tfrac{1-q}{z_2}\big]\Omega[-\epsilon z_2X]\Big|_{z_2^a}\\ &= \sum_{r_1,r_2=0}^{d}\sum_{s=0}^{b+r_1} P^{(r_1,r_2)}h_{b+r_1-s}[X]h_s\big[\epsilon(1-q)\big]\Omega[-\epsilon z_2X]\Big|_{z_2^{a+r_2+s}}\\ &= \sum_{r_1,r_2=0}^{d}\sum_{s=0}^{b+r_1} P^{(r_1,r_2)}[X]h_{b+r_1-s}[X](-1)^s h_s\big[(1-q)\big]h_{a+r_2+s}[-\epsilon X]\end{aligned}$$

Now note that (2.24) of [5] for $r=0$ and $u=q$ gives

$$h_s\big[1-q\big] \;=\; \begin{cases}1 & \text{if } s=0\\ 1-q & \text{if } s>0\end{cases} \tag{1.13}$$

We can thus write

$$\begin{aligned}(-q)^{b-1}\mathbf{B}_a\mathbf{C}_bP[X] &= \sum_{r_1,r_2=0}^{d} P^{(r_1,r_2)}[X]h_{b+r_1}[X]h_{a+r_2}[-\epsilon X]+\\ &\quad+(1-q)\sum_{r_1,r_2=0}^{d}\sum_{s=1}^{b+r_1} P^{(r_1,r_2)}[X]h_{b+r_1-s}[X](-1)^s h_{a+r_2+s}[-\epsilon X]\end{aligned}$$

and the change of summation index $u=a+r_2+s$ gives

$$\begin{aligned}(-q)^{b-1}\mathbf{B}_a\mathbf{C}_bP[X] &= \sum_{r_1,r_2=0}^{d} P^{(r_1,r_2)}[X]h_{b+r_1}[X]h_{a+r_2}[-\epsilon X]+\\ &\quad+(1-q)\sum_{r_1,r_2=0}^{d}\sum_{u=a+r_2+1}^{a+b+r_1+r_2} P^{(r_1,r_2)}[X]h_{a+b+r_1+r_2-u}[X](-1)^{u-a-r_2}h_u[-\epsilon X]\end{aligned} \tag{1.14}$$

Similarly we get (with $E_1=\epsilon(1-q)$)

$$\mathbf{B}_aP[X] \;=\; \sum_{r_2=0}^{d} P^{r_2}[X](\tfrac{1}{z_2})^{r_2}\sum_{u\ge 0} z_2^u h_u[-\epsilon z_2X]\Big|_{z_2^a} \;=\; \sum_{r_2=0}^{d} P^{r_2}[X]h_{r_2+a}[-\epsilon X]$$

Thus (with $E_2=\epsilon(1-q)$)

$$\begin{aligned}(-q)^{b-1}\mathbf{C}_b\mathbf{B}_aP[X] &= \sum_{r_2=0}^{d} P^{r_2}\big[X-\tfrac{1-1/q}{z}\big]h_{r_2+a}\big[-\epsilon\big(X-\tfrac{1-1/q}{z_1}\big)\big]\,\Omega[z_1X]\Big|_{z_1^b}\\ &= \sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X](\tfrac{1}{z_1})^{r_1}\sum_{s=0}^{r_2+a} h_{r_2+a-s}[-\epsilon X]\,(\tfrac{1}{z_1})^s h_s\big[\epsilon(1-1/q)\big]\,\Omega[z_1X]\Big|_{z_1^b},\\ &= \sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X]\sum_{s=0}^{r_2+a} h_{r_2+a-s}[-\epsilon X]\,(-1)^s h_s\big[1-1/q\big]\,h_{r_1+s+b}[X].\end{aligned}$$

Note that now 1.13 gives

$$h_s\big[1-1/q\big] \;=\; \begin{cases} 1 & \text{if } s=0 \\ 1-1/q & \text{if } s>0 \end{cases}$$

Thus

$$\begin{aligned}(-q)^{b-1}\mathbf{C}_b\mathbf{B}_aP[X] \;&=\; \big(1-(1-1/q)\big)\sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X]h_{r_2+a}[-\epsilon X]\,h_{r_1+b}[X] \\ &\quad+\;\big(1-1/q\big)\sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X]\sum_{s=0}^{r_2+a} h_{r_2+a-s}[-\epsilon X]\,(-1)^s\,h_{r_1+s+b}[X]\end{aligned}$$

and the change of summation index $u=r_2+a-s$ gives

$$\begin{aligned}(-q)^{b-1}\mathbf{C}_b\mathbf{B}_aP[X] \;&=\; \frac{1}{q}\sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X]h_{r_2+a}[-\epsilon X]\,h_{r_1+b}[X] \\ &\quad+\;\big(1-1/q\big)\sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X]\sum_{u=0}^{a+r_2} h_u[-\epsilon X]\,(-1)^{r_2+a-u}\,h_{a+b+r_1+r_2-u}[X]\end{aligned}$$

In summary we get

$$\begin{aligned}(-q)^{b-1}\;q\,\mathbf{C}_b\mathbf{B}_aP[X] \;&=\; \sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X]h_{r_2+a}[-\epsilon X]\,h_{r_1+b}[X] \\ &\quad+\;\big(q-1\big)(-1)^a\sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X]\sum_{u=0}^{a+r_2} h_u[-X]\,(-1)^{r_2}\,h_{a+b+r_1+r_2-u}[X]\end{aligned}$$

On the other hand 1.14 can also be written as

$$\begin{aligned}(-q)^{b-1}\mathbf{B}_a\mathbf{C}_bP[X] \;&=\; \sum_{r_1,r_2=0}^{d} P^{(r_1,r_2)}[X]h_{r_2+a}[-\epsilon X]\,h_{r_1+b}[X]\,+ \\ &\quad+\;(-1)^a(1-q)\sum_{r_1,r_2=0}^{d}\;\sum_{u=a+r_2+1}^{a+b+r_1+r_2} P^{(r_1,r_2)}[X]h_{a+b+r_1+r_2-u}[X](-1)^{r_2}h_u[-X]\end{aligned}$$

and thus subtraction gives

$$\begin{aligned}(-q)^{b-1}\;\big(q\,\mathbf{C}_b\mathbf{B}_aP[X]\,-\,\mathbf{B}_a\mathbf{C}_b\big)P[X] \;&=\; \big(q-1\big)(-1)^a\sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X]\sum_{u=0}^{a+b+r_1+r_2} h_u[-X]\,(-1)^{r_2}\,h_{a+b+r_1+r_2-u}[X] \\ &=\; \big(q-1\big)(-1)^a\sum_{r_1,r_2=0}^{d} P^{r_1,r_2}[X](-1)^{r_2}\,h_{a+b+r_1+r_2}[X-X]\end{aligned}$$

Carrying out the summations and using the definition of $P^{r_1,r_2}[X]$ we finally obtain

$$(-q)^{b-1}\;\big(q\,\mathbf{C}_b\mathbf{B}_aP[X]\,-\,\mathbf{B}_a\mathbf{C}_b\big)P[X] \;=\; (q-1)(-1)^a\times\begin{cases} 0 & \text{if } a+b>0 \\ P[X] & \text{if } a+b=0 \\ \sum_{r_1+r_2=-(a+b)} P^{r_1,r_2}[X] & \text{if } a+b<0 \end{cases}$$

which is easily seen to be 1.6, completing the proof.

In particular we have shown that

Theorem 1.1(Haglund-Morse-Zabrocki)

For all $a+b>0$, our Hall-Littlewood operators have the following commutativity property

$$\mathbf{B}_a \mathbf{C}_b \;=\; q\,\mathbf{C}_b \mathbf{B}_a \tag{1.15}$$

An important ingredient in Macdonald Polynomial Theory is a modified symmetric function scalar product we will refer to as the "$*$-*scalar product*" which makes the basis $\{\tilde{H}_\mu[X;q,t]\}_\mu$ an orthogonal set. More precisely, we have the basic identities

$$\big\langle \tilde{H}_\lambda, \tilde{H}_\mu \big\rangle_* \;=\; \begin{cases} 0 & \text{if } \lambda\neq\mu \\ w_\mu(q,t) & \text{if } \lambda=\mu \end{cases} \tag{1.16}$$

where the $w_\mu(q,t)$ are polynomials in $\mathbf{N}[q,t]$ whose precise definition can be found in section 2 of [5].

The $*$-scalar product and the Hall scalar product are related by the identity ([5] (2.16)),

$$\big\langle f\,,\,g\big\rangle \;=\; \big\langle f\,,\,\omega g^*\big\rangle_* \tag{1.17}$$

where for convenience, for any symmetric function $g[X]$ we set

$$g^*[X] \;=\; g\big[\tfrac{X}{M}\big] \qquad (\text{with } M=(1-q)(1-t)) \tag{1.18}$$

To compute the action of ∇ on a symmetric function we need to expand that function in terms of the basis $\{\tilde{H}_\mu[X;q,t]\}_\mu$ and 1.16 is the tool we need to carry this out. In the sequel we will make use of the following expansions

Proposition 1.5

For all $n\geq 1$ and $0<r<n$ we have

$$\begin{gathered} a)\;\; e_n\big[\tfrac{X}{M}\big]=\sum_{\mu\vdash n}\frac{\tilde{H}_\mu[X;q.t]}{w_\mu}\,, \qquad b)\;\; h_n\big[\tfrac{X}{M}\big]=\sum_{\mu\vdash n}\frac{T_\mu\tilde{H}_\mu[X;q,t]}{w_\mu} \\ c)\;\; e_r\big[\tfrac{X}{M}\big]e_{n-r}\big[\tfrac{X}{M}\big] \;=\; \sum_{\mu\vdash n}\frac{\tilde{H}_\mu[X;q.t]}{w_\mu}F_r[MB_\mu-1] \end{gathered} \tag{1.19}$$

with $F_r[X]$ given by 1.3

Proof

Using 1.16 and 1,17 we obtain

$$e_n^* \;=\; \sum_{\mu\vdash n}\frac{\tilde{H}_\mu[X;q.t]}{w_\mu}\big\langle \tilde{H}_\mu\,,\,h_n\big\rangle$$

and 1.19 a) follows since it is well known, ([5] (2.25)), that

$$\big\langle \tilde{H}_\mu\,,\,h_n\big\rangle \;=\; 1 \tag{1.20}$$

Similarly we get

$$h_n^* \;=\; \sum_{\mu\vdash n} \frac{\tilde H_\mu[X;q.t]}{w_\mu} \big\langle \tilde H_\mu \,,\, e_n \big\rangle$$

and 1.19 b) follows since it is well known, ([5] (2.25)), that

$$\big\langle \tilde H_\mu \,,\, e_n \big\rangle \;=\; T_\mu \tag{1.21}$$

Formula 1.19 c) is less immediate. We can again start by writing

$$e_r^* e_{n-r}^* \;=\; \sum_{\mu\vdash n} \frac{\tilde H_\mu[X;q.t]}{w_\mu} \big\langle \tilde H_\mu \,,\, h_r h_{n-r} \big\rangle$$

However, we have no simple evaluation for the scalar product $\big\langle \tilde H_\mu \,,\, h_r h_{n-r} \big\rangle$ other than resorting to the identity in 1.2 which gives

$$e_r^* e_{n-r}^* \;=\; \sum_{\mu\vdash n} \frac{\tilde H_\mu[X;q.t]}{w_\mu} F_r[MB_\mu - 1]$$

with F_r given by 1.3, This proves 1.19 c) and completes our proof.

Remark 1.1

As we will shortly see our proof of Theorem I.1 will require working with polynomial $\nabla e_r^* e_{n-r}^*$. Since, ∇ is defined [1], by setting for the Macdonald basis

$$\nabla \tilde H_\mu \;=\; T_\mu \tilde H_\mu \tag{1.22}$$

formula 1.19 c) gives

$$\nabla e_r\big[\tfrac{X}{M}\big] e_{n-r}\big[\tfrac{X}{M}\big] \;=\; \sum_{\mu\vdash n} \frac{T_\mu \tilde H_\mu[X;q.t]}{w_\mu} F_r[MB_\mu - 1] \tag{1.23}$$

Introducing the operator θ_r by setting for the Macdonald basis

$$\theta_r \tilde H_\mu \;=\; F_r[MB_\mu - 1] \tilde H_\mu \tag{1.24}$$

Formula 1.19 b) allows us to write 1.23 in the form

$$\nabla e_r^* e_{n-r}^* \;=\; \theta_r h_n^* \tag{1.25}$$

We surprised ourselves to discover that such a simple idea allows us to get around the unavailability of a simple evaluation for the scalar product $\big\langle \tilde H_\mu \,,\, h_r h_{n-r} \big\rangle$. By delivering an expression for $\nabla e_r^* e_{n-r}^*$ that we can work with in our calculations, this idea made possible all the results of the present paper.

2. Proof of the symmetric function recursion

Our point of departure is the following basic reduction

Theorem 2.1

The identity

$$\begin{aligned}\big\langle \nabla C_{p_1} C_{p_2} \cdots C_{p_k} 1\,,\, h_r h_{n-r}\big\rangle \;&=\; t^{p_1-1}\big\langle \nabla B_{p_1-2} C_{p_2} \cdots C_{p_k} 1\,,\; h_{r-1} h_{n-1-r}\big\rangle + \\ &+\; \chi(p_1=1)\big(\big\langle \nabla C_{p_2} \cdots C_{p_k} 1\,,\; h_r h_{n-1-r}\big\rangle \,+\, \big\langle \nabla C_{p_2} \cdots C_{p_k} 1\,,\; h_{r-1} h_{n-r}\big)\big\rangle\end{aligned} \qquad 2.1$$

holds for all compositions $(p_1, p_2, \ldots, p_k) \models n$ *and all* $0 < r < n$*, if and only if the following symmetric function identity holds for all* $0 < r < n$ *and* $a \geq 1$

$$C_a^* \theta_r h_n^*[X] \,=\, t^{a-1} B_{a-2}^* \theta_{r-1} h_{n-2}^*[X] \,+\, \chi(a=1)\Big(\theta_r h_{n-1}^*[X] \,+\, \theta_{r-1} h_{n-1}^*[X]\Big) \qquad 2.2$$

where the operators $\mathbf{C}_a^*$ *and* $\mathbf{B}_a^*$ *are the* $*$*-scalar product adjoints of* $\mathbf{C}_a$ *and* $\mathbf{B}_a$

Proof

Note first that since the polynomials $C_{p_2} \cdots C_{p_k} 1$ are essentially only a rescaled version of the Hall-Littlewood polynomials the span the space $\Lambda^{=(n-p_1)}$. Thus (2,1) can hold true as asserted if and only if for all $F[X] \in \Lambda^{=(n-p_1)}$ we have

$$\begin{aligned}\big\langle \nabla C_{p_1} F[X]\,,\, h_r h_{n-r}\big\rangle \;&=\; t^{p_1-1}\big\langle \nabla B_{p_1-2} F[X]\,,\; h_{r-1} h_{n-1-r}\big\rangle + \\ &+\; \chi(p_1=1)\Big(\big\langle \nabla F[X]\,,\; h_r h_{n-1-r}\big\rangle \,+\, \big\langle \nabla F[X]\,,\; h_{r-1} h_{n-r}\big\rangle\Big)\end{aligned}$$

Now passing to $*$-scalar products we may rewrite this identity in the form

$$\begin{aligned}\big\langle \nabla C_{p_1} F[X]\,,\, e_r^* e_{n-r}^*\big\rangle_* \;&=\; t^{p_1-1}\big\langle \nabla B_{p_1-2} F[X]\,,\; e_{r-1}^* e_{n-1-r}^*\big\rangle_* + \\ &+\; \chi(p_1=1)\Big(\big\langle \nabla F[X]\,,\; e_r^* e_{n-1-r}^*\big\rangle_* \,+\, \big\langle \nabla F[X]\,,\; e_{r-1}^* e_{n-r}^*\big\rangle_*\Big)\end{aligned}$$

Next we move all the operators acting on $F[X]$ to the other side of their respective $*$-scalar products and obtain

$$\begin{aligned}\big\langle F[X]\,,\, C_{p_1}^* \nabla e_r^* e_{n-r}^*\big\rangle_* \;&=\; t^{p_1-1}\big\langle F[X]\,,\; B_{p_1-2}^* \nabla e_{r-1}^* e_{n-1-r}^*\big\rangle_* + \\ &+\; \chi(p_1=1)\Big(\big\langle F[X]\,,\; \nabla e_r^* e_{n-1-r}^*\big\rangle_* \,+\, \big\langle F[X]\,,\; \nabla e_{r-1}^* e_{n-r}^*\big\rangle_*\Big)\end{aligned} \qquad 2.3$$

Of course ∇ does not get a "$*$" since, by I.16, all Macdonald polynomials eigen-operators are necessarily self-adjoint with respect to the $*$-scalar product.

But now the arbitrariness of $F[X]$ shows that 2.3 can be true if and only if we have the following symmetric function equality

$$C_{p_1}^* \nabla e_r^* e_{n-r}^* \;=\; t^{p_1-1} B_{p_1-2}^* \nabla e_{r-1}^* e_{n-1-r}^* \,+\, \chi(p_1=1)\big(\nabla e_r^* e_{n-1-r}^* \,+\, \nabla e_{r-1}^* e_{n-r}^*\big) \qquad 2.3$$

Replacing p_1 by a and using 1.25 for various values of r and n yields 2.2 and completes our proof.

Our next task is now to prove 2.2. To begin we will need the following expansion

Proposition 1.1

$$\theta_r h_n^*[X] \;=\; \sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}](-1)^k \sum_{\nu\vdash k} \frac{1}{w_\nu} h_n\big[X(\tfrac{1}{M}-B_\nu)\big] \tag{2.4}$$

Proof

Note first that 1.23 and 1.25 give

$$\theta_r h_n^*[X] \;=\; \sum_{\mu\vdash n} \frac{T_\mu \tilde H_\mu[X;q.t]}{w_\mu} F_r[MB_\mu - 1] \tag{2.5}$$

Recall from 1.3 that

$$F_r[X] \;=\; \sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}] \nabla^{-1} e_k[\tfrac{X}{M}]$$

and since 1.19 a) and 1.22 give

$$\nabla^{-1} e_k[\tfrac{X}{M}] \;=\; \sum_{\nu\vdash k} \frac{T_\nu^{-1}\tilde H_\nu[X;q,t]}{w_\nu}$$

we can write

$$F_r[X] \;=\; \sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}] \sum_{\nu\vdash k} \frac{T_\nu^{-1}\tilde H_\nu[X;q,t]}{w_\nu}$$

and 2.5 becomes

$$\begin{aligned}\theta_r h_n^*[X] \;&=\; \sum_{\mu\vdash n} \frac{T_\mu \tilde H_\mu[X;q,t]}{w_\mu} \sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}] \sum_{\nu\vdash k} \frac{T_\nu^{-1}\tilde H_\nu[MB_\mu-1];q,t]}{w_\nu} \\ &=\; \sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}] \sum_{\nu\vdash k} \frac{1}{w_\nu} \sum_{\mu\vdash n} \frac{T_\mu \tilde H_\mu[X;q,t]}{w_\mu} \frac{\tilde H_\nu[MB_\mu-1];q,t]}{T_\nu}\end{aligned} \tag{2.6}$$

We now use the Macdonald reciprocity formula (2.21) of [5],

$$\frac{\tilde H_\nu[MB_\mu-1;q,t]}{T_\nu} \;=\; (-1)^{n-k} \frac{\tilde H_\mu[MB_\nu-1];q,t]}{T_\mu}$$

and 2.6 becomes

$$\theta_r h_n^*[X] \;=\; \sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}](-1)^{n-k} \sum_{\nu\vdash k} \frac{1}{w_\nu} \sum_{\mu\vdash n} \frac{\tilde H_\mu[X;q,t]\tilde H_\mu[MB_\nu-1;q,t]}{w_\mu}$$

and a use of the Macdonald Cauchy identity (2.17) of [5]

$$\sum_{\mu\vdash n} \frac{\tilde H_\mu[X;q,t]\tilde H_\mu[Y;q,t]}{w_\mu} \;=\; e_n\big[\tfrac{XY}{M}\big]$$

gives

$$\theta_r h_n^*[X] \;=\; \sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}](-1)^{n-k} \sum_{\nu\vdash k} \frac{1}{w_\nu} e_n\big[X(B_\nu - \tfrac{1}{M})\big]$$

which is 2.4 because of the relation (see (2.6) of [5])

$$e_n\big[X(B_\nu - \tfrac{1}{M})\big] \;=\; (-1)^n h_n[X\tfrac{1}{M} - B_\nu]$$

and our proof s now complete.

We are now ready to start working on the identity in 2.2. We will start with the term

$$B^*_{a-2}\theta_{r-1}h^*_{n-2}[X]$$

which, using 2.4 with $r\rightarrow r-1$ and $n\rightarrow n-2$ becomes

$$B^*_{a-2}\theta_{r-1}h^*_{n-2}[X] \;=\; \sum_{k=0}^{r-1} h_{r-1-k}[\tfrac{1}{M}](-1)^k \sum_{\nu\vdash k}\frac{1}{w_\nu} B^*_{a-2}h_{n-2}\big[X(\tfrac{1}{M}-B_\nu)\big] \tag{2.7}$$

Let us now recall that in [5] (Theorem 3.6) it was shown that the action of the operators $\mathbf{B}^*_a$ and $\mathbf{C}^*_a$ on a symmetric polynomial $P[X]$ may be computed by means of the two plethystic formulas

$$\mathbf{B}^*_a P[X] \;=\; P\big[X+\tfrac{M}{z}\big]\sum_{m\geq 0} z^m h_m\big[\tfrac{-X}{1-t}\big]\Big|_{z^{-a}} \tag{2.9}$$

and

$$\mathbf{C}^*_a P[X] \;=\; (\tfrac{-1}{q})^{a-1} P\big[X-\tfrac{\epsilon M}{z}\big]\sum_{m\geq 0} (-\tfrac{z}{q})^m h_m\big[\tfrac{-X}{1-t}\big]\Big|_{z^{-a}} \tag{2.10}$$

we can thus use 2.9 to get

$$\begin{aligned}
B^*_{a-2}h_{n-2}\left[X(\tfrac{1}{M}-B_\nu)\right] \;&=\; h_{n-2}\left[(X+M/z)(\tfrac{1}{M}-B_\nu)\right]\sum_{m\geq 0} z^m h_m\big[\tfrac{-X}{1-t}\big]\Big|_{z^{-a+2}}\\
&=\; \sum_{s=0}^{n-2} h_{n-2-s}\left[X(\tfrac{1}{M}-B_\nu)\right] h_s\left[M(\tfrac{1}{M}-B_\nu)\right]\tfrac{1}{z^s}\sum_{m\geq 0} z^m h_m\big[\tfrac{-X}{1-t}\big]\Big|_{z^{-a+2}}\\
&=\; \sum_{s=a-2}^{n-2} h_{n-2-s}\left[X(\tfrac{1}{M}-B_\nu)\right] h_s\left[1-MB_\nu\right] h_{s-a+2}[-X/(1-t)]
\end{aligned}$$

Using this in 2.7 gives

$$\begin{aligned}
B^*_{a-2}\theta_{r-1}h^*_{n-2}[X] \;&=\; \sum_{k=0}^{r-1} h_{r-1-k}[\tfrac{1}{M}](-1)^k \times\\
&\qquad \times \sum_{\nu\vdash k}\frac{1}{w_\nu}\sum_{s=a-2}^{n-2} h_{n-2-s}\left[X(\tfrac{1}{M}-B_\nu)\right] h_s\left[1-MB_\nu\right] h_{s-a+2}[-X/(1-t)]\\
&=\; \sum_{k=0}^{r-1} h_{r-1-k}[\tfrac{1}{M}](-1)^k \times\\
&\qquad \times \sum_{s=a-2}^{n-2} h_{s-a+2}[-X/(1-t)]\sum_{\nu\vdash k}\frac{1}{w_\nu} h_{n-2-s}\left[X(\tfrac{1}{M}-B_\nu)\right] h_s\left[1-MB_\nu\right]
\end{aligned}$$

and a change $s\rightarrow s-2$ of summation index finally gives

$$B^*_{a-2}\theta_{r-1}h^*_{n-2}[X] \;=\; \sum_{k=0}^{r-1} h_{r-1-k}[\tfrac{1}{M}](-1)^k \;\sum_{s=a}^{n} h_{s-a}[\tfrac{-X}{1-t}]\sum_{\nu\vdash k}\frac{1}{w_\nu}h_{n-s}\left[X(\tfrac{1}{M}-B_\nu)\right] h_{s-2}\left[1-MB_\nu\right] \tag{2.11}$$

Let us now work on the left hand side of 2.2, which, using 2.4 is simply

$$C_a^*\theta_r h_n^*[X] \;=\; \sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}](-1)^k \sum_{\nu\vdash k} \frac{1}{w_\nu}\, C_a^* h_n\big[X(\tfrac{1}{M}-B_\nu)\big] \tag{2.12}$$

Now 2.10 gives

$$\begin{aligned}(-q)^{a-1}C_a^* h_n\left[X(\tfrac{1}{M}-B_\nu)\right] &= h_n\left[(X-\epsilon M/z)(\tfrac{1}{M}-B_\nu)\right]\sum_{m\geq 0}(-\tfrac{z}{q})^m h_m\left[\tfrac{-X}{1-t}\right]\Big|_{z^{-a}}\\ &= \sum_{s=0}^{n} h_{n-s}\left[X(\tfrac{1}{M}-B_\nu)\right] h_s\left[-\epsilon M(\tfrac{1}{M}-B_\nu)\right]\tfrac{1}{z^s}\sum_{m\geq 0}(-\tfrac{z}{q})^m h_m\left[\tfrac{-X}{1-t}\right]\Big|_{z^{-a}}\\ &= \sum_{s=0}^{n} h_{n-s}\left[X(\tfrac{1}{M}-B_\nu)\right](-1)^s h_s\left[-1+MB_\nu\right](-\tfrac{1}{q})^{s-a}h_{s-a}\left[\tfrac{-X}{1-t}\right]\end{aligned}$$

Thus

$$C_a^* h_n\left[X(\tfrac{1}{M}-B_\nu)\right] \;=\; (-q)\sum_{s=0}^{n} h_{n-s}\left[X(\tfrac{1}{M}-B_\nu)\right] h_s\left[-1+MB_\nu\right](-\tfrac{1}{q})^s h_{s-a}\left[\tfrac{-X}{1-t}\right]$$

and 2.12 becomes

$$\begin{aligned}C_a^*\theta_r h_n^*[X] &= \sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}](-1)^k \times\\ &\qquad \times \sum_{\nu\vdash k}\frac{1}{w_\nu}\,(-q)\sum_{s=0}^{n} h_{n-s}\left[X(\tfrac{1}{M}-B_\nu)\right] h_s\left[-1+MB_\nu\right](-\tfrac{1}{q})^s h_{s-a}\left[\tfrac{-X}{1-t}\right]\\ &= \sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}](-1)^k \times\\ &\qquad \times\;(-q)\sum_{s=a}^{n}(-\tfrac{1}{q})^s h_{s-a}\left[\tfrac{-X}{1-t}\right]\sum_{\nu\vdash k}\frac{1}{w_\nu}\,h_{n-s}\left[X(\tfrac{1}{M}-B_\nu)\right] h_s\left[-1+MB_\nu\right]\end{aligned} \tag{2.13}$$

We are now going to make use of the following two summation formulas ((2.28) and (2.29) of [3], see also [10])

$$\sum_{\nu\to\mu} c_{\mu\nu}(q,t)\,(T_\mu/T_\nu)^k \;=\; \begin{cases}\frac{tq}{M}\,h_{k+1}\big[(-1+MB_\nu)/tq\big] & \text{if } k\geq 1\ ,\\ B_\mu(q,t) & \text{if } k=0\ .\end{cases} \tag{2.14}$$

$$\sum_{\mu\leftarrow\nu} d_{\mu\nu}(q,t)\,(T_\mu/T_\nu)^k \;=\; \begin{cases}(-1)^{k-1}\,e_{k-1}\big[-1+MB_\nu\big] & \text{if } k\geq 1\ ,\\ 1 & \text{if } k=0\ .\end{cases} \tag{2.15}$$

We will start by using 2.14 in the form

$$h_s\left[-1+MB_\nu)\right] \;=\; (tq)^{s-1}M\sum_{\tau\to\nu} c_{\nu\tau}(\tfrac{T_\nu}{T_\tau})^{s-1} \,-\, \chi(s=1)$$

and obtain

$$\begin{aligned}
C_a^*\theta_r h_n^*[X] \;&=\; \sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}](-1)^k(-q)\sum_{s=a}^{n}(-\tfrac{1}{q})^s h_{s-a}\big[\tfrac{-X}{1-t}\big]\times \\
&\qquad\times\sum_{\nu\vdash k}\frac{1}{w_\nu}\,h_{n-s}\,\big[X(\tfrac{1}{M}-B_\nu)\big]\Big((tq)^{s-1}M\sum_{\tau\to\nu}c_{\nu\tau}(\tfrac{T_\nu}{T_\tau})^{s-1}\;-\;\chi(s=1)\Big)\\
&=\sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}](-1)^{k-1}\sum_{s=a}^{n}t^{s-1}h_{s-a}[\tfrac{-X}{1-t}]\sum_{\nu\vdash k}\frac{1}{w_\nu}\,h_{n-s}\,\big[X(\tfrac{1}{M}-B_\nu)\big]\Big(M\sum_{\tau\to\nu}c_{\nu\tau}(\tfrac{T_\nu}{T_\tau})^{s-1}\Big)\\
&\qquad+\;\chi(a=1)\sum_{k=0}^{r}h_{r-k}[\tfrac{1}{M}](-1)^k\;\sum_{\nu\vdash k}\frac{1}{w_\nu}\,h_{n-1}\,\big[X(\tfrac{1}{M}-B_\nu)\big] && 2.16\\
&=\sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}](-1)^{k-1}\sum_{s=a}^{n}t^{s-1}h_{s-a}[\tfrac{-X}{1-t}]\sum_{\nu\vdash k}\frac{1}{w_\nu}\,h_{n-s}\,\big[X(\tfrac{1}{M}-B_\nu)\big]\Big(M\sum_{\tau\to\nu}c_{\nu\tau}(\tfrac{T_\nu}{T_\tau})^{s-1}\Big)\\
&\qquad\text{(by 2.4)}\qquad+\;\chi(a=1)\theta_r h_{n-1}^*[X]
\end{aligned}$$

Now, changing the order of ν and τ summations and using the relation

$$\frac{w_\tau}{w_\nu}Mc_{\nu\tau}=d_{\nu\tau}\qquad\qquad ((2.30)\text{ of }[5]),$$

we may rewrite 2.16 as

$$\begin{aligned}
&C_a^*\theta_r h_n^*[X]\;-\;\chi(a=1)\theta_r h_{n-1}^*[X]\;\;=\\
&\quad=\sum_{k=0}^{r}h_{r-k}[\tfrac{1}{M}](-1)^{k-1}\sum_{s=a}^{n}t^{s-1}h_{s-a}[-\tfrac{X}{1-t}]\sum_{\tau\vdash k-1}\frac{1}{w_\tau}\sum_{\nu\leftarrow\tau}\frac{w_\tau}{w_\nu}\,h_{n-s}\,\big[X(\tfrac{1}{M}-B_\nu)\big]\,Mc_{\nu\tau}(\tfrac{T_\nu}{T_\tau})^{s-1} && 2.17\\
&\quad=\sum_{k=0}^{r}h_{r-k}[\tfrac{1}{M}](-1)^{k-1}\sum_{s=a}^{n}t^{s-1}h_{s-a}[-\tfrac{X}{1-t}]\sum_{\tau\vdash k-1}\frac{1}{w_\tau}\sum_{\nu\leftarrow\tau}\;h_{n-s}\,\big[X(\tfrac{1}{M}-B_\nu)\big]\,d_{\nu\tau}(\tfrac{T_\nu}{T_\tau})^{s-1}
\end{aligned}$$

Next we split $B_\nu(q,t)$ into the sum $B_\nu(q,t)=B_\tau(q,t)+\frac{T_\nu}{T_\tau}$ to get

$$\begin{aligned}
&\sum_{\nu\leftarrow\tau}d_{\nu\tau}(\tfrac{T_\nu}{T_\tau})^{s-1}h_{n-s}\,\big[X(\tfrac{1}{M}-B_\nu)\big]\;\;=\\
&\qquad\qquad=\;\sum_{u=0}^{n-s}h_{n-u-s}\,\big[X(\tfrac{1}{M}-B_\tau)\big]\,h_u[-X]\sum_{\nu\leftarrow\tau}d_{\nu\tau}\big(\tfrac{T_\nu}{T_\tau}\big)^{u+s-1}\\
&\text{(by (2.15))}\;\;=\;\sum_{u=0}^{n-s}h_{n-u-s}\,\big[X(\tfrac{1}{M}-B_\tau)\big]\,h_u[-X]\Big((-1)^{u+s-2}e_{u+s-2}\big[MB_\tau-1\big]\;+\;\chi(u+s=1)\Big)\\
&\qquad\qquad=\;\sum_{v=s}^{n}h_{n-v}\,\big[X(\tfrac{1}{M}-B_\tau)\big]\,h_{v-s}[-X]\Big(h_{v-2}\big[1-MB_\tau\big]\;+\;\chi(v=1)\Big)
\end{aligned}$$

Using this in 2.17 gives

$$\begin{aligned} C_a^*\theta_r h_n^*[X] \,-\, \chi(a=1)\theta_r h_{n-1}^*[X] \;&=\; \sum_{k=0}^{r} h_{r-k}[\tfrac{1}{M}](-1)^{k-1}\sum_{s=a}^{n} t^{s-1}h_{s-a}[-\tfrac{X}{1-t}]\times \\ &\times \sum_{\tau\vdash k-1}\frac{1}{w_\tau}\sum_{v=s}^{n} h_{n-v}\,\big[X(\tfrac{1}{M}-B_\tau)\big]\,h_{v-s}[-X]\Big(h_{v-2}\big[1-MB_\tau\big]\,+\,\chi(v=1)\Big)\end{aligned}$$

Since there are no partitions of $k-1$ for $k=0$ we make the change of variable $k{\rightarrow}k+1$ and obtain

$$\begin{aligned} C_a^*\theta_r h_n^*[X] \,-\, \chi(a=1)\theta_r h_{n-1}^*[X] \;&=\; \sum_{k=0}^{r-1} h_{r-1-k}[\tfrac{1}{M}](-1)^{k}\sum_{s=a}^{n} t^{s-1}h_{s-a}[-\tfrac{X}{1-t}]\times \\ &\times \sum_{\tau\vdash k}\frac{1}{w_\tau}\sum_{v=s}^{n} h_{n-v}\,\big[X(\tfrac{1}{M}-B_\tau)\big]\,h_{v-s}[-X]\Big(h_{v-2}\big[1-MB_\tau\big]\,+\,\chi(v=1)\Big)\end{aligned} \qquad 2.18$$

Now the term multiplying $\chi(v=1)$ on the right hand side is

$$\sum_{k=0}^{r-1} h_{r-1-k}[\tfrac{1}{M}](-1)^{k}\sum_{s=a}^{n} t^{s-1}h_{s-a}[-\tfrac{X}{1-t}]\sum_{\tau\vdash k}\frac{1}{w_\tau}\sum_{v=s}^{n} h_{n-v}\,\big[X(\tfrac{1}{M}-B_\tau)\big]\,h_{v-s}[-X]$$

Now $v=1$ forces $s=1$ which in turn forces $a=1$. So this term reduces to

$$\sum_{k=0}^{r-1} h_{r-1-k}[\tfrac{1}{M}](-1)^{k}\sum_{\tau\vdash k}\frac{1}{w_\tau}h_{n-1}\,\big[X(\tfrac{1}{M}-B_\tau)\big]$$

which we recognize as $\theta_{r-1}h_{n-1}^*[X]$. Thus 2.18 reduces to

$$\begin{aligned} C_a^*\theta_r h_n^*[X] - \chi(a=1)\Big(\theta_r h_{n-1}^*[X] \,+\, \theta_{r-1}h_{n-1}^*[X]\Big) \;&=\; \sum_{k=0}^{r-1} h_{r-1-k}[\tfrac{1}{M}](-1)^{k}\times \\ &\times \sum_{s=a}^{n} t^{s-1}h_{s-a}[-\tfrac{X}{1-t}]\sum_{\tau\vdash k}\frac{1}{w_\tau}\sum_{v=s}^{n} h_{n-v}\,\big[X(\tfrac{1}{M}-B_\tau)\big]\,h_{v-s}[-X]h_{v-2}\big[1-MB_\tau\big]\end{aligned} \qquad 2.19$$

Calling this last factor LF we have

$$\begin{aligned} LF \;&=\; t^{a-1}\sum_{s=a}^{n}\sum_{v=s}^{n} h_{v-s}[-X]h_{s-a}[-\tfrac{tX}{1-t}]\sum_{\tau\vdash k}\frac{1}{w_\tau}h_{n-v}\,\big[X(\tfrac{1}{M}-B_\tau)\big]\,h_{v-2}\big[1-MB_\tau\big] \\ &=\; t^{a-1}\sum_{v=a}^{n}\sum_{s=a}^{v} h_{v-s}[-X]h_{s-a}[-\tfrac{tX}{1-t}]\sum_{\tau\vdash k}\frac{1}{w_\tau}h_{n-v}\,\big[X(\tfrac{1}{M}-B_\tau)\big]\,h_{v-2}\big[1-MB_\tau\big]\end{aligned}$$

But making the substitution $s-a=u$ we get

$$\sum_{s=a}^{v} h_{v-s}[-X]h_{s-a}[-\tfrac{tX}{1-t}] \;=\; \sum_{u=0}^{v-a} h_{v-a-u}[-X]h_u[-\tfrac{tX}{1-t}] \;=\; h_{v-a}[-\tfrac{X}{1-t}]$$

This gives

$$LF \;=\; t^{a-1}\sum_{v=a}^{n} h_{v-a}[-\tfrac{X}{1-t}]\sum_{\tau\vdash k}\frac{1}{w_\tau}h_{n-v}\left[X(\tfrac{1}{M}-B_\tau)\right]h_{v-2}\big[1-MB_\tau\big]$$

and 2.19 becomes

$$\begin{aligned} C_a^*\theta_r h_n^*[X]-\chi(a=1)\Big(\theta_r h_{n-1}^*[X]\,+\,\theta_{r-1}h_{n-1}^*[X]\Big) \;&=\\ \;=\; t^{a-1}\sum_{k=0}^{r-1}h_{r-1-k}[\tfrac{1}{M}](-1)^k\sum_{v=a}^{n}h_{v-a}[-\tfrac{X}{1-t}]&\sum_{\tau\vdash k}\frac{1}{w_\tau}h_{n-v}\left[X(\tfrac{1}{M}-B_\tau)\right]h_{v-2}\big[1-MB_\tau\big]\end{aligned}$$

and a look at 2.11 reveals that this last expression is none other than $t^{a-1}B_{a-2}^*\theta_{r-1}h_{n-2}^*[X]$. In other words we have proved the identity

$$C_a^*\theta_r h_n^*[X] \;=\; t^{a-1}B_{a-2}^*\theta_{r-1}h_{n-2}^*[X]\,+\,\chi(a=1)\Big(\theta_r h_{n-1}^*[X]\,+\,\theta_{r-1}h_{n-1}^*[X]\Big)$$

and our proof of Theorem I is thus complete.

3. Combinatorial Consequences

Let us denote by $\mathcal{PF}_{p_1,p_2,\ldots,p_k}(r)$, for $0<r<n$, the collection of parking functions with composition $(p_1,p_2,\ldots,p_k)\models n$ and diagonal word a shuffle of $12\cdots r$ with $r+1\cdots n$. In symbols

$$\mathcal{PF}_{p_1,p_2,\ldots,p_k}(r) \;=\; \Big\{{\scriptstyle PF\in\mathcal{PF}_n\,:\,p(PF)=(p_1,p_2,\ldots,p_k)\ \&\ \sigma(PF)\in 12\cdots r\,\cup\!\!\!\cup\, r+1\cdots n}\Big\}$$

and set

$$\Pi_{(p_1,p_2,\ldots,p_k)}(r;q,t) \;=\; \sum_{PF\in\mathcal{PF}_{p_1,p_2,\ldots,p_k}(r)} t^{area(PF)}q^{dinv(PF)} \tag{3.1}$$

Our basic goal in this section is to prove the identity in I.9 which can be written as

$$\Pi_{(p_1,p_2,\ldots,p_k)}(r;q,t) \;=\; \big\langle\nabla\mathbf{C}_{p_1}\mathbf{C}_{p_2}\cdots\mathbf{C}_{p_k}1\,,\;h_rh_{n-r}\big\rangle. \tag{3.2}$$

Our plan is to verify that both sides satisfy the same recursion and that they are equal for all the base cases. Now we proved (Theorem I.1) that the right hand side satisfies

$$\begin{aligned}\big\langle\nabla C_{p_1}C_{p_2}\cdots C_{p_k}1\,,h_rh_{n-r}\big\rangle \;&=\; t^{p_1-1}\big\langle\nabla B_{p_1-2}C_{p_2}\cdots C_{p_k}1\,,\;h_{r-1}h_{n-1-r}\big\rangle+\\ &+\;\chi(p_1=1)\big(\big\langle\nabla C_{p_2}\cdots C_{p_k}1\,,\;h_rh_{n-1-r}\big\rangle\,+\,\big\langle\nabla C_{p_2}\cdots C_{p_k}1\,,\;h_{r-1}h_{n-r}\big)\big\rangle\end{aligned} \tag{3.3}$$

To extract information from this recursion i we need to rewrite it in a combinatorially more revealing form. Proposition 1.4 was included precisely for this purpose. In fact, the Haglund-Morse-Zabrocki conjectures suggest that the operator B_{p_1-2} in the expression

$$B_{p_1-2}C_{p_2}\cdots C_{p_k}1 \tag{3.4}$$

must be moved to the right passed all operators C_{p_i} to act on 1. This requires using 1.6, but only for $b\geq 1$ and $a\geq -1$. This reduces it to the following two cases

$$\begin{aligned} &a)\quad B_aC_b \;=\; qC_bB_a \qquad (\text{for } a\geq 0\ \&\ b\geq 1)\\ &b)\quad B_{-1}C_1 \;=\; qC_1B_{-1}\,+\,I \qquad (\text{for } a=-1\ \&\ b=1)\end{aligned} \tag{3.5}$$

with "I" the identity operator.

We are thus led to the following version of 3.3.

Proposition 3.1

The right hand side of 3.2 satisfies the following recursions

$$
\begin{array}{ll}
a) & (\text{when } p_1 > 1) \\
& \big\langle \nabla C_{p_1} C_{p_2} \cdots C_{p_k} 1 \,,\, h_r h_{n-r} \big\rangle = t^{p_1-1} q^{k-1} \big\langle \nabla C_{p_2} \cdots C_{p_k} B_{p_1-2} 1 \,,\, h_{r-1} h_{n-1-r} \big\rangle \\
b) & (\text{when } p_1 = 1) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 3.6 \\
& \big\langle \nabla C_1 C_{p_2} \cdots C_{p_k} 1 \,,\, h_r h_{n-r} \big\rangle = \big\langle \nabla C_{p_2} \cdots C_{p_k} 1 \,,\, h_r h_{n-1-r} + h_{r-1} h_{n-r} \big\rangle + \\
& \qquad\qquad + (q-1) \sum_{i=2}^{k}{}^{(p_i=1)} q^{i-2} \big\langle \nabla C_{p_2} \cdots \cancel{C_{p_i}} \cdots C_{p_k} 1 \,,\, h_{r-1} h_{n-1-r} \big\rangle
\end{array}
$$

Proof

Note that when $p_1 > 1$ then $p_1 - 2 \geq 0$ and since all parts of a composition are ≥ 1 we can use 3.5 a) $k-1$ times and immediately obtain 3.6 a) from 3.3. Next note that for $p_1 = 1$ we need to move B_{-1} passed all C_{p_i} in the expression

$$B_{-1} C_{p_2} \cdots C_{p_k} 1$$

To see how 3.6 b) comes out of this operation we need only work it out in a special case. Let us take $k = 4$. This given, we have, by repeated uses of 3.5

$$
\begin{aligned}
B_{-1} C_{p_2} C_{p_3} C_{p_4} 1 &= q C_{p_2} B_{-1} C_{p_3} C_{p_4} 1 + \chi(p_2 = 1)(q-1) C_{p_3} C_{p_4} 1 \\
&= q^2 C_{p_2} C_{p_3} B_{-1} C_{p_4} 1 + \chi(p_3 = 1) q (q-1) C_{p_2} C_{p_4} 1 + \\
&\qquad\qquad\qquad + \chi(p_2 = 1)(q-1) C_{p_3} C_{p_4} 1 \\
&= q^2 C_{p_2} C_{p_3} C_{p_4} B_{-1} 1 + \chi(p_4 = 1) q^2 (q-1) C_{p_2} C_{p_3} 1 \\
&\qquad\qquad\qquad \chi(p_3 = 1) q (q-1) C_{p_2} C_{p_4} 1 + \\
&\qquad\qquad\qquad\qquad + \chi(p_2 = 1)(q-1) C_{p_3} C_{p_4} 1
\end{aligned}
$$

But the first term vanishes, since the operator B_{-a} decreases degrees by a, and we get that

$$\big\langle \nabla B_{-1} C_{p_2} C_{p_3} C_{p_4} 1 \,,\, h_r h_{n-r} \big\rangle \;=\; (q-1) \sum_{i=2}^{4}{}^{(p_i=1)} q^{i-2} \big\langle \nabla C_{p_2} \cdots \cancel{C_{p_i}} \cdots C_{p_k} 1 \,,\, h_{r-1} h_{n-1-r} \big\rangle$$

Of course we can complete the proof of 3.6 b) by an induction argument, but it wouldn't add anything to what we have just seen.

Now it was shown in [9] that

$$B_a 1 \;=\; \sum_{z \models a} C_{z_1} C_{z_2} \cdots C_{z_{l(r)}} 1 \qquad 3.7$$

where "$l(z)$" denotes the length of the composition z.

This given, by combining Proposition 3.1 and 3.7 we obtain

Theorem 3.1

The two sides of 3.2 satisfy the same recursion if for all $(p_1, p_2, \ldots, p_k) \models n$, *and* $0 < r < n$

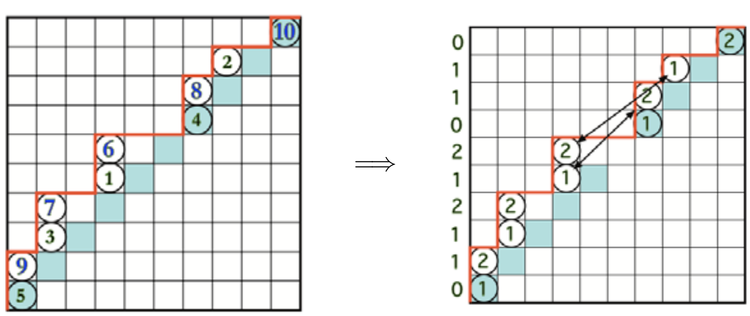
$$
\begin{aligned}
&a) \quad \text{(when } p_1 > 1) \\
&\qquad \Pi_{(p_1,p_2,\ldots,p_k)}(r;q,t)\rangle = t^{p_1-1}q^{k-1} \sum_{z \models p_1-2} \Pi_{(p_2,\ldots,p_k,z_1,z_2,\ldots,z_{l(z)})}(r-1;q,t) \\
&b) \quad \text{(when } p_1 = 1) \\
&\qquad \Pi_{1,p_2,\ldots,p_k)}(r;q,t)\rangle = \Pi_{(p_2,\ldots,p_k)}(r;q,t) + \Pi_{(p_2,\ldots,p_k)}(r-1;q,t) + \\
&\qquad\qquad + (q-1)\sum_{i=2}^{k}{}^{(p_i=1)} q^{i-2}\Pi_{(p_2,\ldots,\cancel{p_i}\cdots p_k)}(r-1;q,t)
\end{aligned}
\tag{3.8}
$$

To verify these two identities we need some observations. To begin, for a $PF \in \mathcal{PF}_{p_1,p_2,\ldots,p_k}(r)$, it will be convenient to refer to $1, 2, \cdots r$ as the "small cars" and to $r+1, \ldots, n$ as "big cars". Now the condition that $\sigma(PF)$ is a shuffle of increasing small cars with increasing big cars forces small cars as well as big cars to be increasing from higher to lower diagonals and from right to left along diagonals. Thus there will never be a small car on top of a small car nor a big car on top of a big car. This implies that the Dyck paths supporting our parking functions will necessarily have only columns of NORTH steps of length at most 2. For the same reason, primary diagonal inversions will occur only when a small car is to the left of a big car in the same diagonal. Likewise a secondary diagonal inversion occurs only when a big casr is to the left of a small car in the adjacent lower diagonal.

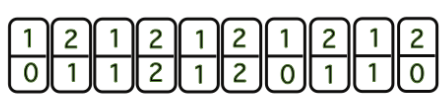
This given, it will be convenient to represent a $PF \in \mathcal{PF}_{p_1,p_2,\ldots,p_k}(r)$ by the "*reduced*" tableau obtained by replacing all the small cars by a "1" and all big cars by a "2". Clearly, to recover PF from such a tableau we need only replace all the "$1's$" by $1, 2, \ldots, r$ and all the "$1's$" by $r+1, \ldots, n$ proceeding by diagonals, from the highest to the lowest and within diagonals from right to left.

More precisely, we will work directly with the corresponding two-line array viewed as a sequence of columns which we will call "*dominos*" and refer to it as "$dom(PF)$".

For instance on the left, in the display below, we have a $PF \in \mathcal{PF}_{6,3,1}(5)$. We purposely depicted the big cars $6, 7, 8, 9, 10$ in bigger size than the small cars $1, 2, 3, 4, 5$. On the right we have its reduced tableau with the adjacent column of diagonal numbers. On the bottom we display $dom(PF)$.



3.9



3.10

It may be good to say a few words on the manner in which the parking functions of a family $\mathcal{PF}_{p_1,p_2,\ldots,p_k}(r)$ can be constructed. First, we create all the Dyck paths which hit the diagonal according to the composition $(p_1, p_2, \ldots, p_k)$ and have no more than $min(r, n-r)$ columns of length two and all remaining columns of length one. Then, for each of these Dyck paths, we fill the lattice cells adjacent to its columns of length two with a 1 below a 2 then place, along the columns of length one, the remaining $r - min(r, n-r)$ $1's$ and $n - r - min(r, n-r)$ $2's$ in all possible ways.

Now that we are familiarized with these parking functions we can proceed to establish the identities in 3.8. To verify 3.8 a) we need only construct a bijection

$$\Phi : \mathcal{PF}_{p_1,p_2,\ldots,p_k}(r) \iff \bigcup_{z \models p_1-2} \mathcal{PF}_{(p_2,\ldots,p_k,z_1,z_2,\ldots,z_{l(z)})}(r-1) \qquad 3.10$$

such that

$$area(PF) = p_1 - 1 + area(\Phi(PF)) \quad \text{and} \quad dinv(PF) = k - 1 + dinv(\Phi(PF)) \qquad 3.11$$

The combinatorial interpretation of these equalities is very suggestive

- some NORTH steps of the supporting Dyck path must be shifted to the right to cause a loss of area of $p_1 - 1$.
- Note that $p_1 > 1$ forces a $PF \in \mathcal{PF}_{p_1,p_2,\ldots,p_k}(r)$ to start with a column of length 2. If we could remove this column we will cause a loss of one diagonal inversion for each of the remaining cars in the main diagonal, thereby satisfying the required dinv loss of $k - 1$.

Led by these two observations and the experience gained in previous work [5] we construct the map Φ as follows .

Given a $PF \in \mathcal{PF}_{p_1,p_2,\ldots p_k}(r)$ with $p_1 > 1$, we apply to $dom(PF)$ the following 4 step procedure, and then let $\Phi(PF)$ be the parking function corresponding to the resulting domino sequence.

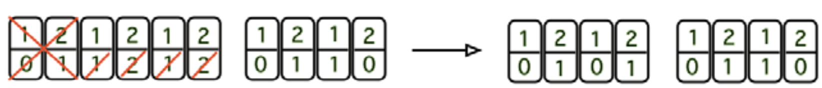
Step 1 *Cut $dom(PF)$ in two sections, the first containing its first p_1 dominos and the second containing the remaining $n - p_1$.*

Step 2 *Remove from the first section the first two dominos.*

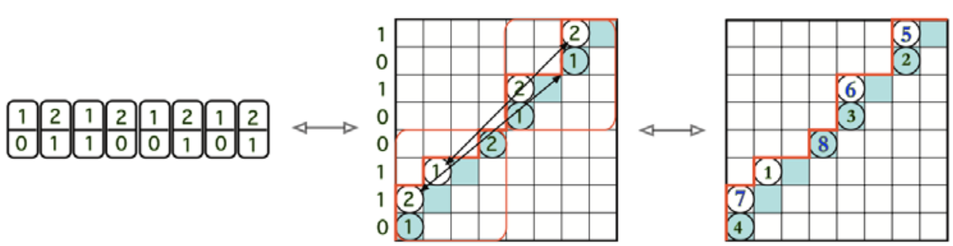
Step 3 *Decrease by 1 the diagonal number of every domino remaining in the first section.*

Step 4 *Cycle the processed first section to the end of the second section.*

For instance in the display below, we have first the result of applying Steps 1,2,3 to the domino sequence in 3.10 and then below it we give the domino sequence resulting from Sep 4 together with the corresponding reduced tableau and the image by Φ of the parking function in 3.9.



3.12



3.13

To complete the proof of 3.8 a) we need to show that Φ is a bijection as stated in 3.10 and that the requirements in 3.11 are satisfied.

To begin, $\Phi(PF)$ is in $\mathcal{PF}_{(p_2,\ldots,p_k,z_1,z_2,\ldots,z_{l(z)})}(r-1)$ with $z=(z_1,z_2,\ldots,z_{l(z)}\models p_1-2$, since decreasing by 1 the diagonal numbers in step 3 will cause the p_1-2 last cars of $\Phi(PF)$ to have a supporting Dyck path that hits the diagonal according to a composition of p_1-2. Conversely, given a $PF'\in\mathcal{PF}_{(p_2,\ldots,p_k,z_1,z_2,\ldots,z_{l(z)})}(r-1)$ we can reconstruct the domino sequence of the parking function PF that Φ maps to PF', by applying the following sequence of steps to $dom(PF')$

Step -1 *Cut $dom(PF')$ into two successive sections of respective lengths $p_2+\cdots+p_k$ and p_1-2.*
Step -2 *Add 1 to the area numbers of the dominos in the second section.*
Step -3 *Cycle back the resulting second section to precede the first section.*
Step -4 *Prepend the resulting domino sequence by the pair $\left[{1\atop 0}\right]\left[{2\atop 1}\right]$*

It is not difficult to see that this construction always yields a legitimate domino sequence of a reduced parking function. For instance, note that since the second section of $dom(PF')$ will necessarily start with one $\left[{1\atop 0}\right]$ or $\left[{2\atop 0}\right]$, then after Step -2 these will become $\left[{1\atop 1}\right]$ or $\left[{2\atop 1}\right]$ thus we are always able to precede any one of them by the pair $\left[{1\atop 0}\right]\left[{2\atop 1}\right]$ and obtain a domino sequence of a parking function with diagonal composition $(p_1,p_2,\ldots p_k)$. This should make it clear that the Φ is a bijection as stated in 3.10.

It remains to verify the equalities in 3.11.

It is quite evident that the area equality is guaranteed by Step 3 together with the removal of the domino $\left[{2\atop 1}\right]$ in Step 2. Moreover, the dinv equality holds true for two reasons:

- Every domino $\left[{1\atop 0}\right]$ following the pair $\left[{1\atop 0}\right]\left[{2\atop 1}\right]$ in $dom(PF)$ used to contribute a secondary diagonal inversion with the removed $\left[{2\atop 1}\right]$ and every $\left[{2\atop 0}\right]$ used to contribute a primary diagonal inversion with the removed $\left[{1\atop 0}\right]$.
- No dinv gains or losses are produced by the reversal of the sequence orders in Step 4, since the combination of Step 3 and Step 4 causes all the primary diagonal inversions to become secondary and all the secondary to become primary as is it is illustrated in 3.9 and 3.13 by the corresponding arrows in the reduced tableaux.

This completes our proof of 3.8 a).

Our next task is to verify 3.8 b). We will start with some auxiliary observations. Note first that for any $PF\in\mathcal{PF}_{p_1,p_2,\ldots,p_k}(r)$ we may regard $dom(PF)$ as a sequence of sections of lengths $p_1,p_2,\ldots,p_k$. Each section starts with a "small car" domino $\left[{1\atop 0}\right]$ or a "big car" domino $\left[{2\atop 0}\right]$. We will call them "*main diagonal dominos* ". Note further that, for any $p_i=1$, its corresponding section reduces to a single main diagonal domino. Conversely, each $\left[{2\atop 0}\right]$ occurring in $dom(PF)$ must be the sole element of a sections of length 1. This is due to the fact each section of length greater than 1 must start with the pair of dominos $\left[{1\atop 0}\right]\left[{2\atop 1}\right]$.

Now let $PF\in\mathcal{PF}_{1,p_2,\ldots,p_k}(r)$ and let PF' be the parking function whose domino sequence is $dom(PF)$ with its initial domino removed. Suppose first that $dom(PF)$ starts with a "big car" domino $\left[{2\atop 0}\right]$. In that case we set $\Phi(PF)=PF'$ and we are done, since $PF'\in\mathcal{PF}_{p_2,\ldots,p_k}(r)$. Moreover, there is no area loss, and since the removed $\left[{2\atop 0}\right]$ did not make any diagonal inversions with any of the succeeding dominos, we have

$$t^{area(PF)}q^{dinv(PF)} \;=\; t^{area(\Phi(PF))}q^{dinv(\Phi(PF))} \qquad 3.14$$

Suppose next that $dom(PF)$ starts with a $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. Note that in this case $PF' \in \mathcal{PF}_{p_2,\ldots,p_k}(r-1)$. However, here the removed $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ used to make a diagonal inversion with every main diagonal domino $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ of $dom(PF)$. Thus in this case we have

$$t^{area(PF)}q^{dinv(PF)} \;=\; t^{area(PF')}q^{dinv(PF')}q^m \qquad 3.15$$

where m gives the number of $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ in $dom(PF)$. We certainly cannot set $\Phi(PF) = PF'$ here, since the weight of PF' occurs with coefficient 1 in the second term on the right hand side of 3.8, b). It turns out that the sum on the right hand side of 3.8 b) is precisely what is needed to perform the necessary correction when $m > 0$. To see how this comes about, we start by writing q^m as the sum

$$q^m = 1 + (q-1) + (q-1)q + (q-1)q^2 + \cdots + (q-1)q^{m-1},$$

so that 3.15 may be rewritten as

$$t^{area(PF)}q^{dinv(PF)} \;=\; t^{area(PF')}q^{dinv(PF')} + (q-1)\sum_{s=1}^{m} t^{area(PF')}q^{dinv(PF')+s-1}. \qquad 3.16$$

Now, suppose that the dominos $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ occur in $dom(PF)$ in positions

$$1 < i_1 < i_2 < \cdots < i_m \leq n \qquad 3.17$$

Note that, by one of our prior observations, we must have $p_{i_s} = 1$ for all $1 \leq s \leq m$. This given, let $PF^{(i_s)}$ be the parking function whose domino sequence is obtained by removing from $dom(PF)$ the initial domino $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ together with the domino $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ in position i_s. Now since every main diagonal domino $dom(PF)$ located between the two removed dominos used to make a primary or secondary diagonal inversion with one or the other of the removed dominos, and the initial domino $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ made a diagonal inversion with the removed $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ as well as all the big car dominos in position $i_{s+1}, \ldots, i_m$, we derive that

$$dinv(PF') + m \;=\; dinv(PF) \;=\; dinv(PF^{(i_s)}) \,+\, i_s - 2 \,+\, 1 \,+\, m - s$$

or better

$$dinv(PF') + s - 1 \;=\; dinv(PF) \;=\; dinv(PF^{(i_s)}) \,+\, i_s - 2$$

which allows us to rewrite 3.6 in the suggestive form

$$t^{area(PF)}q^{dinv(PF)} \;=\; t^{area(PF')}q^{dinv(PF')} + (q-1)\sum_{s=1}^{m} q^{i_s-2}t^{area(PF^{(i_s)})}q^{dinv(PF^{(i_s)})} \qquad 3.16$$

Let us now set for any $PF \in \mathcal{PF}_{1,p_2,\ldots,p_k}(r)$

$$\Phi(PF) \;=\; \begin{cases} PF' & \text{if } dom(PF) \text{ starts with a } \left[\begin{smallmatrix}2\\0\end{smallmatrix}\right] \\[2ex] \big(PF', PF^{(i_1)}, PF^{(i_2)} \ldots, PF^{(i_m)}\big) & \text{if } dom(PF) \text{ starts with a } \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \end{cases}$$

and note that this defines a bijective map of $\mathcal{PF}_{1,p_2,\ldots,p_k}(r)$ onto a disjoint family of subsets covering the union

$$\mathcal{PF}_{p_2,\ldots,p_k}(r)\bigcup \mathcal{PF}_{p_2,\ldots,p_k}(r-1)\bigcup_{f_i=1}\mathcal{PF}_{(p_2,\ldots,\cancel{p_i},\ldots p_k)}(r-1)$$

Indeed this map is onto since

(1) if $PF'\in\mathcal{PF}_{p_2,\ldots,p_k}(r)$ then it is the image by Φ of the PF whose domino sequence is obtained by prepending $dom(PF')$ by a $\left[{2\atop 0}\right]$

(2) if $PF'\in\mathcal{PF}_{p_2,\ldots,p_k}(r-1)$ then it is in the image by Φ of the PF whose domino sequence is obtained by prepending $dom(PF')$ by a $\left[{1\atop 0}\right]$

(3) if $PF^{(i)}\in\mathcal{PF}_{(p_2,\ldots,\cancel{p_i},\ldots p_k)}(r-1)$ with $p_i=1$ then it is in the image by Φ of the PF whose domino sequence is obtained by prepending $dom(PF^{(i)})$ by a $\left[{1\atop 0}\right]$ and inserting a $\left[{2\atop 0}\right]$ in position i.

This given, the identity in 3.8 b) is simply obtained by summing 3.16 over all $PF\in\mathcal{PF}_{1,p_2,\ldots,p_k}(r)$. This completes our proof that the two sides of 3.2 satisfy the same recursion.

We are left to verify the equality in the base cases. To this end note that since at each use of the recursion one or more of the following happens

- r is decreased.
- the composition p is getting finer,
- the number of parts of p decreases.

From the beginning we have required that $0<r<n$, simply because when $r=0$ or $r=n$ the family $\mathcal{PF}_{p_1,p_2,\ldots,p_k}(r)$ reduces to a triviality. In fact, if there are no small cars, or no big cars, the family is empty unless p reduces to a string of $1's$ and if that happens then there is only one parking function with no area and no dinv. Thus the polynomial $\Pi_{p_1,p_2,\ldots,p_k}(r)$ either vanishes or it is equal to 1. This not withstanding, the recursion forces us to include all the degenerate cases. Omitting some trivial cases in which both the combinatorial side as well as the symmetric function side are easily shown to vanish. The only significant basic cases are when p reduces to a string of $1's$ and $0<r<n$. In this case the family $\mathcal{PF}_{1^n}(r)$ consists of all the parking functions obtained by placing along the main diagonal and from right to left all the shuffles of $12\cdots r$ with $r+1\cdots n$. In this case there is no area and the dinv statistic reduces to an inversion count. The corresponding polynomial then is none other than the q-binomial coefficient

$$\Pi_{1^n}(r)\;\;=\;\;\begin{bmatrix}n\\ r\end{bmatrix}_q. \qquad 3.17$$

We need to show that the symmetric function side yields the same result. That is with n occurrences of C_1 we have

$$\left\langle\nabla C_1C_1\cdots C_1 1\;,\;h_r n_{n-r}\right\rangle\;\;=\;\;\begin{bmatrix}n\\ r\end{bmatrix}_q \qquad 3.18$$

Now we have shown (Proposition 1.3) that with n occurrences of C_1 we have

$$\nabla\mathbf{C}_1\mathbf{C}_1\cdots\mathbf{C}_1 1\;\;=\;\;(q,q)_n h_n\left[\tfrac{X}{1-q}\right] \qquad 3.19$$

However we obtain from the Cauchy identity that

$$\left\langle h_n\left[\tfrac{X}{1-q}\right]\;,\;h_\mu[X]\right\rangle\;\;=\;\;h_\mu\left[\tfrac{1}{1-q}\right]$$

which combined with 3.19 gives

$$\Big\langle \nabla C_1 C_1 \cdots C_1 1\, ,\, h_r h_{n-r} \Big\rangle \;=\; \frac{(q,q)_n}{(q,q)_r (q,q)_{n-r}} \qquad 3.20$$

which is another way of writing 3.18.

This completes our proof of I.9.

Remark 3.1

We must note that the very nature of case $p_1 = 1$ of 3.8 makes it stand apart from any of the recursions encountered in all previous parking function literature. For this reason there is no way we could have discovered what to do with our parking functions in this degenerate case without help from the symmetric function side. What is fascinating is that the intricacy of this case and the parking function magics that take place, is none other but a side product of the commutativity relations afforded by the C and B operators. In this context it is interesting to see that 3.8 b) tells us about q-binomial coefficients. In fact, when p reduces to a string of $1's$, using 3.20 3.8 b) states that

$$\frac{(q,q)_n}{(q,q)_r (q,q)_{n-r}} = \frac{(q,q)_{n-1}}{(q,q)_r (q,q)_{n-1-r}} + \frac{(q,q)_{n-1}}{(q,q)_{r-1} (q,q)_{n-r}} + (q-1)\sum_{i=2}^{n} q^{i-2} \frac{(q,q)_{n-2}}{(q,q)_{r-1}(q,q)_{n-1-r}} \qquad 3.21$$

Since

$$(q-1)\sum_{i=2}^{n} q^{i-2} \;=\; (q-1)\sum_{i=0}^{n-2} q^{i} \;=\; (q-1)\frac{1-q^{n-1}}{1-q} \;=\; q^{n-1}-1$$

3.21 becomes

$$\frac{(q,q)_n}{(q,q)_r (q,q)_{n-r}} = \frac{(q,q)_{n-1}}{(q,q)_r (q,q)_{n-1-r}} + \frac{(q,q)_{n-1}}{(q,q)_{r-1} (q,q)_{n-r}} + (q^{n-1}-1) \frac{(q,q)_{n-2}}{(q,q)_{r-1}(q,q)_{n-1-r}} \qquad 3.22$$

or better

$$\frac{(q,q)_n}{(q,q)_r (q,q)_{n-r}} = \frac{(q,q)_{n-1}}{(q,q)_r (q,q)_{n-1-r}} + \frac{(q,q)_{n-1}}{(q,q)_{r-1} (q,q)_{n-r}} + \frac{-(1-q^{n-r})(q,q)_{n-1}}{(q,q)_{r-1}(q,q)_{n-r}}$$

which is just another way of writing the classical recursion

$$\frac{(q,q)_n}{(q,q)_r (q,q)_{n-r}} = \frac{(q,q)_{n-1}}{(q,q)_r (q,q)_{n-1-r}} + q^{n-r} \frac{(q,q)_{n-1}}{(q,q)_{r-1} (q,q)_{n-r}}$$

A fact that bring us to view these ramifications of the shuffle conjecture as a parking function versions of q-binomial identities.

sa

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