Kronecker Coefficients via Symmetric Functions and Constant Term identities

by

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Abstract. This work lies across three areas of investigation that are by themselves of independent interest. A problem that arose in quantum computing led us to a link that tied these areas together. This link led to the calculation of some Kronecker coefficients by computing constant terms and conversely the computations of certain constant terms by computing Kronecker coefficients by symmetric function methods. This led to results as well as methods for solving numerical problems in each of these separate areas.

Introduction

An outstanding yet unsolved problem is to obtain a combinatorial rule for the computation of the integers

$$c^{\lambda}_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}} \;=\; \frac{1}{n!}\sum_{\sigma\in S_n}\chi^{\lambda^{(1)}}(\sigma)\chi^{\lambda^{(2)}}(\sigma)\cdots\chi^{\lambda^{(k)}}(\sigma)\chi^{\lambda}(\sigma) \qquad \text{I.1}$$

where χ^λ and each $\chi^{\lambda^{(i)}}$ are irreducible Young characters of S_n. Let us recall that the pointwise product of any number of characters $\chi^{(1)},\chi^{(2)},\ldots,\chi^{(k)}$ of the symmetric group S_n is also a character of S_n, and we shall denote it here by $\chi^{(1)}*\chi^{(2)}*\cdots*\chi^{(k)}$. This is usually called the '*Kronecker*' product of $\chi^{(1)},\chi^{(2)},\ldots,\chi^{(k)}$. Thus I.1 may written as

$$c^{\lambda}_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}} \;=\; \big\langle \chi^{\lambda^{(1)}}*\chi^{\lambda^{(2)}}*\cdots*\chi^{\lambda^{(k)}},\chi^{\lambda}\big\rangle. \qquad \text{I.2}$$

This integer gives the multiplicity of χ^λ in the Kronecker product $\chi^{\lambda^{(1)}}*\chi^{\lambda^{(2)}}*\cdots*\chi^{\lambda^{(k)}}$. Using the Frobenius map $\mathbf{F}$ that sends the irreducible character χ^λ onto the Schur function s_λ, we can define the Kronecker product of two homogeneous symmetric functions of the same degree f and g by setting

$$f*g \;=\; \mathbf{F}\big((\mathbf{F}^{-1}f)*(\mathbf{F}^{-1}g)\big)$$

With this notation the coefficient in I.1 may also be written in the form

$$c^{\lambda}_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}} \;=\; \big\langle s_{\lambda^{(1)}}*s_{\lambda^{(2)}}*\cdots*s_{\lambda^{(k)}}\,,\,s_\lambda\big\rangle \qquad \text{I.3}$$

where $\langle\,,\,\rangle$ denotes the customary Hall scalar product of symmetric polynomials. This is the vehicle that reduces the computation of Kronecker coefficients to symmetric function manipulations.

A problem which arose in quantum computing (see [6], [7], [11] and [12]) requires the explicit evaluation of the following generating function of Kronecker products

$$W_k(q) \;=\; \sum_{d\geq 0} q^{2d}\big\langle s_{d,d}*s_{d,d}*\cdots*s_{d,d}\,,\,s_{2d}\big\rangle \qquad \text{I.4}$$

where, in each term, the Kronecker product has k factors.

Here and after we will refer to the task of constructing $W_k(q)$ as the '*Sdd Problem*'.

It is well known (see [6] and [7]) and it is an easy consequence of Molien's theorem (see [3]) that all these series can (in principle) be obtained from the following constant term identity.

$$W_k(q) \;=\; \frac{\prod_{i=1}^{k}(1-a_i^2)}{\prod_{S\subseteq[1,k]}\Big(1-q\prod_{i\in S}a_i/\prod_{j\notin S}a_j\Big)}\Bigg|_{a_1^0a_2^0\cdots a_k^0} \qquad \text{I.5}$$

To this date these series have only been obtained for $1\leq k\leq 5$. They are as follows

$$W_2(q) \;=\; \frac{1}{1-q^2}\,, \qquad W_3(q) \;=\; \frac{1}{1-q^4}\,, \qquad W_4(q) \;=\; \frac{1}{(1-q^2)(1-q^4)^2(1-q^6)},$$

and

$$W_5(\sqrt{q}) \;=\; \frac{P_5(q)}{(1-q^2)^4(1-q^3)(1-q^4)^6(1-q^5)(1-q^6)^5},$$

with

$$\begin{aligned} P_5(q) = {} & q^{54}+q^{52}+16q^{50}+9q^{49}+98q^{48}+154q^{47}+465q^{46}+915q^{45}+2042q^{44}+3794q^{43}+7263q^{42}\\ & +12688q^{41}+21198q^{40}+34323q^{39}+52205q^{38}+77068q^{37}+108458q^{36}+147423q^{35}+191794q^{34}\\ & +241863q^{33}+292689q^{32}+342207q^{31}+386980q^{30}+421057q^{29}+443990q^{28}+451398q^{27}\\ & +443990q^{26}+421057q^{25}+386980q^{24}+342207q^{23}+292689q^{22}+241863q^{21}+191794q^{20}\\ & +147423q^{19}+108458q^{18}+77068q^{17}+52205q^{16}+34323q^{15}+21198q^{14}+12688q^{13}\\ & +7263q^{12}+3794q^{11}+2042q^{10}+915q^{9}+465q^{8}+154q^{7}+98q^{6}+9q^{5}+16q^{4}+q^{2}+1\;. \end{aligned}$$

Clearly, the result for $W_2(q)$ is immediate from the definition in I.4. Moreover $W_3(q)$, $W_4(q)$ can be easily obtained by computing the constant term in I.5 with '*Omega*' Package of Andrews et. al. However, the explosion of complexity from $k=4$ to $k=5$ required more powerful machinery. The calculation of $W_5(q)$ using I.5 was first carried out by J-G. Luque and J. Y. Thibon (see [7]) by the partial fraction algorithm of the second author ([13] and [14]). We understand (personal communication by J. Y. Thibon) that the original calculation took a few hours with the computers they used at that time. With current technology, by means of some combinatorial reductions (see [3]), the computation of $W_5(q)$ can be reduced to a few minutes. Nevertheless to this date, the evaluation of $W_6(q)$ by I.5, appears out of reach of our computers.

The present paper resulted from a continuing effort to determine these series by symmetric function methods. We cover here a number of results and techniques that have emerged from this effort.

Our first result in this direction may be stated as follows.

Theorem I.1

$$s_{d,d}*s_{d,d} \;=\; \sum_{\lambda\vdash 2d} s_\lambda\,\chi(\lambda\in EO_4) \qquad \text{I.6}$$

where EO_4 denotes the set of partitions of length 4 whose parts are ≥ 0 and all even or all odd.

It is easily seen that the expressions for $W_3(q)$ and $W_4(q)$ are immediate consequences of this identity. We should mention that a combinatorial proof of I.5 was obtained by J. Remmel (personal communication).

The results and techniques that have emerged from this effort, led us to further uses of the partial fraction algorithm in the computation of generating functions of Kronecker products.

These computations are based on the following surprisingly simple identity

Proposition I.1

For any $k \geq 1$ we have

$$\frac{\prod_{i=1}^{k}(1-y_i)}{\prod_{S\subseteq[1,k]}(1-q\prod_{i\in S}y_i)} = \sum_{m\geq 0} q^m \sum_{r_1\geq 0}\sum_{r_2\geq 0}\cdots\sum_{r_k\geq 0} y_1^{r_1}y_2^{r_2}\cdots y_k^{r_k}\big\langle s_{m-r_1,r_1}*s_{m-r_2,r_2}*\cdots*s_{m-r_k,r_k}\,,\,s_m\big\rangle \qquad \text{I.7}$$

where a factor s_{m-r_i,r_i} is to be interpreted as $-s_{r_i-1,m-r_i+1}$ if $m+1<2r_i$ and 0 if $m+1=2r_i$.

In particular we will see that I.5 is an immediate consequence of this identity. The following is a list of the results we will derive from I.7.

Theorem I.2

For any given $r_1, r_2, \ldots, r_k$ the Kronecker coefficient

$$\big\langle s_{m-r_1,r_1}*s_{m-r_2,r_2}*\cdots*s_{m-r_k,r_k}\,,\,s_m\big\rangle$$

stabilizes after a finite number of terms and the stable value is given by the coefficient

$$P_{r_1,r_2,\ldots,r_k}(1) \;=\; \frac{\prod_{i=1}^{k}(1-y_i)}{\prod_{\phi\neq S\subseteq[1,k]}(1-\prod_{i\in S}y_i)}\bigg|_{y_1^{r_1}y_2^{r_2}\cdots y_k^{r_k}}\,.$$

Theorem I.3

If we set

$$F_3(y_1,y_2,y_3;q) = \sum_{r_1\geq 0}\sum_{r_2\geq 0}\sum_{r_3\geq 0}\sum_{m\geq 2max(r_1,r_2,r_3)} q^m y_1^{r_1}y_2^{r_2}y_3^{r_3}\big\langle s_{m-r_1,r_1}*s_{m-r_2,r_2}*s_{m-r_3,r_3},s_m\big\rangle \qquad \text{I.8}$$

and

$$G_3(y_1,y_2,y_3;q) = \sum_{r_3\geq 0}\sum_{r_2\geq r_3}\sum_{r_1\geq r_2}\sum_{m\geq 2r_1} q^m y_1^{r_1}y_2^{r_2}y_3^{r_3}\big\langle s_{m-r_1,r_1}*s_{m-r_2,r_2}*s_{m-r_3,r_3},s_m\big\rangle \qquad \text{I.9}$$

then

$$F_3(y_1,y_2,y_3;q) = \frac{1+q^3y_1y_2y_3}{(1-q)(1-q^2y_1y_2)(1-q^2y_1y_3)(1-q^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)} \qquad \text{I.10}$$

and

$$G_3(y_1,y_2,y_3;q) = \frac{1}{(1-q)(1-q^2y_1y_2)(1-q^3y_1y_2y_3)(1-q^4y_1^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)}\,. \qquad \text{I.11}$$

Theorem I.4

For given integers $r_1 \geq r_2 \geq r_3 \geq 0$ set

$$\Phi_{r_1,r_2,r_3}(q) \;=\; \sum_{m\geq 2max(r_1,r_2,r_3)} q^m \big\langle s_{m-r_1,r_1} * s_{m-r_2,r_2} * s_{m-r_3,r_3}, s_m \big\rangle \tag{I.12}$$

then provided $r_2 + r_3 \geq r_1$ we have

$$\Phi_{r_1,r_2,r_3}(q) \;=\; \frac{q^{2r_1}}{(1-q)(1-q^2)} \begin{cases} 1 - q^{r_2+r_3-r_1+2} & \textit{if } r_1+r_2+r_3 \textit{ is even} \\ q - q^{r_2+r_3-r_1+2} & \textit{if } r_1+r_2+r_3 \textit{ is odd} \end{cases} \tag{I.13}$$

otherwise $\Phi_{r_1,r_2,r_3}(q)$ vanishes identically.

In this paper we use methods and algorithms from several areas. In an effort to make our presentation self contained, we have included brief tutorials developing the tools we are about to use. Some of this material may be well known to the experts in each particular field, this will be compensated by making our writing readily accessible to the wider audience of researchers who may not be simultaneously proficient in all these disparate areas. In particular in section 1 we have a brief introduction to plethystic notation and use it to derive some basic tools for the computation of Kronecker products and use them to prove Proposition I.1. We will also include in this section a remarkably slick proof of Proposition I.1 kindly provided to us by J. Y. Thibon (personal communication). In section 2 we use these tools to compute the Schur function expansion of $s_{dd} * s_{dd}$ and obtain a proof of Theorem I.1. In section 3 we develop the setup for computing Kronecker coefficients via constant terms. The section terminates with a tutorial on the use of the partial fraction algorithm of G. Xin. In section 4, we use the Xin algorithm to compute the constant terms yielding Theorems I.2, I.3 and I.4.

1. Symmetric function methods

As we stated in the introduction the first three series $W_2(q)$, $W_3(q)$ and $W_4(q)$ can be easily computed. Indeed, for $k = 2$ we have

$$W_2(q) = \sum_{d\geq 0} q^{2d}\big\langle s_{d,d} * s_{d,d}\,,\, s_{2d}\big\rangle = \sum_{d\geq 0} q^{2d}\big\langle s_{d,d}\,,\, s_{d,d}\big\rangle = \sum_{d\geq 0} q^{2d} \;=\; \frac{1}{(1-q^2)}\,.$$

For $k = 3$ we may write

$$W_3(q) = \sum_{d\geq 0} q^{2d}\big\langle s_{d,d} * s_{d,d} * s_{d,d}\,,\, s_{2d}\big\rangle = \sum_{d\geq 0} q^{2d}\big\langle s_{d,d} * s_{d,d}\,,\, s_{d,d}\big\rangle$$

and Theorem I.1 forces d to be even, yielding

$$W_3(q) = \sum_{d\geq 0} q^{4d} \;=\; \frac{1}{1-q^4}.$$

For $k = 4$ we start by writing

$$W_4(q) = \sum_{d\geq 0} q^{2d}\big\langle s_{d,d} * s_{d,d} * s_{d,d} * s_{d,d}\,,\, s_{2d}\big\rangle = \sum_{d\geq 0} q^{2d}\big\langle s_{d,d} * s_{d,d}\,,\, s_{d,d} * s_{d,d}\big\rangle$$

and Theorem I.1 gives

$$\begin{aligned} W_4(q) &= \sum_{d\geq 0} q^{2d} \sum_{\lambda\vdash 2d} \langle s_\lambda\,,\, s_\lambda\rangle\,\chi(\lambda\in EO_4) \\ &= \sum_{\lambda\vdash 2d} q^{|\lambda|}\chi\big(\lambda\in E_{\leq 4}\big) \,+\, \sum_{\lambda\vdash 2d} q^{|\lambda|}\chi\big(\lambda\in O_{=4}\big) \end{aligned} \qquad 1.4$$

where $E_{\leq 4}$ denotes the collection of partitions with at most four parts all of which are even and $O_{=4}$ denotes the collection of partitions with with exactly 4 odd parts. Since to obtain a partition of $2d$ in $E_{\leq 4}$ we need only double the parts of any partition of d with at most 4 parts we see that we have

$$\sum_{\lambda\vdash 2d} q^{|\lambda|}\chi\big(\lambda\in E_{\leq 4}\big) \;=\; \frac{1}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)}. \qquad 1.5$$

Similarly, we can obtain each partition of $2d$ in $O_{=4}$ by taking a partition of $d-1$ with at most 4 parts, doubling each part and adding a column of length 4. This gives

$$\sum_{\lambda\vdash 2d} q^{|\lambda|}\chi\big(\lambda\in O_{=4}\big) \;=\; \frac{q^4}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)}\;. \qquad 1.6$$

Combining 1.4, 1.5 and 1.6 gives

$$\begin{aligned} W_4(q) \;&=\; \frac{1+q^4}{(1-q^2)(1-q^4)(1-q^6)(1-q^8)} \\ &=\; \frac{1}{(1-q^2)(1-q^4)(1-q^6)(1-q^4)} \;=\; \frac{1}{(1-q^2)(1-q^4)^2(1-q^6)} \end{aligned}$$

as desired.

We will prove Theorem I.1 in the next section. In this section we will gather the background needed for this proof.

It will be good to begin by a brief introduction to plethystic substitutions. The convenience of this notational device in the theory of symmetric functions is often overlooked for, in principle, everything that can be done with it can also be done without it. Witness Macdonald's treatise that manages to avoid it almost in its entirety. We say 'almost' since many of the computations in Chapter IV are in fact 'plethystic' in disguise (for instance in page 310).

We make extensive use of the notion of plethystic substitution of a formal power series $E = E(t_1, t_2, \ldots)$ into a symmetric function P, denoted $P[E]$. This operation, which can be easily implemented on a computer, consists of two steps.

(1) Expand P as a polynomial $P = Q_P(p_1, p_2, \ldots, p_k, \ldots)$ in the power symmetric functions.
(2) Set $P[E] = Q_P(p_1, p_2, \ldots, p_k, \ldots)\Big|_{p_k=E(t_1^k, t_2^k, \ldots)}$.

The power of this notation results from the fact that simple operations within the plethystic bracket result in transformations of significant complexity outside the bracket. But the real significance of this statement can only be appreciated through experimentation. For this we will have ample opportunity within this writing.

A crucial ingredient in plethystic calculus is the '*kernel*'

$$\Omega \;=\; exp\Big(\sum_{k\geq 1}\frac{p_k}{k}\Big) \tag{1.10}$$

which may be also be viewed as the ordinary generating function of the ordinary '*homogeneous*' symmetric function. More precisely

$$\Omega \;=\; \sum_{m\geq 0} h_m, \tag{1.11}$$

where the equality of 1.10 and 1.11 results from the familiar expansion

$$h_m \;=\; \sum_{\rho\vdash m}\frac{p_\rho}{z_\rho} \tag{1.12}$$

(recalling that if $\rho = 1^{a_1}2^{a_2}3^{a_3}\cdots$ then $z_\rho \;=\; \prod_i i^{a_i}a_i!$).

We shall also make extensive use here of the Frobenius formula (for $\rho\vdash m$)

$$p_\rho \;=\; \sum_{\lambda\vdash m}\chi^\lambda_\rho\, s_\lambda \tag{1.13}$$

where as customary χ^λ_ρ denotes the Young character indexed by λ at the conjugacy class of permutations of S_m of cycle structure ρ.

We shall also make use of the operation of skewing by a symmetric function. For a symmetric function f, the notation $f^\perp$ will represent an operation on symmetric functions which is dual to multiplication in the sense that

$$\left\langle f^\perp g, h\right\rangle = \langle g, fh\rangle\,.$$

In particular, to compute how $f^\perp$ acts on a symmetric function g, we may use the scalar product to expand g in the power or Schur basis,

$$g = \sum_\lambda \langle g, p_\lambda\rangle\, p_\lambda/z_\lambda = \sum_\lambda \langle g, s_\lambda\rangle\, s_\lambda$$

and conclude that

$$f^\perp g = \sum_\lambda \langle g, fp_\lambda\rangle\, p_\lambda/z_\lambda = \sum_\lambda \langle g, fs_\lambda\rangle\, s_\lambda\;.$$

This given, we have now all the ingredients needed to give, as a warm up exercise, a plethystic proof of the identity in I.5. To begin, we simply note that we have

$$\begin{aligned}\frac{1}{\prod\limits_{S\subseteq[1,k]}\Big(1-q\prod_{i\in S}a_i/\prod_{j\notin S}a_j\Big)} \;&=\; \Omega\Big[q\big(a_1+1/a_1\big)\big(a_2+1/a_2\big)\cdots\big(a_k+1/a_k\big)\Big]\;.\\ \text{(by 1.11)} \;&=\; \sum_{m\geq 0}q^m h_m\big[\big(a_1+1/a_1\big)\big(a_2+1/a_2\big)\cdots\big(a_k+1/a_k\big)\big]\;.\end{aligned}$$

Thus

$$\frac{\prod_{i=1}^k(1-a_i^2)}{\prod\limits_{S\subseteq[1,k]}\Big(1-q\prod_{i\in S}a_i/\prod_{j\notin S}a_j\Big)}\Bigg|_{a_1^0a_2^0\cdots a_k^0} \;=\; \sum_{m\geq 0}q^m h_m\Big[\prod_{i=1}^k\big(a_i+1/a_i\big)\Big]\prod_{i=1}^k(1-a_i^2)\Bigg|_{a_1^0a_2^0\cdots a_k^0}\;. \tag{1.14}$$

Now 1.12 gives

$$h_m\Big[\prod_{i=1}^k \big(a_i+1/a_i\big)\Big] \;=\; \sum_{\rho\vdash m}\frac{1}{z_\rho}\prod_{i=1}^k p_\rho\big[a_i+1/a_i\big]\ . \tag{1.15}$$

Setting for convenience $A_i=a_i+1/a_i$, a multiple use of 1.13 in 1.14 gives

$$h_m\big[A_1A_2\ldots A_k\big] \;=\; \sum_{\alpha_1\vdash m}\sum_{\alpha_2\vdash m}\cdots\sum_{\alpha_k\vdash m} s_{\alpha_1}[A_1]s_{\alpha_2}[A_2]\cdots s_{\alpha_k}[A_k]\sum_{\rho\vdash m}\frac{\chi_\rho^{\alpha_1}\chi_\rho^{\alpha_2}\cdots\chi_\rho^{\alpha_k}}{z_\rho}$$

and the definition of Kronecker product of symmetric functions then gives

$$h_m\big[A_1A_2\ldots A_k\big] \;=\; \sum_{\alpha_1\vdash m}\sum_{\alpha_2\vdash m}\cdots\sum_{\alpha_k\vdash m} s_{\alpha_1}[A_1]s_{\alpha_2}[A_2]\cdots s_{\alpha_k}[A_k]\big\langle s_{\alpha_1}*s_{\alpha_2}*\cdots*s_{\alpha_k}\,,\,s_m\big\rangle \tag{1.16}$$

Since Schur functions in a two variable alphabet vanish at partitions with more than two parts, it follows that we may take here $\alpha_i=(m-r_i,r_i)$ with $r_i\le m/2$ reducing 1.16 to

$$h_m\Big[\prod_{i=1}^k A_i\Big]=\sum_{r_1=0}^{\lfloor m/2\rfloor}\sum_{r_2=0}^{\lfloor m/2\rfloor}\cdots\sum_{r_k=0}^{\lfloor m/2\rfloor}\Big(\prod_{i=1}^k s_{m-r_i,r_i}[A_i]\Big)\big\langle s_{m-r_1,r_1}*s_{m-r_2,r_2}*\cdots*s_{m-r_k,r_k}\,,\,s_m\big\rangle\ . \tag{1.17}$$

Now note that

$$s_{m-r_i,r_i}[a_i+1/a_i] \;=\; \sum_{k=0}^{m-2r_i} a_i^k(1/a_i)^{m-2r_i-k} \;=\; \frac{a_i^{-(m-2r_i)}-a_i^{m-2r_i+2}}{1-a_i^2}\ . \tag{1.18}$$

Thus

$$(1-a_i^2)s_{m-r_i,r_i}[a_i+1/a_i]\Big|_{a_i^0} \;=\; \begin{cases}1 & \text{if } m-2r_i=0\\ 0 & \text{otherwise}\end{cases}$$

which forces m to be an even number. Thus for $m=2d$, 1.17 gives that

$$h_{2d}\Big[\prod_{i=1}^k(a_i+1/a_i)\Big]\prod_{i=1}^k(1-a_i^2)\Big|_{a_1^0a_2^0\cdots a_k^0} \;=\; \big\langle s_{dd}*s_{dd}*\cdots*s_{dd}\,,\,s_{2d}\big\rangle$$

and 1.14 yields

$$\frac{\prod_{i=1}^k(1-a_i^2)}{\prod_{S\subseteq[1,k]}\Big(1-q\prod_{i\in S}a_i/\prod_{j\notin S}a_j\Big)}\Bigg|_{a_1^0a_2^0\cdots a_k^0} \;=\; \sum_{d\ge 0}q^{2d}\big\langle s_{dd}*s_{dd}*\cdots*s_{dd}\,,\,s_{2d}\big\rangle$$

as desired.

The same kinds of manipulations yield us our first

Proof of Proposition I.1

Let

$$G(q;y_1,y_2,\ldots y_k;a_1,a_2,\ldots,a_k) \;=\; \Omega\Big[q\prod_{i=1}^k(1+\tfrac{1}{a_i})\Big]\prod_{i=1}^k\frac{1-\frac{1}{a_i}}{1-a_iy_i} \tag{1.20}$$

We will compute the constant term

$$G(q;y_1,y_2,\ldots y_k;a_1,a_2,\ldots,a_k)\Big|_{a_1^0a_2^0\cdots a_k^0}\ . \qquad 1.21$$

in two different ways. To begin note that for any monomial $a_1^{p_1}a_2^{p_2}\cdots a_k^{p_k}$ we have

$$\frac{1}{a_1^{p_1}a_2^{p_2}\cdots a_k^{p_k}}\prod_{i=1}^k\frac{1}{1-a_iy_i}\Big|_{a_1^0a_2^0\cdots a_k^0} \;=\; y_1^{p_1}y_2^{p_2}\cdots y_k^{p_k}\ .$$

Thus the coefficient of q^m in 1.20 gives

$$h_m\Big[\textstyle\prod_{i=1}^k(1+\frac{1}{a_i})\Big]\prod_{i=1}^k(1-\frac{1}{a_i})\displaystyle\prod_{i=1}^k\frac{1}{1-a_iy_i}\Big|_{a_1^0a_2^0\cdots a_k^0} \;=\; h_m\Big[\textstyle\prod_{i=1}^k(1+y_i)\Big]\prod_{i=1}^k(1-y_i),$$

from which we derive that

$$G(q;y_1,y_2,\ldots y_k;a_1,a_2,\ldots,a_k)\Big|_{a_1^0a_2^0\cdots a_k^0} = \Omega\Big[q\textstyle\prod_{i=1}^k(1+y_i)\Big]\prod_{i=1}^k(1-y_i) \;=\; \dfrac{\prod_{i=1}^k(1-y_i)}{\prod_{S\subseteq[1,k]}(1-q\prod_{i\in S}y_i)}.$$

On the other hand, as we noted before, we can write

$$h_m\Big[\textstyle\prod_{i=1}^k(1+\frac{1}{a_i})\Big] \;=\; \displaystyle\sum_{l_1=0}^{\lfloor\frac{m}{2}\rfloor}\sum_{l_2=0}^{\lfloor\frac{m}{2}\rfloor}\cdots\sum_{l_k=0}^{\lfloor\frac{m}{2}\rfloor}\prod_{i=1}^k s_{m-l_i,l_i}[1+1/a_i]\big\langle s_{m-l_1,l_1}*s_{m-l_2,l_2}*\cdots*s_{m-l_k,l_k}\,,\,s_m\big\rangle\ .$$

Note further that the identity

$$s_{m-l,l}[1+1/a] \;=\; \frac{1}{a^l}\sum_{k=0}^{m-2l}a^{-k} \;=\; \frac{1}{a^l}\frac{1-a^{-(m-2l+1)}}{1-1/a}$$

gives, for $l\le\lfloor m/2\rfloor$ and any integer $r\ge 0$

$$(1-1/a)s_{m-l,l}[1+1/a]a^r\Big|_{a^0} = a^{r-l}\,-\,a^{r-m+l-1}\Big|_{a^0} = \begin{cases}\chi(l=r) & \text{if } r\le\lfloor m/2\rfloor\\ 0 & \text{if } \lfloor m/2\rfloor<r\le m-\lfloor m/2\rfloor\\ -\chi(l=m-r+1) & \text{if } r>m-\lfloor m/2\rfloor\end{cases}\ .$$

This implies that

$$\textstyle\prod_{i=1}^k(1-\frac{1}{a_i})h_m\Big[\prod_{i=1}^k(1+\frac{1}{a_i})\Big]a_1^{r_1}a_2^{r_2}\cdots a_k^{r_k}\Big|_{a_1^0a_2^0\cdots a_k^0} \;=\; \big\langle s_{m-r_1,r_1}*s_{m-r_2,r_2}*\cdots*s_{m-r_k,r_k}\,,\,s_m\big\rangle \qquad 1.22$$

with the proviso that we must take

$$s_{m-r_i,r_i} \;=\; \begin{cases}s_{m-r_i,r_i} & \text{if } r_i\le\lfloor m/2\rfloor\\ 0 & \text{if } \lfloor m/2\rfloor<r_i\le m-\lfloor m/2\rfloor\\ -s_{r_i-1,m-r_i+1} & \text{if } r_i>m-\lfloor m/2\rfloor\end{cases}\ .$$

Taking this into account, multiplying 1.22 by $y_1^{r_1}y_2^{r_2}\cdots y_k^{r_k}$ and summing for all $r_i\geq 0$ gives

$$\prod_{i=1}^k(1-\tfrac{1}{a_i})h_m\Big[\textstyle\prod_{i=1}^k(1+\tfrac{1}{a_i})\Big]\displaystyle\prod_{i=1}^k\frac{1}{1-a_iy_i}\Big|_{a_1^0a_2^0\cdots a_k^0}$$
$$=\sum_{r_1\geq 0}\sum_{r_2\geq 0}\cdots\sum_{r_k\geq 0}y_1^{r_1}y_2^{r_2}\cdots y_k^{r_k}\big\langle s_{m-r_1,r_1}*s_{m-r_2,r_2}*\cdots*s_{m-r_k,r_k}\,,\,s_m\big\rangle\ .$$

Multiplying by q^m and summing proves I.7 and completes our argument.

In the remainder of this section we will review a variety of tools for the computation of Kronecker products which will be used in the next section in the proof of Theorem I.1. We will require the use of the following well known basic identities.

Proposition 1.1

$$\begin{array}{rrcl} 1) & p_\alpha * p_\beta & = & \chi(\alpha=\beta)z_\alpha p_\alpha \\ 2) & p_\alpha * s_\lambda & = & \chi_\alpha^\lambda\, p_\alpha \\ 3) & h_m * f & = & f \end{array} \qquad 1.23$$

The last equality holding true for all homogeneous symmetric functions of degree m.

Proof

Recalling that the definition of the Kronecker product of two homogeneous symmetric functions f,g is defined by means of the Frobenius map by setting

$$f*g \;=\; \mathbf{F}((\mathbf{F}^{-1}f)*(\mathbf{F}^{-1}g))$$

then 1.23 is an immediate consequence of the fact that

$$a)\quad \mathbf{F}^{-1}p_\alpha \;=\; z_\alpha C_\alpha \quad\text{and}\quad b)\quad \mathbf{F}C_\alpha \;=\; p_\alpha/z_\alpha$$

where C_α is the conjugacy class of permutations with cycle structure α. The identity in 1.23 2) then follows by linearity from 1.23 1) and the Frobenius formula

$$s_\lambda \;=\; \sum_\beta \chi_\beta^\lambda\, p_\beta/z_\beta\ .$$

Finally 1.23 is a simple consequence of the fact that the symmetric function h_m is the Frobenius image of the trivial character, and therefore it must act as the identity in a Kronecker product. That is for any homogeneous symmetric polynomial of degree m we have

$$h_m*f \;=\; f.$$

An important tool for reducing the computation of Kronecker products to ordinary products is provided by the following basic identity of D. E. Littlewood [5] (see also [2] for some useful corollaries).

Proposition 1.2

For any k-tuple of homogeneous symmetric functions $f_1, f_2, \ldots, f_k$ of degrees $a_1, a_2, \ldots, a_k$ and any symmetric function H homogeneous of degree $a_1 + a_2 + \cdots + a_k$ we have

$$f_1 f_2 \ldots f_k * H \;=\; \sum_{\alpha_1 \vdash a_1} \sum_{\alpha_2 \vdash a_2} \cdots \sum_{\alpha_k \vdash a_k} \big\langle s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}, H \big\rangle \, f_1 * s_{\alpha_1} \, f_2 * s_{\alpha_2} \cdots f_k * s_{\alpha_k}. \qquad 1.24$$

Proof

We need only verify 1.24 for $f_i = p_{\beta^{(i)}}$ with $\beta^{(i)} \vdash a_i$ and $H = s_\lambda$ with $\lambda \vdash a_1 + a_2 + \cdots + a_k$. This given, the left hand side of 1.24 becomes, using 1.23 2)

$$\begin{aligned} LHS \;=\; p_{\beta^{(1)} \vee \beta^{(2)} \vee \ldots \vee \beta^{(k)}} * s_\lambda \;&=\; \chi^\lambda_{\beta^{(1)} \vee \beta^{(2)} \vee \ldots \vee \beta^{(k)}} \;\; p_{\beta^{(1)} \vee \beta^{(2)} \vee \ldots \vee \beta^{(k)}} \\ &=\; \big\langle p_{\beta^{(1)} \vee \beta^{(2)} \vee \ldots \vee \beta^{(k)}}, s_\lambda \big\rangle \;\; p_{\beta^{(1)} \vee \beta^{(2)} \vee \ldots \vee \beta^{(k)}} \, . \end{aligned}$$

where the symbol $\vee$ denotes coalescing of partitions. On the other hand, using again 1.23 2), the right hand side becomes

$$RHS \;=\; \sum_{\alpha_1 \vdash a_1} \sum_{\alpha_2 \vdash a_2} \cdots \sum_{\alpha_k \vdash a_k} \big\langle s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}, s_\lambda \big\rangle \, \chi^{\alpha_1}_{\beta^{(1)}} p_{\beta^{(1)}} \chi^{\alpha_2}_{\beta^{(2)}} p_{\beta^{(2)}} \cdots \chi^{\alpha_k}_{\beta^{(k)}} p_{\beta^{(k)}}$$

and 1.24 in this case immediately follows from the Frobenius expansion

$$\sum_{\alpha \vdash a} s_\alpha \chi^\alpha_\beta \;=\; p^\beta \, .$$

We can now derive the following useful corollary.

Proposition 1.3

For any homogeneous symmetric function H of degree $a_1 + a_2 + \ldots + a_k$ we have

$$\begin{aligned} h_{a_1} h_{a_2} \cdots h_{a_k} * H \;&=\; \sum_{\alpha_1 \vdash a_1} \sum_{\alpha_2 \vdash a_2} \cdots \sum_{\alpha_k \vdash a_k} \big\langle s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}, H \big\rangle \, s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k} \\ &=\; \sum_{\alpha_2 \vdash a_2} \cdots \sum_{\alpha_k \vdash a_k} s_{\alpha_2} \cdots s_{\alpha_k} \times s^\perp_{\alpha_2} \cdots s^\perp_{\alpha_k} H \, . \end{aligned} \qquad 1.25$$

Proof

The first equality follows by setting $f_i = h_{a_i}$ in 1.24 and using 1.23 3). Note further that the first equality may be also rewritten as

$$h_{a_1} h_{a_2} \cdots h_{a_k} * H \;=\; \sum_{\alpha_1 \vdash a_1} \sum_{\alpha_2 \vdash a_2} \cdots \sum_{\alpha_k \vdash a_k} \big\langle s_{\alpha_1} \, , \, s^\perp_{\alpha_2} \cdots s^\perp_{\alpha_k} s_\lambda \big\rangle \, s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k}$$

and thus the second equality is obtained by carrying out the sum over all $\alpha_1 \vdash a_1$.

Proposition 1.4

For any triplet of homogeneous symmetric functions of the same degree f, g, h we have

$$\big\langle f * g, h \big\rangle \;=\; \big\langle f[X] g[Y] \, , \, h[XY] \big\rangle_{XY} \, . \qquad 1.26$$

Proof

Clearly we need only verify this for power basis elements. In this case 1.26 reduces to

$$\langle p_\alpha * p_\beta \, , \, p_\gamma \rangle \;=\; \langle p_\alpha[X] p_\beta[Y] \, , \, p_\gamma[X] p_\gamma[Y] \rangle_{XY}$$

Using 1.23 1) this becomes

$$\chi(a=b=c)\, z_\alpha^2 \;=\; \langle p_\alpha, p_\gamma \rangle \langle p_\beta, p_\gamma \rangle \;=\; \chi(a=c) z_\alpha \chi(b=c) z_\beta .$$

This shows that the two sides of 1.26 are equal and completes the proof.

We will use two remarkable consequences of 1.26. More precisely

Proposition 1.5

For any $1 \le k \le n$ and any three homogeneous symmetric functions f, g, h of degrees $n-k$, n and n respectively we have

$$\begin{aligned} a) \quad \langle h_k f \, , \, g * h \rangle \;&=\; \sum_{\alpha \vdash k} \langle f \, , \, s_\alpha^\perp g * s_\alpha^\perp h \rangle \\ b) \quad \langle e_k f \, , \, g * h \rangle \;&=\; \sum_{\alpha \vdash k} \langle f \, , \, s_\alpha^\perp g * s_{\alpha'}^\perp h \rangle \; . \end{aligned} \qquad 1.27$$

Proof

From 1.26 with $f \to h_k f$ it follows that

$$\langle h_k f \, , \, g * h \rangle \;=\; \langle h_k[XY] f[XY] \, , \, g[X] h[Y] \rangle_{XY}$$

and the Cauchy formula gives

$$\begin{aligned} \langle h_k f \, , \, g * h \rangle \;&=\; \sum_{\alpha \vdash k} \langle s_\alpha[X] s_\alpha[Y] f[XY] \, , \, g[X] h[Y] \rangle_{XY} \\ &=\; \sum_{\alpha \vdash k} \langle f[XY] \, , \, \langle s_\alpha[X]^\perp g[X] s_\alpha[Y]^\perp h[Y] \rangle_{XY} \\ \text{(by 1.26 again)} \;&=\; \sum_{\alpha \vdash k} \langle f \, , \, s_\alpha^\perp g * s_\alpha^\perp h \rangle \; . \end{aligned}$$

This proves 1.27 a). The proof of 1.27 b) is entirely analogous except that we set $f \to e_k f$ and use the dual Cauchy formula

$$e_k[XY] \;=\; \sum_\alpha s_\alpha[X] s_{\alpha'}[Y].$$

Finally we must point out that the two well known *row* and *column* adding formulas for Schur functions are immediate consequences of the Jacobi-Trudi identities.

Proposition 1.6

For any partition λ with largest part $\le m$ and any μ with at most m parts we have

$$\begin{aligned} a) \quad s_{m,\lambda} \;&=\; \sum_{i \ge 0} (-1)^i h_{m+i} \, e_i^\perp s_\lambda \\ b) \quad s_{\mu+1^m} \;&=\; \sum_{i \ge 0} (-1)^i e_{m+i} \, h_i^\perp s_\mu \end{aligned} \qquad 1.28$$

where $\mu + 1^m$ means adding a column of length m to μ.

Proof

We have

$$s_{m,\lambda} \;=\; \det \begin{pmatrix} h_m & h_{m+1} & h_{m+2} & \cdots & h_{m+k} \\ h_{\lambda_1-1} & h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+k-1} \\ h_{\lambda_2-2} & h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_k-k} & h_{\lambda_k-k+1} & h_{\lambda_k-k+2} & \cdots & h_{\lambda_k} \end{pmatrix}.$$

Expanding this determinant with respect to the first row gives

$$s_{m,\lambda} \;=\; \sum_{i=0}^{k}(-1)^k h_{m+i}s_{\lambda/1^i} \qquad 1.29$$

which is another way of writing 1.28 a). To prove 1.28 b) we simply note that applying the ω transformation to 1.29 we obtain

$$s_{\lambda'+1^m} \;=\; \sum_{i=0}^{k}(-1)^k e_{m+i}s_{\lambda'/i}$$

which is 1.28 b) for $\mu = \lambda'$.

It is important to note the following property of the Kronecker product

Proposition 1.7

If λ and μ are partitions of lengths k and h respectively then the Schur function expansion of the Kronecker product

$$s_\lambda * s_\mu$$

involves Schur functions indexed by partitions of length at most hk

Proof

Note that from 1.25 it follows that

$$h_{a_1}h_{a_2}\cdots h_{a_k} * s_\lambda \;=\; \sum_{\alpha_1\vdash a_1}\sum_{\alpha_2\vdash a_2}\cdots\sum_{\alpha_k\vdash a_k}\big\langle s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_k}, s_\mu\big\rangle\, s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_k}. \qquad 1.30$$

Now from the Littlewood-Richardson rule it follows that the scalar product $\big\langle s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_k}, s_\mu\big\rangle$ is different from zero only if each of the partitions α_i is contained in μ. Thus if μ has length h then again from the Littlewood-Richardson rule it follows that the Schur function expansion of the product $s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_k}$ will only involve Schur functions indexed by partitions with at most kh parts. Thus the assertion is an immediate consequence of 1.30 and the Jacobi-Trudi identity.

We have now all the ingredients we need to prove Theorem I.1.

But before we terminate this section it is instructive to see the symmetric function tricks that Thibon uses to prove Proposition I.1. His argument is based on the simple, but powerful idea that, when there is a way to force the degree to be the desired one, then the Schur row adder, which in our notation is

$$H_a \;=\; \sum_{i\geq 0} h_{a+i}(-1)^i e_i^\perp$$

may loosely be replaced by

$$\Big(\sum_{n\geq 0} h_n\Big)\Big(\sum_{k\geq 0}(-1)^k e_k^\perp\Big).$$

In particular we can write

$$\Big(\sum_{n\geq 0} h_n\Big)\Big(\sum_{k\geq 0}(-1)^k e_k^\perp\Big)s_r\Big|_m = \Big(\sum_{n\geq 0} h_n\Big)\big(s_r - s_{r-1}\big)\Big|_m = s_{m-r}s_r - s_{m-r+1}s_{r-1} \;=\; s_{m-r,r} \qquad 1.31$$

where '$\Big|_m$' is the operator which selects homogeneous terms of degree m. This given, the following is a rewriting of Thibon's proof of I.5 and I.7 in the present notation. We will start with the coefficient

$$C_k(d) \;=\; \big\langle s_{dd}[X_1]s_{dd}[X_2]\cdots s_{dd}[X_k]\,,\, s_{2d}[X_1X_2\cdots X_k]\big\rangle$$

where the left argument is a k-fold product. Using (1) we can write

$$C_k(d) = \big\langle \Omega[X_1]\Omega^\perp[-X_1]s_d[X_1]\Omega[X_2]\Omega^\perp[-X_2]s_d[X_2]\cdots\Omega[X_k]\Omega^\perp[-X_k]s_d[X_k]\,,\, s_{2d}[X_1X_2\cdots X_k]\big\rangle$$

where the role of '$\Big|_m$' is played by the scalar product with $s_{2d}[X_1X_2\cdots X_k]$. Thus we also have

$$\begin{aligned}
C_k(d) &= \big\langle s_d[X_1]s_d[X_2]\cdots s_d[X_k], \Omega[-X_1]\Omega^\perp[X_1]\Omega[-X_2]\Omega^\perp[X_2]\cdots\Omega[-X_k]\Omega^\perp[X_k]s_{2d}[X_1X_2\cdots X_k]\big\rangle\\
&= \big\langle s_d[X_1]s_d[X_2]\cdots s_d[X_k], \Omega[-X_1]\Omega[-X_2]\cdots\Omega[-X_k]]s_{2d}[(X_1+1)(X_2+1)\cdots(X_k+1)]\big\rangle\\
&= \big\langle \Omega[u_1^2X_1]\Omega[u_2^2X_2]\cdots\Omega[u_k^2X_k], \Omega[-X_1]\Omega[-X_2]\cdots\Omega[-X_k]]s_{2d}[(X_1+1)(X_2+1)\cdots(X_k+1)]\big\rangle\Big|_{u_1^{2d}u_2^{2d}\cdots u_k^{2d}}\\
&= \Omega[-u_1^2]\Omega[-u_2^2]\cdots\Omega[-u_k^2]]s_{2d}[({u_1}^2+1)({u_2}^2+1)\cdots({u_k}^2+1)]\big\rangle\Big|_{u_1^{2d}u_2^{2d}\cdots u_k^{2d}}\\
&= \Omega[-u_1^2]\Omega[-u_2^2]\cdots\Omega[-u_k^2]]s_{2d}[(u_1+1/u_1)(u_2+1/u_2)\cdots(u_k+1/u_k)]\big\rangle\Big|_{u_1^0u_2^0\cdots u_k^0}
\end{aligned}$$

the second to last step is due to the reproducing property of the Cauchy kernel.

Multiplying by t^{2d} and summing gives

$$\sum_{d\geq 0} t^{2d}\big\langle s_{dd} * s_{dd} * \cdots * s_{dd}\,,\, s_{2d}\big\rangle \;=\; \frac{(1-u_1^2)(1-u_2^2)\dots(1-u_k^2)}{\prod_{S\subseteq[1,k]}\big(1-t\prod_{i\in S}u_i\prod_{j\notin S}u_j\big)}\Bigg|_{u_1^0u_2^0\cdots u_k^0}$$

and this is I.5.

Now to get I.7 we start with

$$\begin{aligned}
C_{r_1,r_2,\dots,r_k}(m) &= \big\langle s_{m-r_1,r_1}[X_1]\cdots s_{m-r_k,r_k}[X_k], s_m[X_1X_2\cdots X_k]\big\rangle\\
&= \big\langle \Omega[v_1X_1]\Omega^\perp[-X_1/v_1]s_{r_1}[X_1]\cdots\Omega[v_kX_k]\Omega^\perp[-X_k/v_k]s_{r_k}[X_k], s_m[X_1X_2\cdots X_k]\big\rangle\Big|_{v_1^{m-r_1}\cdots v_k^{m-r_k}}\\
&= \big\langle s_{r_1}[X_1]\cdots s_{r_k}[X_k], \Omega[-X_1/v_1]\cdots\Omega[-X_k/v_k]s_m[(X_1+v_1)\cdots(X_k+v_k)]\big\rangle\Big|_{v_1^{m-r_1}\cdots v_k^{m-r_k}}\\
&= \big\langle \Omega[u_1X_1]\cdots\Omega[u_kX_k], \Omega[-X_1/v_1]\cdots\Omega[-X_k/v_k]s_m[(X_1+v_1)\cdots(X_k+v_k)]\big\rangle\Big|_{v_1^{m-r_1}\cdots v_k^{m-r_k}u_1^{r_1}\cdots u_k^{r_k}}\\
&= \Omega[-u_1/v_1]\cdots\Omega[-u_k/v_k]s_m[(u_1+v_1)\cdots(u_k+v_k)]\Big|_{v_1^{m-r_1}\cdots v_k^{m-r_k}u_1^{r_1}\cdots u_k^{r_k}}\\
&= \Omega[-u_1/v_1]\cdots\Omega[-u_k/v_k]s_m[(u_1/v_1+1)\cdots(u_k/v_k+1)]\Big|_{v_1^{-r_1}\cdots v_k^{-r_k}u_1^{r_1}\cdots u_k^{r_k}}\,.
\end{aligned}$$

Multiplying by $t^m(v_1/u_1)^{r_1}\cdots(v_k/u_k)^{r_k}$ and summing gives

$$\sum_{m\geq 0} t^m \sum_{r_1\geq 0}\cdots\sum_{r_k\geq 0} i_{r_1,\ldots,r_k}(m)(v_1/u_1)^{r_1}\cdots(v_k/u_k)^{r_k} \;=\; \prod_{i=1}^{k}(1-u_i/v_i)\Omega[t(u_1/v_1+1)\cdots(u_k/v_k+1)]\;.$$

Making the replacement $v_i/u_i \to y_i$ finally gives

$$\sum_{m\geq 0} t^m \sum_{r_1\geq 0}\cdots\sum_{r_k\geq 0} C_{r_1,\ldots,r_k}(m) y_1^{r_1}\cdots y_k^{r_k} \;=\; \frac{\prod_{i=1}^{k}(1-y_i)}{\prod_{S\subseteq[1,k]}\left(1-t\prod_{i\in S} y_i\right)}$$

as desired.

2. The explicit formula for $\mathbf{s_{d,d} * s_{d,d}}$

We begin this section with the computation of an auxiliary Kronecker coefficient.

Theorem 2.1

For all pairs $a \geq b$, $f \geq e$ with $a+b = f+e = 2d$ we have

$$\big\langle h_a h_b * s_{f,e} \,,\, s_{d,d} \big\rangle \;=\; \begin{cases} 1 + \lfloor \frac{b+e-d}{2} \rfloor & \text{if } b+f-d \geq 0 \\ 0 & \text{otherwise} \end{cases} \;. \qquad 2.1$$

Proof

Using the second equality in 1.25 we derive that

$$\big\langle h_a h_b * s_{f,e} \,,\, s_{d,d} \big\rangle \;=\; \sum_{\alpha \vdash b} \big\langle s_\alpha \times s_\alpha^\perp s_{f,e} \,,\, s_{d,d} \big\rangle \;=\; \sum_{\alpha \vdash b} \big\langle s_\alpha^\perp s_{f,e} \,,\, s_\alpha^\perp s_{d,d} \big\rangle \,. \qquad 2.2$$

Note that the only terms that contribute to 2.2 are those given by partitions $\alpha = (u,v)$ with

$$a) \quad u \leq f \wedge d \qquad\qquad b) \quad v \leq e \wedge d \qquad\qquad c) \quad u+v = b,$$

but since from our assumptions it follows that $f \geq d$, $e \leq d$ and $b \leq d$ it follows that these conditions reduce to

$$a) \quad v \leq e \qquad \text{and} \qquad c) \quad u+v = b. \qquad 2.3$$

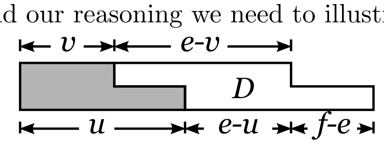
Moreover it is easily seen that for $\alpha = (u,v)$ we have

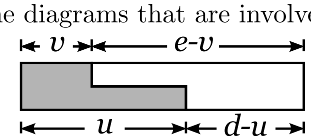
$$s_\alpha^\perp s_{d,d} \;=\; s_{d-v,d-u}.$$

We are thus reduced to the calculation of the scalar products

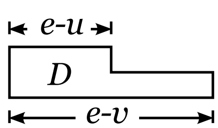
$$\big\langle s_{(e,f)/(u,v)} \,,\, s_{d-v,d-u} \big\rangle. \qquad 2.4$$

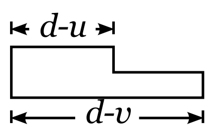
To better understand our reasoning we need to illustrate the diagrams that are involved in this calculation.





We have here on the left the shaded diagram of the partition (u,v) within the partition (e,f) and on the right the shaded diagram of (u,v) with the diagram of (d,d). Note that in applying the Littlewood-Richardson rule to expand the skew Schur function $s_{(e,f)/(u,v)}$ we shall necessarily obtain only diagrams which start with the partition $D = (e-v, e-u)$ (illustrated on the left below) then end with the diagram obtained by draping $f-e$ cells on the right of the last cell of the top row of D.





In order for the scalar product in 2.4 to be different from zero we must be able to drape these $f-e$ cells so as to obtain the diagram of $(d-v, d-u)$, (illustrated above on the right). That is we must place

$d-e$ cells on the second row of D. However, the maximum number of cells we can place on the top row of D is $e-v-(e-u)=u-v$. In conclusion we will be able to obtain $(d-v,d-u)$ if and only if

$$d-e \leq u-v\ .$$

Since $u=b-v$ this inequality is simply

$$d-e \leq b-v-v$$

or better

$$v \leq \frac{b+e-d}{2}.$$

This may be written as

$$a) \quad v \leq \lfloor \tfrac{b+e-d)}{2} \rfloor \qquad \text{or} \qquad b) \quad v \leq \lfloor \tfrac{e-(d-b)}{2} \rfloor\ .$$

Note however that if $b+e<d$ there is no way of producing the diagram of $(d-v,d-u)$ from D and the sum in the right hand side of 2.2 necessarily vanishes. On the other hand if $b+e\geq d$ then since b) is clearly stronger than 2.3 a) we derive that the number of terms that contribute to the right hand side of 2.3 is

$$1+\lfloor \tfrac{b+e-d)}{2} \rfloor.$$

This completes our proof.

As an immediate corollary we obtain that

Theorem 2.2

For all pairs $a\geq b$, $f\geq e$ with $a+b=f+e=2d$ we have

$$\big\langle s_{a,b} * s_{f,e}\ ,\ s_{d,d}\big\rangle \;=\; \begin{cases} 1 & \text{if } b+e-d\geq 0 \text{ and even} \\ 0 & \text{otherwise} \end{cases}\ . \qquad 2.5$$

Proof

Since $s_{a,b}=h_a h_b - h_{a-1}h_{b+1}$ we derive that

$$\big\langle s_{a,b} * s_{f,e}\ ,\ s_{d,d}\big\rangle \;=\; \big\langle h_a h_b * s_{f,e}\ ,\ s_{d,d}\big\rangle - \big\langle h_{a+1}h_{b-1} * s_{f,e}\ ,\ s_{d,d}\big\rangle\ . \qquad 2.6$$

Now note that if $b+e-d<0$ then $b-1+e-d<0$ as well and 2.1 yields that both terms on the right hand side of 2.6 necessarily vanish. In case $b+e-d=2k\geq 0$ then 2.1 gives

$$1+\lfloor \tfrac{b+e-d)}{2} \rfloor \;=\; 1+k \qquad \text{and} \qquad 1+\lfloor \tfrac{b-1+e-d)}{2} \rfloor \;=\; k$$

and 2.1 gives that

$$\big\langle s_{a,b} * s_{f,e}\ ,\ s_{d,d}\big\rangle \;=\; 1+k\ -\ k \;=\; 1.$$

On the other hand if $b+e-d=2k+1\geq 1$ then

$$1+\lfloor \tfrac{b+e-d)}{2} \rfloor \;=\; 1+k \qquad \text{and} \qquad 1+\lfloor \tfrac{b-1+e-d)}{2} \rfloor \;=\; 1+k$$

and 2.1 gives that

$$\big\langle s_{a,b} * s_{f,e}\ ,\ s_{d,d}\big\rangle \;=\; 1+k\ -\ 1-k \;=\; 0.$$

This completes our proof.

We are finally in a position to prove our desired Schur function expansion of $s_{dd} * s_{dd}$.

Theorem 2.3

$$s_{dd} * s_{dd} \;=\; \sum_{\lambda \vdash 2d} \chi(\lambda \in EO_4) s_\lambda \tag{2.7}$$

where EO_4 denotes the set of partitions of length 4 whose parts are ≥ 0 and all even or all odd.

Proof

From Proposition 1.7 it follows that the Schur function expansion of the Kronecker product $s_{dd} * s_{dd}$ involves Schur functions indexed by partitions with at most four parts, thus we need only to show that when s_λ occurs in the expansion of $s_{dd} * s_{dd}$

1) if λ has only one positive part, then $\lambda = 2d$ and $\langle s_{2d}, s_{dd} * s_{dd} \rangle = 1$,
2) if λ has only two positive parts, then all those parts are even and $\langle s_\lambda, s_{dd} * s_{dd} \rangle = 1$,
3) if λ has only three positive parts, then all its parts are even and $\langle s_\lambda, s_{dd} * s_{dd} \rangle = 1$,
4) if λ has four positive parts then its parts are all even or all odd and $\langle s_\lambda, s_{dd} * s_{dd} \rangle = 1$.

Now 1) is entirely trivial since

$$\langle s_{2d} \,,\, s_{dd} * s_{dd} \rangle \;=\; \langle s_{2d} * s_{dd} \,,\, s_{dd} \rangle \;=\; \langle s_{dd} \,,\, s_{dd} \rangle \;=\; 1.$$

Next note that Theorem 2.2 with $\lambda = (a,b)$ $(f,e) = (d,d)$ gives that the scalar product

$$\langle s_{ab} * s_{dd} \,,\, s_{dd} \rangle \;=\; \begin{cases} 1 & \text{if } b \text{ is even} \\ 0 & \text{if } b \text{ is odd} \end{cases}$$

Thus 2) must hold true since

$$\langle s_{ab} \,,\, s_{dd} * s_{dd} \rangle \;=\; \langle s_{ab} * s_{dd} \,,\, s_{dd} \rangle.$$

For the next two cases we will proceed by induction on d. In fact, note that a simple calculation gives that

$$s_{11} * s_{11} \;=\; s_2 \,, \qquad s_{22} * s_{22} \;=\; s_4 + s_{22} + s_{1111}$$

Thus the assertion is clearly true for $d \leq 2$. So we will assume it to be true inductively up to $d - 1 \geq 2$.

To prove 3) we use 1.28 b) and write a Schur function indexed by a partition with exactly three > 0 parts in the form

$$s_{\mu+1^3} \;=\; \sum_{i \geq 0} (-1)^i e_{3+i} h_i^\perp s_\mu$$

with μ a partition with at most three parts. Thus

$$\begin{aligned}
\langle s_{\mu+1^3} \,,\, s_{dd} * s_{dd} \rangle \;&=\; \sum_{i \geq 0} (-1)^i \langle e_{3+i} h_i^\perp s_\mu \,,\, s_{dd} * s_{dd} \rangle \\
&=\; \sum_{i \geq 0} (-1)^i \langle h_i^\perp s_\mu \,,\, e_{3+i}^\perp s_{dd} * s_{dd} \rangle \\
\text{(by Proposition 1.7)} \;&=\; \sum_{i=0}^{1} (-1)^i \langle h_i^\perp s_\mu \,,\, e_{3+i}^\perp s_{dd} * s_{dd} \rangle \qquad\qquad 2.8 \\
&=\; \langle e_3 s_\mu \,,\, s_{dd} * s_{dd} \rangle \;-\; \langle e_4 h_1^\perp s_\mu \,,\, s_{dd} * s_{dd} \rangle \\
\text{(by 1.27 b))} \;&=\; \sum_{\alpha \vdash 3} \langle s_\mu \,,\, (s_\alpha^\perp s_{dd}) * (s_{\alpha'}^\perp s_{dd}) \rangle \;-\; \sum_{\beta \vdash 4} \langle h_1^\perp s_\mu \,,\, (s_\beta^\perp s_{dd}) * (s_{\beta'}^\perp s_{dd}) \rangle \,.
\end{aligned}$$

Since the only $\alpha \vdash 3$ such that both $s_\alpha^\perp s_{dd}$ and $s_{\alpha'}^\perp s_{dd}$ are different from zero is $\alpha = (2,1)$ and likewise the only $\beta \vdash 4$ such that both $s_\beta^\perp s_{dd}$ and $(s_{\beta'}^\perp s_{dd})$ are different from zero is $\beta = (2,2)$, it follows from 2.8 that

$$\begin{aligned} \big\langle s_{\mu+1^3}\,,\, s_{dd} * s_{dd}\big\rangle &= \big\langle s_\mu\,,\,(s_{21}^\perp s_{dd}) * (s_{21}^\perp s_{dd})\big\rangle - \big\langle h_1^\perp s_\mu\,,\,(s_{22}^\perp s_{dd}) * (s_{22}^\perp s_{dd}\big) \\ &= \big\langle s_\mu\,,\,(e_1^\perp s_{d-1,d-1}) * (e_1^\perp s_{d-1,d-1})\big\rangle - \big\langle h_1^\perp s_\mu\,,\, s_{d-2,d-2} * s_{d-2d-2}\big) \\ &= \big\langle e_1 s_\mu\,,\, s_{d-1,d-1} * s_{d-1,d-1}\big\rangle - \big\langle h_1^\perp s_\mu\,,\, s_{d-2,d-2} * s_{d-2d-2}\big)\,. \end{aligned} \tag{2.9}$$

This given, note that μ can only be of the following four types

$$(a,a,a)\,, \quad (a,a,b)\,, \quad (a,b,b)\,, \quad (a,b,c)\,, \tag{2.10}$$

with $a > b > c$. Now from 2.9 we derive that

$$\begin{aligned} \big\langle s_{(a+1,a+1,a+1)}\,,\, s_{dd} * s_{dd}\big\rangle &= \big\langle e_1 s_{aaa}\,,\, s_{d-1,d-1} * s_{d-1,d-1}\big\rangle - \big\langle h_1^\perp s_{aaa}\,,\, s_{d-2,d-2} * s_{d-2d-2}\big\rangle \\ &= \big\langle (s_{aaa1} + s_{a+1,a,a})\,,\, s_{d-1,d-1} * s_{d-1,d-1}\big\rangle - \big\langle s_{a,a,a-1}\,,\, s_{d-2,d-2} * s_{d-2d-2}\big\rangle \\ \text{(by induction)} &= \big\langle s_{aaa1}\,,\, s_{d-1,d-1} * s_{d-1,d-1}\big\rangle \end{aligned}$$

and the inductive hypothesis forces a to be odd and $(a+1, a+1, a+1)$ to have all even parts. In the second case of 2.10, 2.9 gives

$$\begin{aligned} \big\langle s_{(a+1,a+1,b+1)}\,,\, s_{dd} * s_{dd}\big\rangle &= \big\langle e_1 s_{aab}\,,\, s_{d-1,d-1} * s_{d-1,d-1}\big\rangle - \big\langle h_1^\perp s_{aab}\,,\, s_{d-2,d-2} * s_{d-2d-2}\big\rangle \\ &= \big\langle (s_{aab1} + s_{a,a,b+1} + s_{a+1,a,b})\,,\, s_{d-1,d-1} * s_{d-1,d-1}\big\rangle \\ &\qquad - \big\langle (s_{a,a,b-1} + s_{a,a-1,b})\,,\, s_{d-2,d-2} * s_{d-2d-2}\big\rangle \\ \text{(by induction)} &= \big\langle (s_{aab1} + s_{a,a,b+1})\,,\, s_{d-1,d-1} * s_{d-1,d-1}\big\rangle - \big\langle s_{a,a,b-1}\,,\, s_{d-2,d-2} * s_{d-2d-2}\big\rangle\,. \end{aligned} \tag{2.11}$$

If a and b are of different parity the inductive hypothesis reduces this to

$$\big\langle s_{(a+1,a+1,b+1)}\,,\, s_{dd} * s_{dd}\big\rangle = \big\langle s_{a,a,b+1}\,,\, s_{d-1,d-1} * s_{d-1,d-1}\big\rangle - \big\langle s_{a,a,b-1}\,,\, s_{d-2,d-2} * s_{d-2d-2}\big\rangle$$

but again the inductive hypothesis forces both $a, b+1$ to be even as well as

$$\big\langle s_{(a+1,a+1,b+1)}\,,\, s_{dd} * s_{dd}\big\rangle = 1 - 1 = 0.$$

This leaves as the only possibility that a and b have the same parity. But then the inductive hypothesis reduces 2.11 to

$$\big\langle s_{(a+1,a+1,b+1)}\,,\, s_{dd} * s_{dd}\big\rangle = \big\langle s_{aab1}\,,\, s_{dd} * s_{dd}\big\rangle$$

and the inductive hypothesis forces a, b to be both odd, thus $(a+1, a+1, b+1)$ to be all even and

$$\big\langle s_{(a+1,a+1,b+1)}\,,\, s_{dd} * s_{dd}\big\rangle = 1.$$

In the third case 2.11 gives

$$\begin{aligned} \big\langle s_{(a+1,b+1,b+1)}\,,\, s_{dd} * s_{dd}\big\rangle &= \big\langle (s_{abb1} + s_{a,b+1,b} + s_{a+1,b,b})\,,\, s_{d-1,d-1} * s_{d-1,d-1}\big\rangle \\ &\qquad - \big\langle (s_{a,b,b-1} + s_{a-1,b,b})\,,\, s_{d-2,d-2} * s_{d-2d-2}\big\rangle \\ \text{(by induction)} &= \big\langle (s_{abb1} + s_{a+1,b,b})\,,\, s_{d-1,d-1} * s_{d-1,d-1}\big\rangle - \big\langle s_{a-1,b,b}\,,\, s_{d-2,d-2} * s_{d-2d-2}\big\rangle\,. \end{aligned} \tag{2.12}$$

Now if a, b have different parity this reduces to

$$\left\langle s_{(a+1,b+1,b+1)}\,,\, s_{dd} * s_{dd} \right\rangle \;=\; \left\langle s_{a+1,b,b}\,,\, s_{d-1,d-1} * s_{d-1,d-1} \right\rangle - \left\langle s_{a-1,b,b}\,,\, s_{d-2,d-2} * s_{d-2d-2} \right\rangle$$

and the inductive hypothesis yields that both terms vanish if a is even and b is odd. On the other hand if a is odd and b is even the inductive hypothesis gives

$$\left\langle s_{(a+1,b+1,b+1)}\,,\, s_{dd} * s_{dd} \right\rangle \;=\; 1 - 1 \;=\; 0\,.$$

If a and b have the same parity 2.12 reduces to

$$\left\langle s_{(a+1,b+1,b+1)}\,,\, s_{dd} * s_{dd} \right\rangle \;=\; \left\langle s_{abb1}\,,\, s_{d-1,d-1} * s_{d-1,d-1} \right\rangle$$

and the inductive hypothesis forces a and b to be both odd thus $(a+1, b+1, b+1)$ to be even and

$$\left\langle s_{(a+1,b+1,b+1)}\,,\, s_{dd} * s_{dd} \right\rangle \;=\; 1\,.$$

Finally, for the last case of 2.10, 2.9 gives

$$\begin{aligned} \left\langle s_{(a+1,b+1,c+1)}\,,\, s_{dd} * s_{dd} \right\rangle \;=\; & \left\langle (s_{abc1} + s_{a,b,c+1} + s_{a,b+1,c} + s_{a+1,b,c})\,,\, s_{d-1,d-1} * s_{d-1,d-1} \right\rangle \\ & - \left\langle (s_{a,b,c-1} + s_{a,b-1,c} + s_{a-1,b,c})\,,\, s_{d-2,d-2} * s_{d-2d-2} \right\rangle . \end{aligned} \qquad 2.13$$

If a, b and $c+1$ have the same parity, the inductive hypothesis reduces 2.13 to

$$\left\langle s_{(a+1,b+1,c+1)}\,,\, s_{dd} * s_{dd} \right\rangle = \left\langle s_{a,b,c+1}\,,\, s_{d-1,d-1} * s_{d-1,d-1} \right\rangle - \left\langle s_{a,b,c-1}\,,\, s_{d-2,d-2} * s_{d-2d-2} \right\rangle = 1 - 1 = 0\,.$$

If $a, b+1$ and c have the same parity, the inductive hypothesis reduces 2.13 to

$$\left\langle s_{(a+1,b+1,c+1)}\,,\, s_{dd} * s_{dd} \right\rangle = \left\langle s_{a,b+1,c}\,,\, s_{d-1,d-1} * s_{d-1,d-1} \right\rangle - \left\langle s_{a,b-1,c}\,,\, s_{d-2,d-2} * s_{d-2d-2} \right\rangle = 1 - 1 = 0\,.$$

If $a+1, b$ and c have the same parity, the inductive hypothesis reduces 2.13 to

$$\left\langle s_{(a+1,b+1,c+1)}\,,\, s_{dd} * s_{dd} \right\rangle = \left\langle s_{a+1,b,c}\,,\, s_{d-1,d-1} * s_{d-1,d-1} \right\rangle - \left\langle s_{a-1,b,c}\,,\, s_{d-2,d-2} * s_{d-2d-2} \right\rangle = 1 - 1 = 0\,.$$

If a, b and c have the same parity, the inductive hypothesis reduces 2.13 to

$$\left\langle s_{(a+1,b+1,c+1)}\,,\, s_{dd} * s_{dd} \right\rangle \;=\; \left\langle s_{abc1}\,,\, s_{dd} * s_{dd} \right\rangle \;=\; 1$$

with (a, b, c) all odd and thus $(a+1, b+1, c+1)$ all even.

This completes the proof of 3). To prove 4) we note that 1.28 b) yields that we may write s_λ in the form

$$s_\lambda \;=\; \sum_{i\geq 0} (-1)^i e_{4+i} h_i^{\perp} s_\mu$$

with μ a partition with no more than four parts. Thus

$$\begin{aligned} \big\langle s_\lambda\,,\, s_{dd} * s_{dd}\big\rangle &= \sum_{i\geq 0}(-1)^i\big\langle e_{4+i}h_i^\perp s_\mu\,,\, s_{dd} * s_{dd}\big\rangle \\ \text{(by Proposition 1.7)} &= \big\langle e_4 s_\mu\,,\, s_{dd} * s_{dd}\big\rangle \\ \text{(by 1.27 b))} &= \sum_{\alpha\vdash 4}\big\langle s_\mu\,,\, (s_\alpha^\perp s_{dd}) * (s_{\alpha'}^\perp s_{dd})\big\rangle\ . \end{aligned} \qquad 2.14$$

But now again the only $\alpha \vdash 4$ for which both factors $s_\alpha^\perp s_{dd}$ and $s_{\alpha'}^\perp s_{dd}$ do not vanish is $\alpha = (2,2)$. This reduces 2.14 to

$$\big\langle s_\lambda\,,\, s_{dd} * s_{dd}\big\rangle \;=\; \big\langle s_\mu\,,\, (s_{22}^\perp s_{dd}) * (s_{22}^\perp s_{dd})\big\rangle \;=\; \big\langle s_\mu\,,\, s_{d-2,d-2} * s_{d-2,d-2}\big\rangle$$

and the inductive hypothesis yields that this vanishes unless all parts of μ are even or equivalently all parts of λ are odd and in this case

$$\big\langle s_\lambda\,,\, s_{dd} * s_{dd}\big\rangle \;=\; 1.$$

This completes the proof of the Theorem.

3. Kronecker coefficients for two part partitions

The identity of Theorem 2.2 namely,

$$\big\langle s_{a,b} * s_{f,e}\,,\, s_{d,d}\big\rangle \;=\; \begin{cases} 1 & \text{if } b+e-d\geq 0 \text{ and even} \\ 0 & \text{otherwise} \end{cases} ,$$

has a more general form that may be stated as follows.

Theorem 3.1

For given integers $r_1 \geq r_2 \geq r_3 \geq 0$ set

$$\Phi_{r_1,r_2,r_3}(q) \;=\; \sum_{m\geq 2max(r_1,r_2,r_3)} q^m\big\langle s_{m-r_1,r_1} * s_{m-r_2,r_2} * s_{m-r_3,r_3}, s_m\big\rangle\ . \qquad 3.1$$

Provided $r_2 + r_3 \geq r_1$ we have

$$\Phi_{r_1,r_2,r_3}(q) \;=\; \frac{q^{2r_1}}{(1-q)(1-q^2)}\begin{cases} 1 - q^{r_2+r_3-r_1+2} & \text{if } r_1+r_2+r_3 \text{ is even} \\[2ex] q - q^{r_2+r_3-r_1+2} & \text{if } r_1+r_2+r_3 \text{ is odd} \end{cases} , \qquad 3.2$$

otherwise $\Phi_{r_1,r_2,r_3}(q)$ vanishes identically.

The developments that led to a proof of this identity and the mathematics that resulted from it may be as interesting as the result itself. The discovery that the generating function

$$G_k(y_1, y_2, \ldots, y_k; q) = \sum_{m\geq 0} q^m \sum_{r_1\geq 0}\sum_{r_2\geq 0}\cdots\sum_{r_k\geq 0} y_1^{r_1} y_2^{r_2}\cdots y_k^{r_k}\big\langle s_{m-r_1,r_1} * s_{m-r_2,r_2} * \cdots * s_{m-r_k,r_k}\,,\, s_m\big\rangle \qquad 3.3$$

has the simple form

$$G_k(y_1, y_2, \ldots, y_k; q) = \frac{\prod_{i=1}^k (1-y_i)}{\prod_{S\subseteq[1,k]}(1-q\prod_{i\in S} y_i)} \;=\; \Omega\big[q(1+y_1)(1+y_2)\cdots(1+y_k)\big]\prod_{i=1}^k(1-y_i) \qquad 3.4$$

suggested that we should be able to extract from it explicit expressions for the generating functions

$$\Phi_{r_1,r_2,\ldots,r_k}(q) \;=\; \sum_{m\geq 2max(r_1,r_2,\ldots,r_k)} q^m\big\langle s_{m-r_1,r_1} * s_{m-r_2,r_2} * \cdots * s_{m-r_k,r_k}, s_m\big\rangle \qquad 3.5$$

As a starting step we should be able to extract from 3.3 the terms where all the Schur functions have partition indexing. That means getting the sub-series where $m \geq 2max(r_1, r_2, \ldots, r_k)$. Now we can do this again by resorting to a trick from MacMahon partition analysis. More precisely we have the following recipes.

Proposition 3.1

Let

$$G(y_1, y_2, \ldots, y_k; q) \;=\; \sum_{m\geq 0} q^m \sum_{r_1\geq 0}\sum_{r_2\geq 0}\cdots\sum_{r_k\geq 0} y_1^{r_1} y_2^{r_2}\cdots y_k^{r_k} c_{r_1,r_2,\ldots,r_k}(m)\;, \qquad 3.6$$

then

$$\sum_{m\geq 0} q^m \sum_{r_1=0}^{\lfloor\frac{m}{2}\rfloor}\sum_{r_2=0}^{\lfloor\frac{m}{2}\rfloor}\cdots\sum_{r_k=0}^{\lfloor\frac{m}{2}\rfloor} y_1^{r_1}y_2^{r_2}\cdots y_k^{r_k} c_{r_1,r_2,\ldots,r_k}(m) = G\big(\tfrac{y_1}{a_1^2}, \tfrac{y_2}{a_2^2}, \ldots, \tfrac{y_k}{a_k^2}; qa_1a_2\cdots a_n\big)\Big|_{a_1^{\geq}a_2^{\geq}\cdots a_k^{\geq}} \qquad 3.7$$

where the symbol '$a_i^{\geq}$' represents the operator that selects all the terms where a_i appears to a non-negative power and then setting all $a_i = 1$. In the same vein, we also have

$$\sum_{r_1\geq r_2\geq\cdots\geq r_k}\;\sum_{m\geq 2r_1} q^m y_1^{r_1}y_2^{r_2}\cdots y_k^{r_k} c_{r_1,r_2,\ldots,r_k}(m) \;=\; G\big(\tfrac{y_1a_1}{a_0^2}, \tfrac{y_2a_2}{a_1}, \ldots, \tfrac{y_k}{a_{k-1}}; qa_0\big)\Big|_{a_0^{\geq}a_1^{\geq}\cdots a_{k-1}^{\geq}} \qquad 3.8$$

and of course in this manner we can also derive formula 1.1 that is

$$W_k(q) \;=\; \sum_{d\geq 0} q^{2d} c_{d,d,\ldots,d}(2d) \;=\; G\big(\tfrac{1}{a_1^2}, \tfrac{1}{a_2^2}, \ldots, \tfrac{1}{a_k^2}; q\,a_1a_2\cdots a_n\big)\Big|_{a_1^0a_2^0\cdots a_k^0}\,. \qquad 3.9$$

Proof

Note that from 3.6 it follows that

$$\begin{aligned} &G\big(\frac{y_1}{a_1^2}, \frac{y_2}{a_2^2}, \ldots, \frac{y_k}{a_k^2}; qa_1a_2\cdots a_n\big)\\ &\quad= \sum_{m\geq 0} q^m \sum_{r_1\geq 0}\sum_{r_2\geq 0}\cdots\sum_{r_k\geq 0} y_1^{r_1}y_2^{r_2}\cdots y_k^{r_k} c_{r_1,r_2,\ldots,r_k}(m)\, a_1^{m-2r_1}a_2^{m-2r_2}\cdots a_k^{m-2r_k}\end{aligned}$$

and the operator '$\Big|_{a_1^{\geq} a_2^{\geq} \cdots a_k^{\geq}}$' will carry out the desired selection. The same can be seen from the following identity

$$G\big(\tfrac{y_1 a_1}{a_0^2}, \tfrac{y_2 a_2}{a_1}, \ldots, \tfrac{y_k}{a_{k-1}}; q a_0\big) = \sum_{m\geq 0} q^m \sum_{r_1\geq 0}\sum_{r_2\geq 0}\cdots\sum_{r_k\geq 0} y_1^{r_1} y_2^{r_2}\cdots y_k^{r_k} c_{r_1,r_2,\ldots,r_k}(m)\, a_0^{m-2r_1} a_2^{r_1-r_2}\cdots a_{k-1}^{r_{k-1}-r_k}\ .$$

Likewise we can easily see that 3.9 can be obtained,by the constant term operator '$\Big|_{a_1^0 a_2^0 \cdots a_k^0}$,' from the identity

$$G\big(\tfrac{1}{a_1^2}, \tfrac{1}{a_2^2}, \ldots, \tfrac{1}{a_k^2}; q\, a_1 a_2\cdots a_n\big) = \sum_{m\geq 0} q^m \sum_{r_1\geq 0}\sum_{r_2\geq 0}\cdots\sum_{r_k\geq 0} c_{r_1,r_2,\ldots,r_k}(m)\, a_1^{m-2r_1} a_2^{m-2r_2}\cdots a_k^{m-2r_k}\ .$$

Armed with these tools the following two results were obtained in a matter of seconds from the MAPLE package of G. Xin (called *ELL2.mpl*)

Theorem 3.2

If we set

$$F_3(y_1,y_2,y_3;q) = \sum_{r_1\geq 0}\sum_{r_2\geq 0}\sum_{r_3\geq 0}\sum_{m\geq 2max(r_1,r_2,r_3)} q^m y_1^{r_1} y_2^{r_2} y_3^{r_3} \big\langle s_{m-r_1,r_1} * s_{m-r_2,r_2} * s_{m-r_3,r_3}, s_m\big\rangle \qquad 3.10$$

and

$$G_3(y_1,y_2,y_3;q) = \sum_{r_3\geq 0}\sum_{r_2\geq r_3}\sum_{r_1\geq r_2}\sum_{m\geq 2r_1} q^m y_1^{r_1} y_2^{r_2} y_3^{r_3} \big\langle s_{m-r_1,r_1} * s_{m-r_2,r_2} * s_{m-r_3,r_3}, s_m\big\rangle\ , \qquad 3.11$$

then

$$F_3(y_1,y_2,y_3;q) = \frac{1+q^3y_1y_2y_3}{(1-q)(1-q^2y_1y_2)(1-q^2y_1y_3)(1-q^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)} \qquad 3.12$$

and

$$G_3(y_1,y_2,y_3;q) = \frac{1}{(1-q)(1-q^2y_1y_2)(1-q^3y_1y_2y_3)(1-q^4y_1^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)}\ . \qquad 3.13$$

Either of these two identities yield as Corollary

A Proof of Theorem 3.1

The easiest proof of 3.2 is obtained from 3.11. We start with deriving from 3.13 that

$$\begin{aligned} G_3(y_1,y_2,y_3;q) &= \frac{1}{(1-q)}\sum_{a\geq 0}\sum_{b\geq 0}\sum_{c\geq 0}\sum_{d\geq 0} q^{2a+3b+4c+4d}(y_1y_2)^a(y_1y_2y_3)^b(y_1^2y_2y_3)^c(y_1^2y_2^2y_3^2)^d \\ &= \frac{1}{(1-q)}\sum_{a\geq 0}\sum_{b\geq 0}\sum_{c\geq 0}\sum_{d\geq 0} q^{2a+3b+4c+4d} y_1^{a+b+2c+2d} y_2^{a+b+c+2d} y_3^{b+c+2d}\ . \end{aligned} \qquad 3.14$$

Since by definition

$$\Phi_{r_1,r_2,r_3}(q) \;=\; G_3(y_1,y_2,y_3;q)\Big|_{y_1^{r_1}y_2^{r_2}y_3^{r_3}}$$

we need to solve for a, b, c, d in the system of equations

$$\begin{aligned} a+b+2c &= r_1-2d \\ a+b+c &= r_2-2d \\ b+c &= r_3-2d\ . \end{aligned} \qquad 3.15$$

Now subtracting the second from the first and the third from the second gives

$$c = r_1 - r_2 \quad \text{and} \quad a = r_2 - r_3 \qquad 3.16$$

and consequently

$$b \;=\; r_3 - 2d - (r_1 - r_2) \;=\; r_2 + r_3 - r_1 - 2d\ . \qquad 3.17$$

This shows that there are no solutions if $r_2 + r_3 < r_1$ proving the last assertion of the Theorem. Moreover, since $b \geq 0$ we must also require that $2d \leq r_2 + r_3 - r_1$. Thus, when $r_2 + r_3 \geq r_1$ from 3.14 we derive that

$$\begin{aligned} \Phi_{r_1,r_2,r_3}(q) &= \frac{1}{(1-q)} \sum_{d=0}^{(r_2+r_3-r_1)/2} q^{2(r_2-r_3)+3(r_2+r_3-r_1-2d)+4(r_1-r_2)+4d} \\ &= \frac{1}{(1-q)} \sum_{d=0}^{(r_2+r_3-r_1)/2} q^{2r_2-2r_3+3r_2+3r_3-3r_1-6d)+4r_1-4r_2+4d} \\ &= \frac{q^{2r_1}}{(1-q)} \begin{cases} \sum_{d=0}^{k} q^{2k-2d} & \text{if } r_2+r_3-r_1 = 2k \\ \sum_{d=0}^{k} q^{2k+1-2d} & \text{if } r_2+r_3-r_1 = 2k+1 \end{cases} \\ &= \frac{q^{2r_1}}{(1-q)(1-q^2)} \begin{cases} 1-q^{2k+2} & \text{if } r_2+r_3-r_1 = 2k \\ q(1-q^{2k+2}) & \text{if } r_2+r_3-r_1 = 2k+1 \end{cases} . \end{aligned}$$

This proves 3.2.

If we believe in computers and more significantly we believe in the validity of the MAPLE software that yielded 3.13, then Theorem 3.1 is well and done. But for some of us there is something purely emotional that makes it unsatisfactory to have only a computer proof of a Mathematical result. Thus for the benefit of the reader who prefers human proofs, we will indulge in all the manipulations that are necessary to derive both 3.12 and 3.13 almost entirely by hand.

We will begin by deriving 3.12 from 3.11. To this end note that the proof of Theorem 3.1 yields a significant byproduct

Proposition 3.2

If we set

$$G_3^{(1)}(y_1,y_2,y_3;q) = \sum_{r_3\geq 0}\sum_{r_2\geq r_3}\sum_{r_1>r_2}\sum_{m\geq 2r_1} q^m y_1^{r_1} y_2^{r_2} y_3^{r_3} \big\langle s_{m-r_1,r_1} * s_{m-r_2,r_2} * s_{m-r_3,r_3}, s_m \big\rangle \qquad 3.18$$

$$G_3^{(2)}(y_1,y_2,y_3;q) = \sum_{r_3\geq 0}\sum_{r_2>r_3}\sum_{r_1\geq r_2}\sum_{m\geq 2r_1} q^m y_1^{r_1} y_2^{r_2} y_3^{r_3} \big\langle s_{m-r_1,r_1} * s_{m-r_2,r_2} * s_{m-r_3,r_3}, s_m \big\rangle \qquad 3.19$$

$$G_3^{(1,2)}(y_1,y_2,y_3;q) = \sum_{r_3\geq 0}\sum_{r_2>r_3}\sum_{r_1>r_2}\sum_{m\geq 2r_1} q^m y_1^{r_1}y_2^{r_2}y_3^{r_3}\big\langle s_{m-r_1,r_1} * s_{m-r_2,r_2} * s_{m-r_3,r_3}, s_m\big\rangle \qquad 3.20$$

then

$$G_3^{(1)}(y_1,y_2,y_3;q) = \frac{q^4y_1^2y_2y_3}{(1-q)(1-q^2y_1y_2)(1-q^3y_1y_2y_3)(1-q^4y_1^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)} \qquad 3.21$$

$$G_3^{(2)}(y_1,y_2,y_3;q) = \frac{q^2y_1y_2}{(1-q)(1-q^2y_1y_2)(1-q^3y_1y_2y_3)(1-q^4y_1^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)} \qquad 3.22$$

$$G_3^{(1,2)}(y_1,y_2,y_3;q) = \frac{q^2y_1y_2\,\times\,q^4y_1^2y_2y_3}{(1-q)(1-q^2y_1y_2)(1-q^3y_1y_2y_3)(1-q^4y_1^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)}\,. \qquad 3.23$$

Proof

We saw in 3.16 that to assure that $r_1 > r_2$ we need only take $c \geq 1$ in 3.12. That proves 3.21. Similarly to assure $r_2 > r_3$ we need only take $a \geq 1$ in 3.12. That proves 3.22. Finally, we see that taking both $a \geq 1$ and $c \geq 1$ in 3.14 assures both inequalities $r_1 > r_2 > r_3$. This proves 3.23 and completes our argument.

Now we have exactly what we need to show

Proposition 3.3

The validity of 3.13 forces the validity of 3.12

Proof

The sum in 3.10 is over all lattice triplets r_1, r_2, r_3 that lie in the positive octant. Now it is well known in combinatorics that the lattice k-tuples $r_1, r_2, \ldots, r_k$ in the positive octant of k-dimensional space, decomposes into $k!$ disjoint simplicial cones indexed by permutations $\sigma \in S_k$. More precisely the cone C_σ consists of the collection of triplets

$$C_\sigma \;=\; \big\{(r_1,r_2,\ldots,r_k)\,:\, r_{\sigma_i}\geq r_{\sigma_{i+1}} \text{ with } \; r_{\sigma_i} > r_{\sigma_{i+1}} \text{ when } \; \sigma_i>\sigma_{i+1}\;\big\}\,.$$

In particular the lattice triplets r_1, r_2, r_3 that lie in the positive octant will decompose into the 6 cones

$$C_{123} = \{(r_1,r_2,r_3)\,:\, r_1\geq r_2\geq r_3\},\;\; C_{132} = \{(r_1,r_2,r_3)\,:\, r_1\geq r_3> r_2\},$$

$$C_{213} = \{(r_1,r_2,r_3)\,:\, r_2> r_1\geq r_3\},\;\; C_{231} = \{(r_1,r_2,r_3)\,:\, r_2\geq r_3> r_1\},$$

$$C_{312} = \{(r_1,r_2,r_3)\,:\, r_3> r_1\geq r_2\},\;\; C_{321} = \{(r_1,r_2,r_3)\,:\, r_3> r_2> r_1\}.$$

This given, the summation in 3.10 can be accordingly decomposed, and we obtain from that

$$F_3(y_1,y_2,y_3;q) = \sum_{\sigma\in S_3} F_3^{(\sigma)}(y_1,y_2,y_3;q) \qquad 3.24$$

where for convenience we have set

$$F_3^{(\sigma)}(y_1,y_2,y_3;q) \;=\; \sum_{(r_1,r_2,r_3)\in C_\sigma}\;\sum_{m\geq 2max(r_1,r_2,r_3)} q^m y_1^{r_1}y_2^{r_2}y_3^{r_3}\;\big\langle s_{m-r_1,r_1} * s_{m-r_2,r_2} * s_{m-r_3,r_3}, s_m\big\rangle\,.$$

Now it immediately follows from Proposition 3.2 that

$$F_3^{(123)}(y_1,y_2,y_3;q)=G_3(y_1,y_2,y_3;q)\,,\ \ F_3^{(132)}(y_1,y_2,y_3;q)=G_3^{(2)}(y_1,y_3,y_2;q)\ ,$$

$$F_3^{(213)}(y_1,y_2,y_3;q)=G_3^{(1)}(y_2,y_1,y_3;q)\,,\ \ F_3^{(231)}(y_1,y_2,y_3;q)=G_3^{(2)}(y_2,y_3,y_1;q)\ ,$$

$$F_3^{(312)}(y_1,y_2,y_3;q)=G_3^{(1)}(y_3,y_1,y_2;q)\,,\ \ F_3^{(321)}(y_1,y_2,y_3;q)=G_3^{(1,2)}(y_3,y_2,y_1;q)\ .$$

Carrying out these substitutions gives

$$\begin{aligned}
F_3^{(123)} &= \frac{1}{(1-q^2y_1y_2)(1-q^3y_1y_2y_3)(1-q^4y_1^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)}\ ,\\
F_3^{(132)} &= \frac{q^2y_1y_3}{(1-q^2y_1y_3)(1-q^3y_1y_2y_3)(1-q^4y_1^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)}\ ,\\
F_3^{(213)} &= \frac{q^4y_2^2y_1y_3}{(1-q^2y_1y_2)(1-q^3y_1y_2y_3)(1-q^4y_2^2y_1y_3)(1-q^4y_1^2y_2^2y_3^2)}\ ,\\
F_3^{(231)} &= \frac{q^2y_2y_3}{(1-q^2y_2y_3)(1-q^3y_1y_2y_3)(1-q^4y_2^2y_1y_3)(1-q^4y_1^2y_2^2y_3^2)}\ ,\\
F_3^{(312)} &= \frac{q^4y_3^2y_1y_2}{(1-q^2y_1y_3)(1-q^3y_1y_2y_3)(1-q^4y_3^2y_1y_2)(1-q^4y_1^2y_2^2y_3^2)}\ ,\\
F_3^{(321)} &= \frac{q^6y_3^3y_2^2y_1}{(1-q^2y_2y_3)(1-q^3y_1y_2y_3)(1-q^4y_3^2y_1y_2)(1-q^4y_1^2y_2^2y_3^2)}\ .
\end{aligned}$$

Carrying out the sum in 2.24 by hand is a bit tedious. But miraculously, given the presence of so many unwanted denominators, MAPLE still was able to factor the resulting sum to

$$\frac{1+q^3y_1y_2y_3}{(1-q)(1-q^2y_1y_2)(1-q^2y_1y_3)(1-q^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)}$$

as desired. The reader is welcome to verify this by hand.

To complete the proof of Theorems 3.1 and 3.2 we are thus left with proving the identity in 3.11. Our proof of 3.13 uses the partial fraction algorithm of G. Xin. For the benefit of the reader who is not familiar with this computational device the proof will be preceded by a derivation of the basic tools of the algorithm and thus will be postponed until next section.

But before closing this section, we must add that we have tried Xin's partial fraction software to obtain the generating function

$$F_4(y_1,y_2,y_3,y_4;q)=\sum_{r_4\ge 0}\sum_{r_3\ge r_4}\sum_{r_2\ge r_3}\sum_{r_1\ge r_2}\sum_{m\ge 2r_1} q^m y_1^{r_1}y_2^{r_2}y_3^{r_3}y_4^{r_4}\big\langle s_{m-r_1,r_1}*\cdots *s_{m-r_4,r_4},s_m\big\rangle \qquad 3.25$$

using the identity

$$F_4(y_1,y_2,y_3,y_4;q)\ \ =\ \ G_4\big(\tfrac{y_1a_1}{a_0^2},\tfrac{y_2a_2}{a_1},\tfrac{y_3a_3}{a_2},\tfrac{y_4}{a_3};qa_0\big)\Big|_{a_0^{\ge} a_1^{\ge} a_2^{\ge} a_3^{\ge}}$$

with

$$G_4(y_1,y_2,y_3,y_4;q) = \frac{\prod_{i=1}^4 (1-y_i)}{\prod_{S\subseteq[1,4]}(1-q\prod_{i\in S} y_i)}\ . \qquad 3.26$$

The software in a matter of minutes produced the desired generating function. Unfortunately the result lacked the sheer beauty and simplicity of $F_3(y_1,y_2,y_3;q)$ as given by 3.12. In fact, $F_4(y_1,y_2,y_3,y_4;q)$ turns out to have an extremely large numerator containing more than 3000 terms. This circumstance not withstanding, it is still possible that this generating function may have an elegant expression as a sum of simple terms. Indeed, there is plentiful computational evidence that shows that a sum of simple rational fractions can have horrendous expressions when written as a single rational fraction with the least common denominator.

We must add that, if our desire is to obtain the generating functions

$$\Phi_{r_1,r_2,\ldots,r_k}(q) \;=\; \sum_{m\geq 2max(r_1,r_2,\ldots,r_k)} q^m \big\langle s_{m-r_1,r_1} * s_{m-r_2,r_2} * \cdots * s_{m-r_k,r_k}, s_m\big\rangle, \qquad 3.27$$

then they can be obtained by a two step procedure. The first step is to extract the coefficient

$$\Psi_{r_1,r_2,\ldots,r_k}(q) \;=\; \frac{\prod_{i=1}^k (1-y_i)}{\prod_{S\subseteq[1,k]}(1-q\prod_{i\in S} y_i)}\Big|_{y_1^{r_1}y_2^{r_2}\cdots y_k^{r_k}}\ , \qquad 3.28$$

and the next step is to obtain $\Phi_{r_1,r_2,\ldots,r_k}(q)$ by extracting from $\Psi_{r_1,r_2,\ldots,r_k}(q)$ all the terms where $m < 2\ max(r_1,r_2,\ldots,r_k)$. This extraction is also easy to carry out. In fact, note that we may write

$$\Psi_{r_1,r_2,\ldots,r_k}(q) \;=\; \frac{P_{r_1,r_2,\ldots,r_k}(q)}{1-q}$$

with

$$P_{r_1,r_2,\ldots,r_k}(q) \;=\; \frac{\prod_{i=1}^k (1-y_i)}{\prod_{\phi\neq S\subseteq[1,k]}(1-q\prod_{i\in S} y_i)}\Big|_{y_1^{r_1}y_2^{r_2}\cdots y_k^{r_k}} \qquad 3.29$$

an ordinary polynomial. To obtain 3.27 from 3.28 we can use the algorithm given by the following Proposition.

Proposition 3.4

Let

$$F(q) \;=\; \frac{P(q)}{1-q} \;=\; \sum_{m\geq 0} c_m q^m \qquad 3.30$$

with $P(q)$ a polynomial. Then the rational functions

$$F_R(q) \;=\; \sum_{m\geq R} c_m q^m$$

satisfy the recursion

$$F_R(q) \;=\; \sum_{m\geq R} P(q)\Big|_{q^m} q^m \;+\; qF_{R-1}(q) \qquad 3.31$$

with initial condition $F_0(q) = F(q)$.

Proof

Note that from 3.30 we derive that

$$c_m \;=\; P(q)\Big|_{q^m} + c_{m-1} \tag{3.32}$$

Thus

$$\sum_{m\geq R} c_m q^m \;=\; \sum_{m\geq R} P(q)\Big|_{q^m} q^m + \sum_{m\geq R} c_{m-1} q^m$$

and 3.31 follows if we adopt the convention that $c_s = 0$ when $s < 0$.

We derive from this the following surprising fact already noticed by Murnaghan [9] in the case of triple Kronecker products. See also the variety of Kronecker coefficient identities derived by Scharf, Thibon and Wybourne in [10].

Theorem 3.3

For any given $r_1, r_2, \ldots, r_k$ the Kronecker coefficient

$$\big\langle s_{m-r_1,r_1} * s_{m-r_2,r_2} * \cdots * s_{m-r_k,r_k}\, ,\, s_m \big\rangle$$

stabilizes after a finite number of terms and the stable value is given by the coefficient

$$P_{r_1,r_2,\ldots,r_k}(1) \;=\; \frac{\prod_{i=1}^{k}(1-y_i)}{\prod_{\phi\neq S\subseteq[1,k]}(1-\prod_{i\in S} y_i)}\Bigg|_{y_1^{r_1} y_2^{r_2}\cdots y_k^{r_k}} .$$

Proof

We see from 3.32 that we will have $c_m = c_{m-1}$ as soon as m becomes greater than the degree of $P(q)$. This given, we may simply compute the stable value of c_m from 3.30. By taking the limit

$$\lim_{q\to 1}(1-q)F(q) \;=\; P(1) .$$

Thus the assertion follows from 3.29.

For example we calculated in this manner the rational function $\phi_{2,3,4,2,3}(q)$ and obtained first that

$$\psi_{2,3,4,2,3}(q) \;=\; \frac{-9q^6+9q^7+144q^8+197q^9+154q^{10}+71q^{11}+25q^{12}+5q^{13}+q^{14}}{1-q} .$$

This then yielded

$$\phi_{2,3,4,2,3}(q) \;=\; \frac{144q^8+197q^9+154q^{10}+71q^{11}+25q^{12}+5q^{13}+q^{14}}{1-q}$$

whose series expansion is

$$144q^8+341q^9+495q^{10}+566q^{11}+591q^{12}+596q^{13}+597q^{14}+597q^{15}+597q^{16}+597q^{17}+597q^{18}+597q^{19}+O(q^{20}) .$$

In particular this gives that

$$\big\langle s_{m-2,2} * s_{m-3,3} * s_{m-4,4} * s_{m-2,2} * s_{m-3,3}\, ,\, s_m \big\rangle \;=\; 597$$

for all $m \geq 14$.

4. The partial fraction algorithm

To complete the proof of Theorems 3.1 and 3.2 we are left with proving the identity in 3.11. We will do this, via Proposition 3.1. More precisely, we plan to obtain the generating function

$$G(y_1,y_2,y_3;q)=\sum_{m\geq 2r_1}\sum_{r_1\geq r_2\geq r_3\geq 0} q^m y_1^{r_1}y_2^{r_1}y_3^{r_3}\big\langle s_{m-r_1,r_1}*s_{m-r_2,r_2}*s_{m-r_3,r_3}\,,\,s_m\big\rangle \tag{4.1}$$

from the series

$$F(y_1,y_2,y_3;q) \;=\; \frac{(1-y_1)(1-y_2)(1-y_3)}{(1-q)(1-qy_1)(1-qy_2)(1-qy_3)(1-qy_1y_2)(1-qy_1y_3)(1-qy_2y_3)(1-qy_1y_2y_3)}$$

by the formula

$$G(y_1,y_2,y_3;q)=F(y_1a_1/a^2,y_2a_2/a_1,y_3/a_2;qa)\Big|_{a^{\geq}a_1^{\geq}a_2^{\geq}}. \tag{4.2}$$

At the moment, the best tool we have in our possession is the partial fraction algorithm of G. Xin [13].

Since many of the computer results presented here were obtained by software implementing this algorithm and we will also use later for another hand computation, it will be good to include a brief introduction to its basics.

Firstly, to avoid ordinary convergence problems we need to work in the field of iterated formal Laurent series. The definition of this field is recursive and is determined by a chosen total order of all the variables appearing in our given 'kernel' Ω. In the applications we are to compute the constant term (usually denoted $\Omega_{=0}$) or a positive term (usually denoted $\Omega_{\geq 0}$). To be precise these two operations will involve only a specific subset of the variables. Denoting this subset $a_1,a_2,\ldots,a_k$, here we use the notations $\Omega\big|_{a_1^0a_2^0\cdots a_n^0}$ and $\Omega\big|_{a_1^{\geq 0}a_2^{\geq 0}\cdots a_k^{\geq 0}}$ respectively. The first operation consists in expanding Ω as a formal iterated Laurent series and selecting the terms that do not contain any of the variables $a_1,a_2,\ldots,a_k$. This is done by a succession of one variable constant term extractions. To compute the operator $\Omega_{a_1^{\geq 0}a_2^{\geq 0}\cdots a_k^{\geq 0}}$, we again start with the expansion of Ω as a formal iterated Laurent series and proceed with k successive steps. But here, at the time we operate on the variable a_i we delete all the terms which contain a_i to a negative exponent and then set $a_i=1$ in the remaining terms.

It should be emphasized that, in either case, it is not a good strategy to decide before hand in which order the variables $a_1,a_2,\ldots,a_k$ are to be operated upon. The reason for this is that, it is difficult to predict before hand the nature of the rational function remaining after each successive step. Yet as we shall see, there are criteria, based on the nature of this rational function, that suggest which variable should be operated upon in the next step in order to achieve the simplest and shortest path to the final answer. This will be illustrated in the specific calculations we will carry out.

Supposing that our variables, in the chosen total order, are $x_1,x_2,\ldots,x_n$. Then, for a given field of scalars K the initial field is $K((x_1))$ consisting of formal Laurent series in x_1 with coefficients in K, that is the series in which x_1 appears with a negative exponent only in a finite number of terms. In symbols

$$K((x_1)) \;=\; \Big\{\sum_{m\geq M} a_m x_1^m \,:\, a_m\in K\Big\}\,.$$

This given, recursively we define the field of iterated Laurent series $K((x_1))((x_2))\cdots((x_n))$ to be the field of formal Laurent series in x_n with coefficients in $K((x_1))((x_2))\cdots((x_{n-1}))$. The fundamental fact is that the total order allows us to imbed the field of rational functions $K(x_1, x_2, \ldots, x_n)$ as a subfield of $K((x_1))((x_2))\cdots((x_n))$. We shall only describe here how this imbedding is carried out but leave all the matters of consistency to the original works [13], [14]. The important fact is that under this imbedding all the identities in $K(x_1, x_2, \ldots, x_n)$ become identities in $K((x_1))((x_2))\cdots((x_n))$.

We will begin with the recipe for converting each rational function in the given variables into a formal Laurent series. The rational functions we will work with here may all be written in the form

$$F = \frac{P}{(1-m_1)(1-m_2)\cdots(1-m_n)}$$

with P a Laurent polynomial and $m_1, m_2, \ldots, m_k$ monomials. Our first need is to be able to decide whether a given factor $\frac{1}{1-m_i}$ should be converted to

$$a) \quad \sum_{s\geq 0} m_i^s \quad \text{or} \quad b) \quad -\sum_{s\geq 1} \frac{1}{m_i^s} \;\Big(= \frac{-\frac{1}{m_1}}{1-\frac{1}{m_1}}\Big)$$

The decision is based on the idea that the total order forces one of the two '*formal*' inequalities $m_i < 1$ or $m_i > 1$ to be true. In the first case, we choose a) (the '*ordinary form*') and in the second case, we choose b) (the '*dual form*'). The criterion is as follows: we scan through the variables occurring in the monomial m_i. If the smallest variable has positive exponent, then $m_i < 1$. If it has negative exponent, then $m_i > 1$.

For simplicity of notation we will avoid using summations and simply rewrite the given rational function in the form

$$F = P \times \Big(\prod_{m_i<1} \frac{1}{1-m_i}\Big) \times \Big(\prod_{m_j>1} \frac{-\frac{1}{m_j}}{1-\frac{1}{m_j}}\Big). \qquad 4.3$$

We shall refer to this symbolic expression as the '*proper form*' of F.

To compute $F\big|_{a_1^0 a_2^0 \cdots a_k^0}$ as well as $F\big|_{a_1^{\geq} a_2^{\geq} \cdots a_k^{\geq}}$ by the partial algorithm of G. Xin (see [13], [14]), at each step we use a partial fraction expansion to eliminate one of the variables $a_1, a_2, \ldots, a_k$.

To see how this is done, assume that to begin we have chosen to eliminate the variable x. This given, by suitable manipulations we rewrite our rational function in the form

$$F = Q(x) + \frac{R(x)}{(1-xU_1)\cdots(1-xU_h)(x-V_1)\cdots(x-V_k)}$$

with $Q(x)$ a Laurent polynomial, $R(x)$ a polynomial of degree less than $h+k$ and $U_1, U_2, \ldots, U_h$ as well as $V_1, V_2, \ldots, V_k$ are monomials not containing x. The nature of the denominator will be determined by the requirement that

$$xU_i < 1 \quad \text{for } 1 \leq i \leq h \qquad \text{and} \qquad V_j/x < 1 \quad \text{for } 1 \leq j \leq k.$$

The next step is to derive the partial fraction expansion:

$$F = Q(x) + \sum_{i=1}^{h} \frac{A_i}{(1-xU_i)} + \sum_{j=1}^{k} \frac{B_j}{(x-V_j)} \qquad 4.4$$

which, as customary, is obtained by setting

$$A_i = (1 - xU_i)(F(x) - Q(x))\Big|_{x=1/U_1} \qquad \text{and} \qquad B_j = (x - V_j)(F(x) - Q(x))\Big|_{x=V_j}.$$

This immediately yields the equalities

$$F\Big|_{x^0} \;=\; Q(x)\Big|_{x^0} + \sum_{i=1}^{h} A_i \tag{4.5}$$

as well as

$$F\Big|_{x^{\geq}} \;=\; Q(x)\Big|_{x^{\geq 0}} + \sum_{i=1}^{h} \frac{A_i}{(1-U_i)}\;. \tag{4.6}$$

The reason for this is that each term $\frac{B_j}{x-V_j}$ in 4.4 comes from a monomial $m < 1$ in 4.3 which had the factorization $m = V_j/x$, so that the proper form of the last summation in 4.4 will be

$$\sum_{j=1}^{k} \frac{B_j/x}{(1 - V_j/x)}$$

and we see that the corresponding series contains only negative powers of x and thus yields no contribution to either $F\Big|_{x^0}$ or $F\Big|_{x^{\geq}}$.

To help decide which variable must be operated on at the i^{th} step, the next two Propositions show that there are alternate ways to express the same result.

Proposition 4.1

Suppose that the m_i are distinct monomials not containing x, and that F is a rational function of x with partial fraction decomposition

$$F(x) \;=\; P(x) + \sum_{i=1}^{k} \frac{A_i}{x - m_i}$$

where $P(x)$ is a polynomial and $A_i = F(x)(x - m_i)\big|_{x=a_i}$, then

$$\begin{aligned} a)\quad F(x)\Big|_{x^0} \;&=\; P(0) - \sum_{x/m_i<1} A_i/m_i \;=\; F(0) + \sum_{x/m_i>1} A_i/m_i \\ b)\quad F(x)\Big|_{x^{\geq}} \;&=\; P(1) + \sum_{x/m_i<1} \frac{A_i}{1-m_i} \;=\; F(1) - \sum_{x/m_i>1} \frac{A_i}{1-m_i} \end{aligned} \tag{4.7}$$

Of course for b) we must assume that none of the m_i is equal to 1.

Proof

The proper form of $F(x)$ is

$$F(x) \;=\; P(x) + \sum_{x/m_i<1} \frac{-A_i/m_i}{1 - x/m_i} + \sum_{x/m_i>1} \frac{A_i/x}{1 - m_i/x}.$$

Thus both results follow from the same reasons as before. For the two alternate forms we simply use the two identities

$$\begin{aligned} F(0) &= P(0) - \sum_{x/m_i<1} A_i/m_i - \sum_{x/m_i>1} A_i/m_i \\ F(1) &= P(1) + \sum_{x/m_i<1} \frac{A_i}{1-m_i} + \sum_{x/m_i>1} \frac{A_i}{1-m_i} . \end{aligned}$$

In some applications we may need to operate on a kernel given in the form

$$F(x) = \frac{P(x)}{\prod_{i=1}^n (1-(x/m_i)^{e_i})} \qquad (\text{with } e_i \pm 1). \tag{4.8}$$

Though it can be converted in the form given by 4.4 it is convenient to have an answer that may be directly obtained from this form.

Proposition 4.2

Suppose that $F(x)$ is given by 4.8 with the m_i monomials not containing x. Suppose further that

$$\lim_{x\to 0} F(x) = 0. \tag{4.9}$$

Then

$$F(x)\Big|_{x^0} = \sum_{x/m_j<1} e_j \Big(F(x)(1-\big(\tfrac{x}{m_j}\big)^{e_j}\Big)\Big|_{x=m_j} = -\sum_{x/m_j>1} e_j \Big(F(x)(1-\big(\tfrac{x}{m_j}\big)^{e_j}\Big)\Big|_{x=m_j} \tag{4.10}$$

$$F(x)\Big|_{x^{\geq}} = \sum_{x/m_j<1} e_j \frac{-m_j}{1-m_j} \Big(F(x)(1-\big(\tfrac{x}{m_j}\big)^{e_j}\Big)\Big|_{x=m_j} = F(1) + \sum_{x/m_j>1} e_j \frac{m_j}{1-m_j} \Big(F(x)(1-\big(\tfrac{x}{m_j}\big)^{e_j}\Big)\Big|_{x=m_j} . \tag{4.11}$$

Proof

The hypothesis in 4.9 assures that $F(x)$ has a partial fraction of the form

$$F(x) = \sum_{x/m_j<1} \frac{A_j}{1-x/m_j} + \sum_{x/m_j>1} \frac{B_j}{x-m_j} \tag{4.12}$$

where we have

$$A_j = (1-x/m_j)F(x)\Big|_{x=m_j}$$

and

$$B_j = (x-m_j)F(x)\Big|_{x=m_j} .$$

But we see that we have

$$\frac{1-x/m_j}{1-(x/m_j)^{e_j}}\Big|_{x=m_j} = \left\{ \begin{array}{ll} 1 & \text{if } e_j = 1 \\ \frac{1-x/m_j}{x/m_j-1}\, x/m_j\Big|_{x=m_j} = -1 & \text{if } e_j = -1 \end{array} \right\} = e_j$$

and thus we may also write

$$A_j = e_j \times \big(1-(x/m_j)^{e_j}\big)F(x)\Big|_{x=m_j} . \tag{4.13}$$

Similarly we see that

$$\frac{x-m_j}{1-(x/m_j)^{e_j}}\Big|_{x=m_j} \;=\; \left\{\begin{array}{ll} \frac{x-m_j}{m_j-x}\Big|_{x=m_j} m_j = -m_j & \text{if } e_j=1 \\ \frac{x-m_j}{x-m_j}x\Big|_{x=m_j} = m_j & \text{if } e_j=-1 \end{array}\right\} \;=\; -e_j m_j$$

and we may thus also write

$$B_j \;=\; -e_j m_j \times (1-(x/m_j)^{e_j})F(x)\Big|_{x=m_j}. \tag{4.14}$$

Now from 4.12 we derive (as we have seen in the proof of 4.5)

$$F(x)\Big|_{x^0} \;=\; \sum_{x/m_j<1} A_j$$

and 4.15 gives

$$F(x)\Big|_{x^0} \;=\; \sum_{x/m_j<1} e_j \times (1-(x/m_j)^{e_j})F(x)\Big|_{x=m_j}.$$

But, using the second identity in 4.8 (and taking also account of 4.9) we can also write

$$F(x)\Big|_{x^0} \;=\; \sum_{x/m_j>1} B_j/m_j$$

and 4.13 gives

$$F(x)\Big|_{x^0} \;=\; \sum_{x/m_j>1} -e_j \times (1-(x/m_j)^{e_j})F(x)\Big|_{x=m_j}.$$

Note further that from 4.12 we derive that

$$F(x)\Big|_{x^{\geq}} \;=\; \sum_{x/m_j<1} \frac{A_j}{1-1/m_j} \;=\; \sum_{x/m_j<1} \frac{-m_j}{1-m_j}A_j \tag{4.15}$$

and 4.13 gives

$$F(x)\Big|_{x^{\geq}} \;=\; \sum_{x/m_j<1} \frac{-m_j}{1-m_j}\; e_j \times (1-(x/m_j)^{e_j})F(x)\Big|_{x=m_j}.$$

This proves the first equality in 4.11. Moreover, setting $x=1$ in 4.12 gives

$$F(1) \;=\; \sum_{x/m_j<1} \frac{-m_j}{1-m_j}\, A_j \;+\; \sum_{x/m_j>1} \frac{B_j}{1-m_j}$$

and from 4.15 we derive that

$$F(x)\Big|_{x^{\geq}} \;=\; F(1) \;-\; \sum_{x/m_j>1} \frac{B_j}{1-m_j}$$

and using 4.14

$$F(x)\Big|_{x^{\geq}} \;=\; F(1) \;+\; \sum_{x/m_j>1} \frac{1}{1-m_j}\, e_j m_j \times (1-(x/m_j)^{e_j})F(x)\Big|_{x=m_j}.$$

This yields the second equality in 4.11 and completes our proof.

Remark 4.1

It will be convenient to say that a denominator factor $1-(x/m_i)^{e_i}$ is '*contributing*' if $x/m_i<1$ and say that it is '*dually contributing*' if $x/m_i>1$. In using Proposition 4.2, our choice of applying 4.10 or 4.11 should be dictated by which is the smaller of the two: the number of contributing factors or the number of dually contributing factors.

It should be mentioned that in the more general applications of the partial fraction algorithm, the given kernel may be of the form

$$F=\frac{P}{(1-m_1)^{a_1}(1-m_2)^{a_2}\cdots(1-m_n)^{a_k}}$$

with some $a_i\neq 1$, and worse yet the variable x to be eliminated may appear, in some of the m_i, also to a power $\neq 1$. The reader will find in the original papers how to deal with kernels of the most general form. Here, the only additional cases we need are covered by the following auxiliary result.

Proposition 4.3

Suppose that our kernel is of the form

$$F(x)\;=\;\frac{1}{(1-Ux)}\times G(x)$$

where $G(x)$ is a rational function whose corresponding iterated formal Laurent series expands as a sum of monomials which contain x only to a negative power. For instance if

$$G(x)\;=\;\frac{P(\frac{1}{x})}{\prod_{i=1}^{k}(1-m_i/x^{a_i})}$$

with P a polynomial, $m_1,m_2,\ldots,m_k$ and U monomials not containing x, $a_i\geq 1$ and

$$Ux<1\,,\qquad m_i/x^{a_i}<1\quad \textit{for}\quad 1\leq i\leq k$$

then

$$F(x)\Big|_{x^0}=G(x)\Big|_{x=1/U},\qquad F(x)\Big|_{x\geq}=\frac{1}{(1-U)}\times G(x)\Big|_{x=1/U}\,.$$

Proof

Recall that $\big|_{x^0}$ deletes every monomial that contains x, and $\big|_{x\geq}$ deletes every monomial that contains x to a negative power and sets $x=1$ otherwise.

In summary, since both these operators act separately on each individual monomial we need only establish these identities in the case that $G(x)$ itself is a monomial. But in this case we have

$$G(x)\;=\;\frac{m}{x^a}$$

with m a monomial and $a>0$. Thus

$$\frac{1}{(1-Ux)}\times G(x)\Big|_{x^0}\;=\;\frac{1}{(1-Ux)}\times\frac{m}{x^a}\Big|_{x^0}\;=\;m\,U^a\;=\;G(x)\Big|_{x=1/U}$$

and

$$\frac{1}{(1-Ux)} \times G(x)\Big|_{x^{\geq}} \;=\; \frac{1}{(1-Ux)} \times \frac{m}{x^a}\Big|_{x^{\geq}} \;=\; m\,\frac{U^a}{1-U} \;=\; \frac{1}{1-U} \times G(x)\Big|_{x=1/U}$$

as desired. This completes our proof.

Armed with this package of identities we can now proceed with our proof of the identity in 3.11. Our goal is to obtain the rational function

$$\begin{aligned} G(y_1,y_2,y_3;q) \;&=\; \frac{(1-y_1a_1/a^2)(1-y_2a_2/a_1)(1-y_3/a_2)}{(1-qa)(1-qy_1a_1/a)(1-qay_2a_2/a_1)(1-qay_3/a_2)(1-qy_1y_2a_2/a)} \times \\ &\qquad \times \frac{1}{(1-qy_1a_1y_3/aa_2)(1-qay_2y_3/a_1)(1-qy_1y_2y_3/a)}\Big|_{a^{\geq}a_1^{\geq}a_2^{\geq}} . \end{aligned} \tag{4.16}$$

To begin, we choose our variable order by the requirement that we must have

$$q < y_1 < y_2 < y_3 < a < a_1 < a_2 \; .$$

Under this order, all the monomials

$$y_1a_1/a^2\,,\quad y_2a_2/a_1\,,\quad y_3/a_2\,,\quad qa$$

will be *formally* less than 1. This is consistent with the fact that we have made the replacements

$$y_1 \to y_1a_1/a^2\,,\quad y_2 \to y_2a_2/a_1\,,\quad y_3 \to y_3/a_2\,,\quad q \to qa$$

in the formal power series

$$G_3(y_1,y_2,y_3;q) = \frac{1}{(1-q)(1-q^2y_1y_2)(1-q^3y_1y_2y_3)(1-q^4y_1^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)} \; . \tag{4.17}$$

We are now ready to carry out our calculation.

The result will be obtained by repetitive uses of Proposition 4.2. Noting that there are only two contributing factors containing a_1 in 4.16 and the same is true for a_2 while there are 4 contributing factors containing a, it appears more economical to eliminate first a_1.

This given, separating the factors containing a_1 we will rewrite 4.16 in the form

$$G(y_1,y_2,y_3;q) \;=\; \Big(F\Big|_{a_1^{\geq}}\Big)\frac{(1-y_3/a_2)}{(1-qa)(1-qay_3/a_2)(1-qy_1y_2a_2/a)(1-qy_1y_2y_3/a)}\Big|_{a_2^{\geq}a^{\geq}} \tag{4.18}$$

with

$$F = \frac{(1-y_1a_1/a^2)(1-y_2a_2/a_1)}{(1-qy_1a_1/a)(1-qay_2a_2/a_1)(1-qy_1a_1y_3/aa_2)(1-qay_2y_3/a_1)} \; .$$

Since our choice of total order makes F in proper form, to compute $F\Big|_{a_1^{\geq}}$ we can use Proposition 4.2 with

$$m_1 = \frac{a}{qy_1}\,,\quad m_2 = qay_2a_2\,,\quad m_3 = \frac{aa_2}{qy_1y_3}\,,\quad m_4 = qay_2y_3 \; .$$

Using this notation we may write

$$F = \frac{(1-y_1a_1/a^2)(1-y_2a_2/a_1)}{(1-a_1/m_1)(1-(a_1/m_2)^{-1})(1-a_1/m_3)(1-(a_1/m_4)^{-1})} \; .$$

This gives under the conventions of Proposition 4.2

$$e_1 = 1\,, \quad e_2 = -1\,, \quad e_3 = 1\,, \quad e_4 = -1\;.$$

Selecting the first equality in 4.12 and noting that the contributing factors are those containing m_1 and m_3 we derive that

$$F\Big|_{a_1^{\geq}} \;=\; \frac{-m_1}{1-m_1}(1-a_1/m_1)\,F\Big|_{a_1=m_1} + \frac{-m_3}{1-m_3}(1-a_1/m_3)\,F\Big|_{a_1=m_3} \;=\; Q_1 + Q_3$$

with

$$\begin{aligned} Q_1 &= \frac{-\frac{a}{qy_1}}{1-\frac{a}{qy_1}}\,\frac{(1-\frac{a}{qy_1}y_1/a^2)(1-y_2a_2/\frac{a}{qy_1})}{(1-qay_2a_2/\frac{a}{qy_1})(1-\frac{a}{qy_1}qy_1y_3/aa_2)(1-qay_2y_3/\frac{a}{qy_1})} \\ &= \frac{1}{1-qy_1/a}\,\frac{-(1-qa)(1-qy_1y_2a_2/a)/qa}{(1-q^2y_1y_2a_2)(1-y_3/a_2)(1-q^2y_1y_2y_3)} \end{aligned}$$

and

$$\begin{aligned} Q_3 &= \frac{-\frac{aa_2}{qy_1y_3}}{1-\frac{aa_2}{qy_1y_3}}\,\frac{(1-y_1\frac{aa_2}{qy_1y_3}/a^2)(1-y_2a_2/\frac{aa_2}{qy_1y_3})}{(1-qy_1\frac{aa_2}{qy_1y_3}/a)(1-qay_2a_2/\frac{aa_2}{qy_1y_3})(1-qay_2y_3/\frac{aa_2}{qy_1y_3})} \\ &= \frac{1}{1-qy_1y_3/aa_2}\,\frac{(1-a_2/aqy_3)(1-qy_1y_2y_3/a)}{(1-a_2/y_3)(1-q^2y_1y_2y_3)(1-q^2y_1y_2y_3^2/a_2)} \; . \end{aligned}$$

We can now rewrite 4.18 as

$$G(y_1,y_2,y_3;q) \;=\; R_1 + R_3 \tag{4.19}$$

with

$$\begin{aligned} R_1 &= Q_1\frac{(1-y_3/a_2)}{(1-qa)(1-qay_3/a_2)(1-qy_1y_2a_2/a)(1-qy_1y_2y_3/a)}\Big|_{a_2^{\geq}a^{\geq}} \\ &= \frac{1}{1-qy_1/a}\,\frac{-(1-qa)(1-qy_1y_2a_2/a)/qa}{(1-q^2y_1y_2a_2)(1-y_3/a_2)(1-q^2y_1y_2y_3)} \\ &\qquad \frac{(1-y_3/a_2)}{(1-qa)(1-qay_3/a_2)(1-qy_1y_2a_2/a)(1-qy_1y_2y_3/a)}\Big|_{a_2^{\geq}a^{\geq}} \\ &= \frac{1}{(1-q^2y_1y_2y_3)}\times \\ &\qquad \frac{1}{(1-q^2y_1y_2a_2)(1-qay_3/a_2)} \times \frac{-1/qa}{(1-qy_1/a)(1-qy_1y_2y_3/a)}\Big|_{a_2^{\geq}a^{\geq}} \end{aligned}$$

and

$$\begin{aligned} R_3 &= Q_3\frac{(1-y_3/a_2)}{(1-qa)(1-qay_3/a_2)(1-qy_1y_2a_2/a)(1-qy_1y_2y_3/a)}\Big|_{a_2^{\geq}a^{\geq}} \\ &= \frac{1}{1-qy_1y_3/aa_2}\,\frac{(1-a_2/aqy_3)(1-qy_1y_2y_3/a)}{(1-a_2/y_3)(1-q^2y_1y_2y_3)(1-q^2y_1y_2y_3^2/a_2)} \end{aligned}$$

$$\frac{-(1-a_2/y_3)y_3/a_2}{(1-qa)(1-qay_3/a_2)(1-qy_1y_2a_2/a)(1-qy_1y_2y_3/a)}\Big|_{a_2^{\geq} a^{\geq}}$$

$$= \frac{1}{1-qy_1y_3/aa_2}\,\frac{-(1-aqy_3/a_2)a_2/aqy_3}{(1-q^2y_1y_2y_3)(1-q^2y_1y_2y_3^2/a_2)}$$

$$\frac{-y_3/a_2}{(1-qa)(1-qay_3/a_2)(1-qy_1y_2a_2/a)}\Big|_{a_2^{\geq} a^{\geq}}$$

$$= \frac{1}{(1-q^2y_1y_2y_3)}\times$$

$$\frac{1}{(1-qy_1y_3/aa_2)(1-qy_1y_2a_2/a)(1-q^2y_1y_2y_3^2/a_2)}\Big|_{a_2^{\geq}} \quad \frac{1/aq}{(1-qa)}\Big|_{a^{\geq}}\ .$$

We may rewrite the result of the last two calculations in the form

$$R_1 \;=\; \frac{1}{(1-q^2y_1y_2y_3)}\times S_1 \quad \frac{-1/qa}{(1-qy_1/a)(1-qy_1y_2y_3/a)}\Big|_{a^{\geq}} \tag{4.20}$$

and

$$R_3 \;=\; \frac{1}{(1-q^2y_1y_2y_3)} \quad \times S_3\ \frac{1/aq}{(1-qa)}\Big|_{a^{\geq}} \tag{4.21}$$

with

$$S_1 \;=\; \frac{1}{(1-q^2y_1y_2a_2)(1-qay_3/a_2)}\Big|_{a_2^{\geq}}$$

and

$$S_3 \;=\; \frac{1}{(1-qy_1y_2a_2/a)(1-qy_1y_3/aa_2)(1-q^2y_1y_2y_3^2/a_2)}\Big|_{a_2^{\geq}}\ .$$

We will now choose to eliminate a_2 next. We can immediately see that both S_1 and S_2 are very special cases of Proposition 4.3 which gives

$$S_1 \;=\; \frac{1}{(1-q^2y_1y_2)}\,\frac{1}{(1-qay_3/a_2)}\Big|_{a_2=1/q^2y_1y_2} \;=\; \frac{1}{(1-q^2y_1y_2)(1-aq^3y_1y_2y_3)}. \tag{4.22}$$

and

$$\begin{aligned} S_3 &= \frac{1}{(1-qy_1y_2/a)}\times\frac{1}{(1-qy_1y_3/aa_2)(1-q^2y_1y_2y_3^2/a_2)}\Big|_{a_2=a/qy_1y_2} \\ &= \frac{1}{(1-qy_1y_2/a)}\,\frac{1}{(1-q^2y_1^2y_2y_3/a^2)(1-q^3y_1^2y_2^2y_3^2/a)}\ . \end{aligned} \tag{4.23}$$

Using 4.22 and 4.23 in 4.20 and 4.21 now gives

$$\begin{aligned} R_1 &= \frac{1}{(1-q^2y_1y_2y_3)(1-q^2y_1y_2)}\times \quad \frac{-1/qa}{(1-aq^3y_1y_2y_3)(1-qy_1/a)(1-qy_1y_2y_3/a)}\Big|_{a^{\geq}} \\ &= \frac{1}{(1-q^2y_1y_2y_3)(1-q^2y_1y_2)}\times T_1 \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} R_3 &= \frac{1}{(1-q^2y_1y_2y_3)} \quad \times\frac{1}{1-qy_1y_2/a}\ \frac{1}{(1-q^2y_1^2y_2y_3/a^2)(1-q^3y_1^2y_2^2y_3^2/a)}\ \frac{1/aq}{(1-qa)}\Big|_{a^{\geq}} \\ &= \frac{1}{(1-q^2y_1y_2y_3)} \quad \times T_3\ . \end{aligned} \tag{4.25}$$

Thus to eliminate a we are left with the computations of

$$T_1 \;=\; \frac{-1/qa}{(1-aq^3y_1y_2y_3)(1-qy_1/a)(1-qy_1y_2y_3/a)}\Big|_{a\geq}$$

and

$$T_3 \;=\; \frac{1/aq}{(1-qa)(1-qy_1y_2/a)(1-q^2y_1^2y_2y_3/a^2)(1-q^3y_1^2y_2^2y_3^2/a)}\;\Big|_{a\geq}\;.$$

Both of them are also immediate applications of Proposition 4.1. Thus

$$\begin{aligned} T_1 &= \frac{1}{(1-q^3y_1y_2y_3)}\times\frac{-1/qa}{(1-qy_1/a)(1-qy_1y_2y_3/a)}\Big|_{a=1/q^3y_1y_2y_3} \\ &= \frac{1}{(1-q^3y_1y_2y_3)}\times\frac{-q^2y_1y_2y_3}{(1-q^4y_1^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)} \end{aligned}$$

and

$$\begin{aligned} T_3 \;&=\; \frac{1}{1-q}\times\frac{1/aq}{(1-qy_1y_2/a)(1-q^2y_1^2y_2y_3/a^2)(1-q^3y_1^2y_2^2y_3^2/a)}\;\Big|_{a=1/q} \\ &= \frac{1}{(1-q)(1-q^2y_1y_2)(1-q^4y_1^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)}\;. \end{aligned}$$

Combining these results with 4.20 and 4.21 gives

$$R_1 \;=\; \frac{1}{(1-q^2y_1y_2y_3)(1-q^2y_1y_2)}\times\frac{-q^2y_1y_2y_3}{(1-q^3y_1y_2y_3)(1-q^4y_1^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)}$$

$$R_3 \;=\; \frac{1}{(1-q^2y_1y_2y_3)}\;\times\frac{1}{(1-q)(1-q^2y_1y_2)(1-q^4y_1^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)}$$

and 4.19 gives

$$\begin{aligned} G(y_1y_2,y_3;q) \;&=\; \frac{-q^2y_1y_2y_3(1-q)\;+\;(1-q^3y_1y_2y_3)}{(1-q^2y_1y_2y_3)(1-q)(1-q^2y_1y_2)(1-q^3y_1y_2y_3)(1-q^4y_1^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)} \\ &=\; \frac{1}{(1-q)(1-q^2y_1y_2)(1-q^3y_1y_2y_3)(1-q^4y_1^2y_2y_3)(1-q^4y_1^2y_2^2y_3^2)} \end{aligned}$$

as desired. This completes our proof of the identity in 3.11.

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