ON A THREE PART CASE OF THE SHUFFLE CONJECTURE

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Abstract. We prove a symmetric function identity related to the operator ∇ of Bergeron-Garsia when it acts on Hall-Littlewood symmetric functions. This identity motivates a combinatorial recurrence on lattice paths with a N, E, blue NE, red NE, and blue/red NNE steps. Our main result is a combinatorial interpretation for certain coefficients in ∇ acting on a form of Hall-Littlewood symmetric functions studied by Haglund-Morse-Zabrocki that generalizes many of the q, t-Catalan, Schröder and lattice path results by restricting to a paths with a fixed touch composition.

1. Introduction

Consider lattice paths in an $n \times n$ square starting at $(0,0)$ and ending at (n,n) that take steps such as those that appear in Figure 1 and that do not pass below the $x = y$ diagonal.

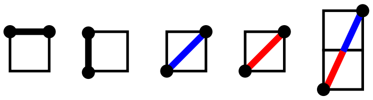

Figure 1. The step types that appear in a walk from $(0,0)$ to (n,n)

A typical path of this type might look like the example in the figure below.

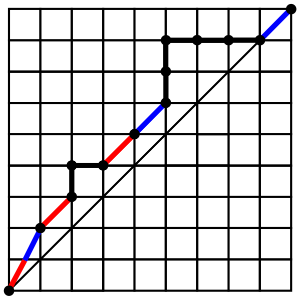

Figure 2. A typical path P with $dinv(P) = 6$ and $area(P) = 8$.

The positions that the path touches the diagonal determines a composition of n. Say that points where the path hits the diagonal are $(0,0), (r_1, r_1), (r_2, r_2), \dots, (r_{\ell-1}, r_{\ell-1}), (n,n)$, then the touch composition of the path will be defined as $(r_1 - 0, r_2 - r_1, \ldots, n - r_{\ell-1})$. If P represents the path then $touch(P)$ will represent the touch composition of the path. In the example above the touch composition is $(8, 1)$.

The area of the path will be denoted by $area(P)$ and represents the number of cells that either pass through or are strictly below the path and are also strictly above the diagonal. Let u_i be the number of cells counted towards the area in the i^{th} row. For instance in the typical path example that we have drawn above the area is equal to 8 and the sequence of u_i is given by $(0, 1, 1, 1, 1, 1, 1, 2, 0)$.

There is another statistic represented by $dinv(P)$ which is a translation of the *dinv* statistic introduced by Haiman (see [10]) to this setting. Let $dinv_1(P)$ be equal to the number of pairs $(i < j)$ with $u_i = u_j$ such that either the i^{th} step is black or the i^{th} step is red and the j^{th} step is blue. Also let $dinv_2(P)$ be the number of pairs $(i < j)$ with $u_i = u_j + 1$ such that either the j^{th} step is black or i^{th} step is blue and the j^{th} step is red. Now set $dinv(P) = dinv_1(P) + dinv_2(P)$. In the case of the example above the pairs $\{(1, 9), (3, 6), (5, 6), (4, 5), (4, 6), (4, 7)\}$ all contribute to $dinv_1$ but no pairs contribute to $dinv_2$, hence in this case $dinv(P) = 6$.

The operator ∇ was introduced by Bergeron-Garsia [1] to generalize the relationship between the conjectured Frobenius series of the space of diagonal harmonics on $2n$ letters and the elementary symmetric function e_n. In [2] Bergeron-Garsia-Haiman-Tesler conjectured that when ∇ acts on several different families of symmetric functions the coefficients are polynomials in $\mathbb{N}[q, t]$. In particular, they attribute to A. Lascoux the conjecture (see [2] Conjecture II and III) that when ∇ acts on certain Hall-Littlewood symmetric functions the result is Schur positive.

Haglund-Morse-Zabrocki [13] introduced a family of symmetric functions $C_\alpha[X; q]$ indexed by compositions α that are closely related to the Hall-Littlewood symmetric functions. They conjectured there that the operator ∇ applied to $C_\alpha[X; q]$ was a generating function for parking functions whose support path touched the diagonal at points indicated by the composition α and their conjecture generalized the shuffle conjecture of [11].

In this paper we provide evidence for this conjecture. Our main theorem is the following combinatorial expression.

Theorem 1. *For $a + b + c = n$ and for $\alpha \models n$ (α is a composition of n),*

$$\langle \nabla(C_\alpha[X; q]), e_a h_b h_c \rangle = \sum_P q^{area(P)} t^{dinv(P)}$$

where the sum is over all paths of the type described above with $touch(P) = \alpha$ that have a black vertical steps, b red steps and c blue steps.

For instance we consider the case when $\alpha = (3, 2)$ and $a = 1$, $b = 2$, $c = 2$ we list all 6 paths of this type.

The first path listed has area 4, the others all have area statistic equal to 3. The dinv statistics are respectively $2, 4, 3, 3, 2, 2$. From the theorem above we may conclude that $\left\langle \nabla(C_{(3,2)}[X; q]), e_1 h_2 h_2 \right\rangle = t^4 q^2 + t^3(q^4 + 2q^3 + 2q^2)$.

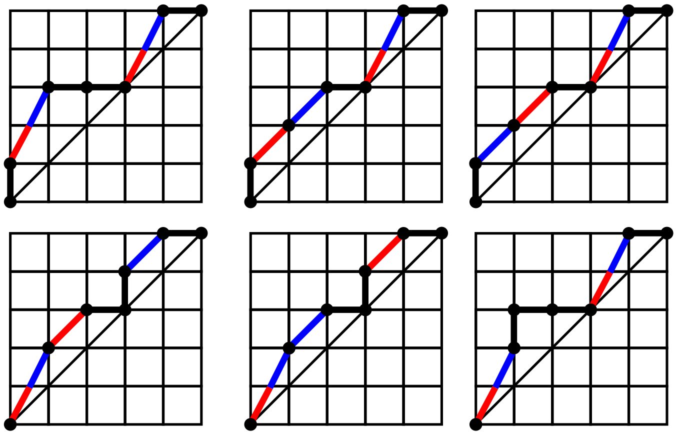

FIGURE 3. The paths with touch composition $(3, 2)$.

This theorem generalizes other results related to q, t-Schröder Theorem of Haglund [9] in two ways. First, since

$$e_n = \sum_{\alpha \models n} C_\alpha[X; q] \ ,$$

Haglund's result follows immediately from ours by summing over all compositions. Moreover, Haglund gives two Theorems, one that is a combinatorial interpretation for $\langle \nabla(e_n), e_a h_b \rangle$ and another a similar result for the special case of the shuffle conjecture $\langle \nabla(e_n), h_b h_c \rangle$. Our Theorem 1 interpolates between the two and generalizes nearly all known special cases of the shuffle conjecture.

The bijection between the paths of this type and parking functions whose reading word is given by a shuffle of $(a(a-1)\ldots 21) \sqcup\sqcup ((a+1)(a+2)\cdots(a+b)) \sqcup\sqcup ((a+b+1)(a+b+2)\cdots n)$ is made explicit in remark 8 in section 3.

The proof proceeds by showing that both the left and the right hand sides of this equation satisfy the same recurrence. It builds on the techniques recently developed in [8, 13, 14] as well as the foundational identities which were developed in [2, 3, 7, 9, 10, 11].

The remainder of this paper is divided into 2 sections. The first deals mainly with symmetric function identities and the second (much shorter) section deals mainly with the combinatorics. The basic tools and identities needed to derive the symmetric function results that are mainly developed in other references are listed in section 2.1. In section 2.2 we develop the symmetric function recurrence and in section 3 we show that this symmetric function recurrence agrees with the combinatorial interpretation given in Theorem 1.

2. THE RECURSION ON SYMMETRIC FUNCTIONS

We will assume that the reader is familiar with basic notions of symmetric functions and refer the reader to the reference [16] for additional information. In the following calculations, we will mainly use the elementary $e_r[X]$ and homogeneous $h_r[X]$ symmetric functions, the Schur symmetric functions $s_\lambda[X]$ along with some basic identities of plethystic notation for symmetric functions.

The most important identities will involve the evaluation of elementary and homogeneous symmetric functions at two sets of variables. We first set $\Omega[X] = \sum_{r\geq 0} h_r[X]$. This given, we then have that for two sets of variables (or expressions) X and Y,

$$h_r[X+Y] = \sum_{a=0}^{r} h_a[X] h_{r-a}[Y], \tag{1}$$

and this may be extended to the generating function so that

$$\Omega[X+Y] = \Omega[X]\Omega[Y]\ .$$

By substituting $h_r[X-X]=0$ in (1) for $r>0$, then we may also conclude that $h_r[-X] = (-1)^r e_r[X]$. Note that $f[-X]$ does not correspond to setting variables x_i to $-x_i$ in the symmetric function. To represent this other type of negation we introduce a symbol ϵ with the property that

$$f[\epsilon X] = (-1)^{deg(f)} f[X]$$

for any symmetric function $f[X]$ of homogeneous degree.

2.1. **Toolkit of Macdonald symmetric function identities.** To begin, we identify the partition λ with the lattice cells in diagram of λ as the set of pairs of integers $\{(i,j) : 0 \leq i < \ell(\lambda), 0 \leq j < \lambda_i\}$. It is appropriate in this case to use the notation $c \in \lambda$ to indicate that c is a pair in this set and c_1 and c_2 to refer to its coordinates. In this case we define

$$\begin{array}{llll} \text{arm of } c & a_\lambda(c) = \lambda_{c_1+1} - c_2 - 1 & \qquad \text{coarm of } c & a'_\lambda(x) = c_2 \\ \text{leg of } c & l_\lambda(c) = \lambda'_{c_2+1} - c_1 - 1 & \qquad \text{coleg of } c & l'_\lambda(c) = c_1\ . \end{array}$$

This given we set

$$\begin{aligned} B_\mu(q,t) &= \sum_{c\in\mu} t^{l'_\mu(c)} q^{a'_\mu(c)} \\ T_\mu &= \prod_{c\in\mu} t^{l'_\mu(c)} q^{a'_\mu(c)} \\ M &= (1-q)(1-t) \\ D_\mu(q,t) &= MB_\mu(q,t) - 1 \\ w_\mu(q,t) &= \prod_{c\in\mu} (q^{a_\mu(c)} - t^{l_\mu(c)+1})(t^{l_\mu(c)} - q^{a_\mu(c)+1}). \end{aligned}$$

The Macdonald symmetric functions $\tilde{H}_\lambda[X;q,t]$ are the unique basis that satisfy

1. $\tilde{H}_\lambda[(1-q)X;q,t] = \sum_{\mu\geq\lambda} r_{\lambda\mu}(q,t) s_\mu[X]$ for some coefficients $r_{\lambda\mu}(q,t)$ and
2. $\tilde{H}_\lambda[(t-1)X;q,t] = \sum_{\mu\leq\lambda} t_{\lambda\mu}(q,t) s_\mu[X]$ for some coefficients $t_{\lambda\mu}(q,t)$ and
3. $\left\langle \tilde{H}_\lambda[X;q,t], h_n \right\rangle = 1$ where $\langle , \rangle$ denotes the usual Hall scalar product.

The operator ∇ is defined to act on symmetric functions so that it has the this basis as eigenfunctions where the eigenvalues are given by $\nabla \tilde{H}_\lambda[X;q,t] = T_\lambda \tilde{H}_\lambda[X;q,t]$. Where it is appropriate to abbreviate to conserve space, we will denote $\tilde{H}_\mu = \tilde{H}_\mu[X;q,t]$, and similarly $B_\mu = B_\mu(q,t)$, $D_\mu = D_\mu(q,t)$ and $w_\mu = w_\mu(q,t)$.

It follows that if we should define the scalar product

$$\langle f[X], g[X]\rangle_* := \langle f[X], g[-\epsilon M X]\rangle$$

with respect to the usual Hall scalar product and refer to this as the $*$-scalar product. Then the family of symmetric functions $\tilde{H}_\lambda[X;q,t]$ are orthogonal with respect to the $*$-scalar product. In fact, we have that

$$\left\langle \tilde{H}_\lambda[X;q,t], \tilde{H}_\mu[X;q,t]\right\rangle_* = \chi(\lambda=\mu) w_\mu$$

where we have use the notation $\chi(true) = 1$ and $\chi(false) = 0$. Since the Macdonald symmetric functions $\tilde{H}_\lambda$ are orthogonal with respect to this scalar product, the operator ∇ is self dual $\langle \nabla f, g\rangle_* = \langle f, \nabla g\rangle_*$. For a symmetric function $f[X]$, set $f^*[X] = f\left[\frac{X}{M}\right]$. In particular e_n^* and h_n^* will appear in many of the calculations in the remainder of this section.

Any symmetric function $F[X]$ of degree n can be expanded in the Macdonald basis by the formula

$$F[X] = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu[X;q,t]}{w_\mu} \left\langle F[X], \tilde{H}_\mu[X;q,t]\right\rangle_* \ . \tag{2}$$

Now if we define the q,t-coefficients $d_{\mu\nu}$ and $c_{\mu\nu}$ as those that appear in the Pieri formulas for Macdonald polynomials

$$e_1 \tilde{H}_\nu = \sum_{\mu \leftarrow \nu} d_{\mu\nu}(q,t)\tilde{H}_\mu \qquad e_1^\perp \tilde{H}_\mu = \sum_{\nu \rightarrow \mu} c_{\mu\nu}(q,t)\tilde{H}_\nu$$

then

$$d_{\mu\nu} = M c_{\mu\nu} \frac{w_\nu}{w_\mu} \ .$$

Moreover, for $k \geq 0$ (see [7] and [2]),

$$\sum_{\nu \rightarrow \mu} c_{\mu\nu}(q,t)(T_\mu/T_\nu)^k = \frac{tq}{M} h_{k+1}\left[D_\mu(q,t)/tq\right] + \chi(k=0)/M \tag{3}$$

$$\sum_{\mu \leftarrow \nu} d_{\mu\nu}(q,t)(T_\mu/T_\nu)^k = (-1)^{k-1} e_{k-1}\left[D_\nu(q,t)\right] + \chi(k=0) \ . \tag{4}$$

The other set of formulas that we will need are the two families of symmetric functions which are indexed by compositions that were introduced by Haglund-Morse-Zabrocki [13]. In the spirit of Jing's [15] creation operators for the Hall-Littlewood symmetric functions, they defined the operators given by their action on a symmetric function $F[X]$ by the expressions

$$\begin{aligned}\mathbb{C}_a F[X] &= (-q)^{-a+1} F\left[X - \frac{1-1/q}{z}\right] \Omega[zX]\Big|_{z^a} = (-q)^{-a+1} \sum_{r\geq 0} q^{-r} h_{a+r}[X] h_r^{q\perp} F[X] \\ \mathbb{B}_a F[X] &= F\left[X - \frac{1-q}{z}\right] \Omega[zX]\Big|_{z^a} = \sum_{r\geq 0} (-1)^r e_{a+r}[X] h_r^{q\perp} F[X] \ .\end{aligned}$$

where we we have used the notation here that $h_m^{q\perp}$ is the operator which is dual to multiplication by $h_m[(1-q)X]$ with respect to the usual Hall scalar product. They then defined a spanning set of the symmetric functions by

$$\begin{aligned}C_\alpha[X;q] &= \mathbb{C}_{\alpha_1}\mathbb{C}_{\alpha_2}\cdots\mathbb{C}_{\alpha_{\ell(\alpha)}}(1) \\ B_\alpha[X;q] &= \mathbb{B}_{\alpha_{\ell(\alpha)}}\cdots\mathbb{B}_{\alpha_2}\mathbb{B}_{\alpha_1}(1) \ .\end{aligned}$$

Haglund-Morse-Zabrocki [13] showed that $\mathbb{C}_a$, $\mathbb{B}_b$ have a simple commutation relation if $a+b>0$. However, we will need a more refined result for our recurrence since we will need to know how $\mathbb{B}_{-1}$ and $\mathbb{C}_1$ interact. To do this we compute the following commutation relation which also explains what happens in the case when $a+b\leq 0$ as well.

Proposition 2. *For $a,b\in\mathbb{Z}$,*

$$\mathbb{B}_a\mathbb{C}_b = q\mathbb{C}_b\mathbb{B}_a + (-1)^{a+b} q^{a+1}(q-1) h_{-a-b}^{q^2\perp} \ .$$

Proof. Using plethystic notation manipulations we note that $\sum_{r\geq 0} z^r h_r^{q\perp} F[X] = F[X + z(1-q)]$ and use this to compute

$$h_r^{q\perp} h_m[X] = \sum_{i\geq 0} h_{m-i}[X] h_i[1-q] h_{r-i}^{q\perp}$$

and

$$h_r^{q\perp} e_m[X] = \sum_{i\geq 0} e_{m-i}[X] e_i[1-q] h_{r-i}^{q\perp} \ .$$

$$\begin{aligned}\mathbb{B}_a\mathbb{C}_b =& (-q)^{1-b} \sum_{r\geq 0}\sum_{s\geq 0}\sum_{i\geq 0} (-1)^r q^{-s} e_{a+r} h_{b+s-i} h_i[1-q] h_{r-i}^{q\perp} h_s^{q\perp} \\ =& (-q)^{1-b} \sum_{r\geq 0}\sum_{s\geq 0} (-1)^r q^{-s} e_{a+r} h_{b+s} h_r^{q\perp} h_s^{q\perp} \\ &+ (-q)^{1-b} \sum_{i\geq 1}\sum_{r\geq 0}\sum_{s\geq 0} (-1)^{r+i} q^{-s} e_{a+r+i} h_{b+s-i}(1-q) h_r^{q\perp} h_s^{q\perp}.\end{aligned}$$

$$\begin{aligned}
q\mathbb{C}_b\mathbb{B}_a =&(-q)^{1-b}\sum_{s\geq 0}\sum_{r\geq 0}\sum_{i\geq 0}(-1)^r q^{-s+1} h_{b+s}e_{a+r-i}e_i[1-q]h_{s-i}^{q\perp}h_r^{q\perp}\\
=&(-q)^{1-b}\sum_{s\geq 0}\sum_{r\geq 0}(-1)^r q^{-s+1}e_{a+r}h_{b+s}h_s^{q\perp}h_r^{q\perp}\\
&+(-q)^{1-b}\sum_{i\geq 1}\sum_{s\geq 0}\sum_{r\geq 0}(-1)^{r+i+1}q^{-s}e_{a+r-i}h_{b+s+i}(1-q)h_s^{q\perp}h_r^{q\perp}.
\end{aligned}$$

If we take the difference between these two expressions we have

$$\begin{aligned}
\mathbb{B}_a\mathbb{C}_b - q\mathbb{C}_b\mathbb{B}_a =&(1-q)(-q)^{1-b}\sum_{s\geq 0}\sum_{r\geq 0}q^{-s}(-1)^r e_{a+r}h_{b+s}h_r^{q\perp}h_s^{q\perp}\\
&+(1-q)(-q)^{1-b}\sum_{r\geq 0}\sum_{s\geq 0}q^{-s}\left(\sum_{i\geq 1}(-1)^{r+i}e_{a+r-i}h_{b+s+i}\right)h_r^{q\perp}h_s^{q\perp}\\
&+(1-q)(-q)^{1-b}\sum_{r\geq 0}\sum_{s\geq 0}q^{-s}\left(\sum_{i\geq 1}(-1)^{r+i}e_{a+r+i}h_{b+s-i}\right)h_r^{q\perp}h_s^{q\perp}\\
=&(1-q)(-q)^{1-b}\sum_{r\geq 0}\sum_{s\geq 0}q^{-s}\left(\sum_{i\in\mathbb{Z}}(-1)^{r+i}e_{a+r-i}h_{b+s+i}\right)h_r^{q\perp}h_s^{q\perp}\ .
\end{aligned}$$

Now by shifting the sum over $i\in\mathbb{Z}$ and using that $\sum_{i\in\mathbb{Z}}(-1)^i h_i[X]e_{m-i}[X]=\chi(m=0)$ we may reduce this expression by

$$\begin{aligned}
&=(1-q)(-q)^{1-b}\sum_{r\geq 0}\sum_{s\geq 0}q^{-s}\left(\sum_{i\in\mathbb{Z}}(-1)^{r+b+s+i}e_{a+r+b+s-i}h_i\right)h_r^{q\perp}h_s^{q\perp}\\
&=(1-q)(-q)^{1-b}(-1)^a\sum_{r+s=-a-b}q^{-s}h_r^{q\perp}h_s^{q\perp}
\end{aligned}$$

Now we can use equation (1) to reduce this to the expression given in the statement of the proposition since

$$\begin{aligned}
\sum_{r+s=-a-b}q^{-s}h_r[(1-q)X]^\perp h_s[(1-q)X]^\perp =& \sum_{r+s=-a-b}q^{-s-r}h_r[(q-q^2)X]^\perp h_s[(1-q)X]^\perp\\
=&\, q^{a+b}h_{-a-b}[(1-q^2)X]^\perp\ .
\end{aligned}$$

□

The commutation relation between the $\mathbb{B}$ and $\mathbb{C}$ operators is sort of half the story which allows us to express to relate the two families of symmetric functions $C_\alpha[X;q]$ and $B_\alpha[X;q]$.

The other property which Proposition 5.2 in [13] which says

$$\mathbb{B}_n(1) = e_n[X] = \sum_{\alpha \models n} C_\alpha[X;q] \ . \tag{5}$$

For our purposes we will also need the dual operators with respect to the $*$-scalar product. If we set,

$$\mathbb{C}_a^* F[X] = (-q)^{-a+1}\, F\left[X - \epsilon M/z\right] \Omega\left[\frac{-\epsilon z X}{q(1-t)}\right]\Big|_{z^{-a}} \tag{6}$$

$$\mathbb{B}_a^* F[X] = F\left[X + M/z\right] \Omega\left[\frac{-zX}{1-t}\right]\Big|_{z^{-a}}, \tag{7}$$

then a straightforward calculation on the reproducing kernel for the $*$-scalar product, $\Omega\left[\frac{-\epsilon XY}{M}\right]$, that shows

$$\mathbb{C}_a^X \Omega\left[\frac{-\epsilon XY}{M}\right] = \mathbb{C}_a^{*Y} \Omega\left[\frac{-\epsilon XY}{M}\right] \ .$$

This implies that for symmetric functions f, g,

$$\begin{aligned} \langle \mathbb{C}_a f[X], g[X] \rangle_* &= \left\langle \left\langle \mathbb{C}_a^X \Omega\left[\frac{-\epsilon XY}{M}\right], f[Y] \right\rangle_*, g[X] \right\rangle_* \\ &= \left\langle \left\langle \mathbb{C}_a^{Y}{}^* \Omega\left[\frac{-\epsilon XY}{M}\right], g[X] \right\rangle_*, f[Y] \right\rangle_* = \langle \mathbb{C}_a^* g[Y], f[Y] \rangle_* \ . \end{aligned}$$

A similar calculation shows that $\mathbb{B}_a$ and $\mathbb{B}_a^*$ are also dual with respect to the $*$-scalar product.

2.2. **The symmetric function recursion.**

The main result of this section is the following symmetric function recursion.

Theorem 3. *For integers $n, a, b \geq 0$ and $m \geq 1$,*

$$\begin{aligned} \mathbb{C}_m^* \nabla h_a^* e_b^* e_{n-a-b}^* &= \mathbb{B}_{m-1}^* \nabla h_{a-1}^* e_b^* e_{n-a-b}^* \ + \ \mathbb{B}_{m-2}^* \nabla h_a^* e_{b-1}^* e_{n-a-b-1}^* \\ &\quad + \ \chi(m=1) \nabla h_a^* (e_{b-1}^* e_{n-a-b}^* \ + \ e_b^* e_{n-a-b-1}^*) \end{aligned} \tag{8}$$

The proof of this theorem occurs after we develop a few expressions that will be useful for our derivation. The consequences of this symmetric function identity are what will help us develop the recurrence which we show will agree with the combinatorial interpretation.

Corollary 4. *If $a + b + c = m + deg\ F[X]$,*

$$\begin{aligned} \langle \nabla \mathbb{C}_m F[X], e_a h_b h_c \rangle &= t^{m-1} \langle \nabla \mathbb{B}_{m-1} F[X], e_{a-1} h_b h_c \rangle + t^{m-1} \langle \nabla \mathbb{B}_{m-2} F[X], e_a h_{b-1} h_{c-1} \rangle \\ &\quad + \chi(m=1) \langle \nabla F[X], e_a (h_{b-1} h_c + h_b h_{c-1}) \rangle \end{aligned}$$

Proof. This result is mainly a corollary of Theorem 3. We note that since $\langle f[X], g[X]\rangle = \langle f[X], g^*[-\epsilon X]\rangle$,

$$\begin{aligned}\left\langle \nabla \mathbb{C}_m F[X], e_a h_b h_{n-a-b}\right\rangle &= \left\langle \nabla \mathbb{C}_m F[X], h^*_a e^*_b e^*_{n-a-b}\right\rangle_* \\ &= \left\langle F[X], \mathbb{C}^*_m \nabla h^*_a e^*_b e^*_{n-a-b}\right\rangle_* \\ &= \left\langle F[X], \mathbb{B}^*_{m-1} \nabla h^*_{a-1} e^*_b e^*_{n-a-b} \;+\; \mathbb{B}^*_{m-2} \nabla h^*_a e^*_{b-1} e^*_{n-a-b-1}\right\rangle_* \\ &\quad + \;\chi(m=1) \left\langle F[X], \nabla h^*_a (e^*_{b-1} e^*_{n-a-b} \;+\; e^*_b e^*_{n-a-b-1})\right\rangle_*\end{aligned}$$

and now the duality of these operators can at this point easily seen to agree with the right hand side in the statement of the theorem. □

The main consequence of this recursion is that if $F[X] = C_\alpha[X;q] = \mathbb{C}_{\alpha_1}\mathbb{C}_{\alpha_2}\cdots\mathbb{C}_{\alpha_\ell}(1)$ then we conclude the symmetric function recursion which forms the basis of the proof of Theorem 1.

Corollary 5. *Let and $a, b, c, m, n \in \mathbb{Z}$ such that $a + b + c = m + n$ and $\alpha \models n$ be a composition. If $m > 1$, then*

$$\begin{aligned}\left\langle \nabla C_{(m,\alpha)}[X;q], e_a h_b h_c\right\rangle &= \sum_{\beta\models m-1} t^{m-1} q^{\ell(\alpha)} \left\langle \nabla C_{(\alpha,\beta)}[X;q], e_{a-1} h_b h_c\right\rangle \\ &\quad + \sum_{\beta\models m-2} t^{m-1} q^{\ell(\alpha)} \left\langle \nabla C_{(\alpha,\beta)}[X;q], e_a h_{b-1} h_{c-1}\right\rangle \;.\end{aligned}$$

If $m = 1$, then

$$\begin{aligned}\left\langle \nabla C_{(1,\alpha)}[X;q], e_a h_b h_c\right\rangle =& q^{\ell(\alpha)} \left\langle \nabla C_\alpha[X;q], e_{a-1} h_b h_c\right\rangle + \left\langle \nabla C_\alpha[X;q], e_a (h_{b-1} h_c + h_b h_{c-1})\right\rangle \\ &+ (q-1) \sum_{i:\alpha_i=1} q^{i-1} \left\langle \nabla C_{\hat\alpha^{(i)}}[X;q], e_a h_{b-1} h_{c-1}\right\rangle\end{aligned}$$

where we denote $\hat\alpha^{(i)} = (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{\ell(\alpha)})$.

Proof. When $r \geq 0$, Proposition 2 have the commutation relation that $\mathbb{B}_r \mathbb{C}_a = q\mathbb{C}_a \mathbb{B}_r$ for all $a > 0$. Using equation (5) we have

$$\begin{aligned}\mathbb{B}_r(C_\alpha[X;q]) &= q^{\ell(\alpha)} \mathbb{C}_{\alpha_1}\mathbb{C}_{\alpha_2}\cdots\mathbb{C}_{\alpha_{\ell(\alpha)}} \mathbb{B}_r(1) \\ &= \sum_{\beta\models r} q^{\ell(\alpha)} \mathbb{C}_{\alpha_1}\mathbb{C}_{\alpha_2}\cdots\mathbb{C}_{\alpha_{\ell(\alpha)}} C_\beta[X;q] \\ &= \sum_{\beta\models r} q^{\ell(\alpha)} C_{(\alpha,\beta)}[X;q] \;.\end{aligned}$$

In the case when $r = -1$, then from Proposition 2, $\mathbb{B}_r\mathbb{C}_a = q\mathbb{C}_a\mathbb{B}_r$ if $a > 1$ and if $a = 1$ we have $\mathbb{B}_r\mathbb{C}_a = q\mathbb{C}_a\mathbb{B}_r + (q-1)$. It follows then that $\mathbb{B}_{-1}$ will commute with all $\mathbb{C}_{\alpha_i}$ if $\alpha_i > 1$ and introduce a power of q, but will 'kill' a term of the form $\mathbb{C}_1$. Since $\mathbb{B}_{-1}(1) = 0$, the only terms that survive this operator are those that do 'kill' a term $\mathbb{C}_1$. As a consequence,

$$\mathbb{B}_{-1}(C_\alpha[X;q]) = (q-1) \sum_{i:\alpha_i=1} q^{i-1} C_{\hat\alpha^{(i)}}[X;q] \;.$$

The result then follows directly from Corollary 4 with $F[X] = C_\alpha[X;q]$. □

Before proceeding with the proof of Theorem 3 we will need to first develop an expansion of the expression $\nabla h_a^* e_b^* e_{n-a-b}^*$ since this appears repeatedly in our calculations.

Proposition 6.

$$\nabla h_a^* e_b^* e_{n-a-b}^* = \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r}[1/M] h_{b-s}[1/M] (-1)^{n-r-s} \sum_{\nu \vdash r+s} \frac{e_r\big[B_\nu\big]}{w_\nu} e_n\big[X D_\nu / M\big] \tag{9}$$

Proof.

$$\nabla h_a^* e_b^* e_{n-a-b}^* = \sum_{\mu \vdash n} \frac{T_\mu \tilde{H}_\mu[X;q,t]}{w_\mu} \Big\langle \tilde{H}_\mu \,,\; e_a h_b h_{n-a-b} \Big\rangle \tag{10}$$

Now Theorem I.2 of [6] gives an expansion of the scalar product $\Big\langle \tilde{H}_\mu, f h_r \Big\rangle$, yielding

$$\begin{aligned} \Big\langle \tilde{H}_\mu \,,\; e_a h_b h_{n-a-b} \Big\rangle &= \nabla^{-1} h_a[(X-\epsilon)/M] e_b[(X-\epsilon)/M] \Big|_{X \to D_\mu} \\ &= \sum_{r=0}^{a} \sum_{s=0}^{b} e_{a-r}[1/M] h_{b-s}[1/M] \nabla^{-1} h_r[X/M] e_s[X/M] \Big|_{X \to D_\mu} . \end{aligned} \tag{11}$$

The same identity also shows,

$$\begin{aligned} \nabla^{-1} h_r[X/M] e_s[X/M] \Big|_{X \to D_\mu} &= \sum_{\nu \vdash r+s} \frac{T_\nu^{-1} \tilde{H}_\nu[D_\mu;q,t]}{w_\nu} \Big\langle \tilde{H}_\nu \,,\; e_r h_s \Big\rangle \\ &= (-1)^{n-r-s} \sum_{\nu \vdash r+s} \frac{T_\mu^{-1} \tilde{H}_\mu[D_\nu;q,t]}{w_\nu} e_r[B_\nu] \end{aligned} \tag{12}$$

where we have derived the expression for the scalar product

$$\begin{aligned} \Big\langle \tilde{H}_\nu \,,\; e_r h_s \Big\rangle = \nabla^{-1} h_r[(X-\epsilon)/M] \Big|_{X \to D_\nu} &= \sum_{i=0}^{r} e_{r-i} \left[\frac{1}{M}\right] \nabla^{-1} h_i[X/M] \Big|_{X \to D_\nu} \\ = \sum_{i=0}^{r} e_{r-i} \left[\frac{1}{M}\right] \sum_{\gamma \vdash i} \frac{\tilde{H}_\gamma[D_\nu]}{w_\gamma} &= \sum_{i=0}^{r} e_{r-i} \left[\frac{1}{M}\right] e_i \left[\frac{D_\nu}{M}\right] = e_r[B_\nu] \end{aligned}$$

Combining (10),(11),(12) we can thus write

$$\begin{aligned}
&\nabla h_a^* e_b^* e_{n-a-b}^* \\
&= \sum_{\mu\vdash n} \frac{T_\mu \tilde{H}_\mu[X;q,t]}{w_\mu} \sum_{r=0}^{a}\sum_{s=0}^{b} e_{a-r}\left[\frac{1}{M}\right] h_{b-s}\left[\frac{1}{M}\right](-1)^{n-r-s} \sum_{\nu\vdash r+s} \frac{T_\mu^{-1}\tilde{H}_\mu[D_\nu;q,t]}{w_\nu} e_r[B_\nu] \\
&= \sum_{r=0}^{a}\sum_{s=0}^{b} e_{a-r}[1/M] h_{b-s}[1/M](-1)^{n-r-s} \sum_{\nu\vdash r+s} \frac{e_r\left[B_\nu\right]}{w_\nu} \sum_{\mu\vdash n} \frac{\tilde{H}_\mu[X;q,t]\tilde{H}_\mu[D_\nu;q,t]}{w_\mu} \\
&= \sum_{r=0}^{a}\sum_{s=0}^{b} e_{a-r}[1/M] h_{b-s}[1/M](-1)^{n-r-s} \sum_{\nu\vdash r+s} \frac{e_r\left[B_\nu\right] e_n\left[XD_\nu/M\right]}{w_\nu}
\end{aligned}$$

□

Now two of the terms in our main theorem are $\mathbb{B}_{m-1}^*\nabla h_{a-1}^* e_b^* e_{n-a-b}^*$ and $\mathbb{B}_{m-2}^*\nabla h_a^* e_{b-1}^* e_{n-a-b}^*$ and we will need an expression for these. Essentially these are shifts of each other so we will start by computing the first one.

Proposition 7. *For* $m \geq 1$,

$$\mathbb{B}_{m-1}^*\nabla h_{a-1}^* e_b^* e_{n-a-b}^* = \sum_{r=0}^{a-1}\sum_{s=0}^{b}\sum_{u=0}^{n-m} (-1)^{n-r-s+u-1} e_{a-r-1}[1/M] h_{b-s}[1/M] e_u\left[\frac{X}{1-t}\right] \tag{13}$$

$$\times \sum_{\nu\vdash r+s} e_{n-u-m}\left[\frac{XD_\nu}{M}\right] e_{u+m-1}[D_\nu] \frac{e_r\left[B_\nu\right]}{w_\nu} \tag{14}$$

Proof. From equation (9) we have an equation for the left hand side. We need first compute $\mathbb{B}_{m-1}^*\left(e_{n-1}\left[\frac{XD_\nu}{M}\right]\right)$ using equation (7).

$$\begin{aligned}
\mathbb{B}_{m-1}^*\left(e_{n-1}\left[\frac{XD_\nu}{M}\right]\right) &= e_{n-1}\left[\frac{(X+M/z)D_\nu}{M}\right]\Omega\left[\frac{-zX}{1-t}\right]\Big|_{z^{-m+1}} \\
&= \sum_{u=0}^{n-1} e_{n-1-u}\left[\frac{XD_\nu}{M}\right] e_u[D_\nu]\Omega\left[\frac{-zX}{1-t}\right]\Big|_{z^{u-m+1}} \\
&= \sum_{u=0}^{n-1} e_{n-u}\left[\frac{XD_\nu}{M}\right] e_u[D_\nu] h_{u-m+1}\left[\frac{-X}{1-t}\right] \\
&= \sum_{u=m-1}^{n-1} (-1)^{u-m+1} e_{n-1-u}\left[\frac{XD_\nu}{M}\right] e_u[D_\nu] e_{u-m+1}\left[\frac{X}{1-t}\right] \\
&= \sum_{u=0}^{n-m} (-1)^{u} e_{n-u-m}\left[\frac{XD_\nu}{M}\right] e_{u+m-1}[D_\nu] e_u\left[\frac{X}{1-t}\right] \qquad (15)
\end{aligned}$$

So now the expression stated in the proposition for $\mathbb{B}^*_{m-1}\nabla h^*_{a-1}e^*_b e^*_{n-a-b}$ follows by combining equation (15) and (9). $\square$

We can also calculate $B^*_{m-2}\nabla h^*_a e^*_{b-1}e^*_{n-a-b-1}$ by replacing $a \to a+1$, $b \to b-1$, $n \to n-1$, $m \to m-1$ in equation (13)-(14).

$$B^*_{m-2}\nabla h^*_a e^*_{b-1}e^*_{n-a-b-1} = \sum_{r=0}^{a}\sum_{s=0}^{b-1}\sum_{u=0}^{n-m}(-1)^{n-r-s+u}e_{a-r}[1/M]h_{b-1-s}[1/M]e_u\left[\frac{X}{1-t}\right]$$

$$\times\sum_{\nu\vdash r+s}e_{n-u-m}\left[\frac{XD_\nu}{M}\right]e_{u+m-2}[D_\nu]\frac{e_r[B_\nu]}{w_\nu} \tag{16}$$

Proof. (of Theorem 3) To develop the left hand side of our identity we begin again with Proposition 6.

(17)

$$\mathbb{C}^*_m\nabla h^*_a e^*_b e^*_{n-a-b} = \sum_{r=0}^{a}\sum_{s=0}^{b}e_{a-r}[1/M]h_{b-s}[1/M](-1)^{n-r-s}\sum_{\nu\vdash r+s}\frac{e_r[B_\nu]}{w_\nu}\mathbb{C}^*_m e_n[XD_\nu/M]$$

Now to proceed further we will calculate the action of $\mathbb{C}^*_m$ on $e_n[XD_\nu/M]$. Recall that the definition of $\mathbb{C}^*_m$ from equation (6),

$$\begin{aligned}
\mathbb{C}^*_m e_n\left[\frac{XD_\nu}{M}\right] &= (-q)^{-m+1}e_n[(X-\epsilon M/z)D_\nu/M]\Omega\left[\frac{-\epsilon zX}{q(1-t)}\right]\Big|_{z^{-m}} \\
&= (-q)^{-m+1}e_n[XD_\nu/M-\epsilon D_\nu/z]\Omega\left[\frac{-\epsilon zX}{q(1-t)}\right]\Big|_{z^{-m}} \\
&= (-q)^{-m+1}\sum_{u=0}^{n}e_{n-u}[XD_\nu/M]e_u[-\epsilon D_\nu/z]\Omega\left[\frac{-\epsilon zX}{q(1-t)}\right]\Big|_{z^{-m}} \\
&= (-q)^{-m+1}\sum_{u=0}^{n}e_{n-u}[XD_\nu/M]h_u[D_\nu]z^{-u}\Omega\left[\frac{-\epsilon zX}{q(1-t)}\right]\Big|_{z^{-m}} \\
&= (-q)^{-m+1}\sum_{u=0}^{n}e_{n-u}[XD_\nu/M]h_u[D_\nu]h_{u-m}\left[\frac{-\epsilon X}{q(1-t)}\right] \\
&= (-q)^{-m+1}\sum_{u=m}^{n}e_{n-u}[XD_\nu/M]h_u[D_\nu]q^{m-u}e_{u-m}\left[\frac{X}{1-t}\right] \\
&= (-q)^{-m+1}\sum_{u=0}^{n-m}q^{-u}e_{n-m-u}[XD_\nu/M]h_{u+m}[D_\nu]e_u\left[\frac{X}{1-t}\right]
\end{aligned} \tag{18}$$

Now we apply the identity (3) for $h_{u+m}[D_\nu]$ so that

$$h_k[D_\nu] = Mt^{k-1}q^{k-1}\sum_{\gamma\to\nu}c_{\nu\gamma}\left(T_\nu/T_\gamma\right)^{k-1} - \chi(k=1)$$

and substituting into the previous equation we have

$$
\begin{aligned}
&= (-q)^{-m+1} \sum_{u=0}^{n-m} q^{-u} e_{n-m-u}\left[\frac{XD_\nu}{M}\right] e_u\left[\frac{X}{1-t}\right] (Mt^{u+m-1} q^{u+m-1} \sum_{\gamma\to\nu} c_{\nu\gamma}\left(\frac{T_\nu}{T_\gamma}\right)^{u+m-1} \\
&\qquad - \chi(u+m=1)) \\
&= (-1)^{m-1} \sum_{u=0}^{n-m} e_{n-m-u}\left[\frac{XD_\nu}{M}\right] e_u\left[\frac{X}{1-t}\right] (Mt^{u+m-1} \sum_{\gamma\to\nu} c_{\nu\gamma}\left(\frac{T_\nu}{T_\gamma}\right)^{u+m-1} \\
&\qquad\qquad - \chi(u+m=1)) \\
&= (-1)^{m-1} \sum_{u=0}^{n-m} e_{n-m-u}\left[\frac{XD_\nu}{M}\right] e_u\left[\frac{X}{1-t}\right] Mt^{u+m-1} \sum_{\gamma\to\nu} c_{\nu\gamma}\left(\frac{T_\nu}{T_\gamma}\right)^{u+m-1} \\
&\qquad\qquad - \chi(m=1) e_{n-1}\left[\frac{XD_\nu}{M}\right]
\end{aligned}
$$

Substituting this back into equation (17), we have

$$
\begin{aligned}
&\mathbb{C}_m^* \nabla h_a^* e_b^* e_{n-a-b}^* \\
&\quad = \sum_{r=0}^{a}\sum_{s=0}^{b} e_{a-r}[1/M] h_{b-s}[1/M] (-1)^{n-r-s+m-1} \sum_{\nu\vdash r+s} \frac{e_r[B_\nu]}{w_\nu} \sum_{u=0}^{n-m} e_{n-m-u}[XD_\nu/M] \\
&\qquad\quad \times e_u\left[\frac{X}{1-t}\right] Mt^{u+m-1} \sum_{\gamma\to\nu} c_{\nu\gamma}\left(T_\nu/T_\gamma\right)^{u+m-1} \\
&\qquad\quad - \chi(m=1) \sum_{r=0}^{a}\sum_{s=0}^{b} e_{a-r}[1/M] h_{b-s}[1/M] (-1)^{n-r-s} \sum_{\nu\vdash r+s} \frac{e_r[B_\nu]}{w_\nu} e_{n-1}[XD_\nu/M]
\end{aligned}
\tag{19}
$$

Notice that (19) is exactly (9) with n replaced by $n-1$ hence we have

$$
\begin{aligned}
&\mathbb{C}_m^* \nabla h_a^* e_b^* e_{n-a-b}^* - \chi(m=1) \nabla h_a^* e_b^* e_{n-1-a-b}^* \\
&\quad = \sum_{r=0}^{a}\sum_{s=0}^{b} e_{a-r}[1/M] h_{b-s}[1/M] (-1)^{n-r-s+m-1} \sum_{\nu\vdash r+s} \frac{e_r[B_\nu]}{w_\nu} \sum_{u=0}^{n-m} e_{n-m-u}[XD_\nu/M] \\
&\qquad\quad \times e_u\left[\frac{X}{1-t}\right] Mt^{u+m-1} \sum_{\gamma\to\nu} c_{\nu\gamma}\left(T_\nu/T_\gamma\right)^{u+m-1}
\end{aligned}
$$

Now the coefficients $c_{\mu\nu}$ and $d_{\mu\nu}$ are related by the identity $w_\nu d_{\nu\gamma} = M c_{\nu\gamma} w_\gamma$. We will do a few intermediate reductions before replacing the $c_{\nu\gamma}$ coefficients:

$$
\begin{aligned}
&= \sum_{r=0}^{a}\sum_{s=0}^{b}\sum_{u=0}^{n-m} e_{a-r}[1/M]h_{b-s}[1/M](-1)^{n-r-s+m-1}e_u\left[\frac{X}{1-t}\right]t^{u+m-1} \\
&\qquad \times \sum_{\nu\vdash r+s}\frac{e_r\left[B_\nu\right]}{w_\nu}e_{n-m-u}[XD_\nu/M]M\sum_{\gamma\to\nu}c_{\nu\gamma}\left(T_\nu/T_\gamma\right)^{u+m-1} \\
&= \sum_{r=0}^{a}\sum_{s=0}^{b}\sum_{u=0}^{n-m} e_{a-r}[1/M]h_{b-s}[1/M](-1)^{n-r-s+m-1}e_u\left[\frac{X}{1-t}\right]t^{u+m-1} \\
&\qquad \times \sum_{\gamma\vdash r+s-1}\frac{1}{w_\gamma}\sum_{\nu\leftarrow\gamma}e_r\left[B_\nu\right]e_{n-m-u}[XD_\nu/M]d_{\nu\gamma}\left(T_\nu/T_\gamma\right)^{u+m-1}
\end{aligned}
\tag{20}
$$

But now $e_r[X+z]=e_r[X]+ze_{r-1}[X]$ and $B_\nu=B_\gamma+T_\nu/T_\gamma$ and $XD_\nu/M=XD_\gamma/M+XT_\nu/T_\gamma$.

Now we will use identity (4) and expand the last part of equation (20) first.

$$
\begin{aligned}
&\sum_{\nu\leftarrow\gamma}e_r\left[B_\nu\right]e_{n-m-u}\left[\frac{XD_\gamma}{M}+X\frac{T_\nu}{T_\gamma}\right]d_{\nu\gamma}\left(\frac{T_\nu}{T_\gamma}\right)^{u+m-1} \\
&= \sum_{i=0}^{n-m-u}e_i[X]\sum_{\nu\leftarrow\gamma}\left(e_r\left[B_\gamma\right]+\frac{T_\nu}{T_\gamma}e_{r-1}[B_\gamma]\right)e_{n-m-u-i}\left[\frac{XD_\gamma}{M}\right]d_{\nu\gamma}\left(\frac{T_\nu}{T_\gamma}\right)^{u+m+i-1} \\
&= \sum_{i=0}^{n-m-u}e_i[X]\Bigg(e_r\left[B_\gamma\right]e_{n-m-u-i}\left[\frac{XD_\gamma}{M}\right]\sum_{\nu\leftarrow\gamma}d_{\nu\gamma}\left(\frac{T_\nu}{T_\gamma}\right)^{u+m+i-1} \\
&\qquad + e_{r-1}[B_\gamma]e_{n-m-u-i}\left[\frac{XD_\gamma}{M}\right]\sum_{\nu\leftarrow\gamma}d_{\nu\gamma}\left(\frac{T_\nu}{T_\gamma}\right)^{u+m+i}\Bigg) \\
&= \sum_{i=0}^{n-m-u}e_i[X]e_r\left[B_\gamma\right]e_{n-m-u-i}\left[\frac{XD_\gamma}{M}\right](-1)^{u+m+i}e_{u+m+i-2}\left[D_\gamma\right]+ \\
&\qquad + \chi(m=1,u=0)e_r\left[B_\gamma\right]e_{n-1}\left[\frac{XD_\gamma}{M}\right] \\
&\qquad + \sum_{i=0}^{n-m-u}e_i[X]e_{r-1}[B_\gamma]e_{n-m-u-i}\left[\frac{XD_\gamma}{M}\right](-1)^{u+m+i-1}e_{u+m+i-1}\left[D_\gamma\right]
\end{aligned}
$$

Now if we substitute this back into (20) we have:

$$
\begin{aligned}
&\mathbb{C}_m^*\nabla h_a^* e_b^* e_{n-a-b}^* - \chi(m=1)\nabla h_a^* e_b^* e_{n-1-a-b}^* \\
&= \sum_{r=0}^{a}\sum_{s=0}^{b}\sum_{u=0}^{n-m}\sum_{i=0}^{n-m} e_i[X]e_{a-r}[1/M]h_{b-s}[1/M](-1)^{n-r-s+u+i-1} e_u\left[\frac{X}{1-t}\right] t^{u+m-1} \\
&\qquad \times \sum_{\gamma\vdash r+s-1} \frac{1}{w_\gamma} e_r\left[B_\gamma\right] e_{n-m-u-i}[XD_\gamma/M] e_{u+m+i-2}\left[D_\gamma\right] \\
&+ \sum_{r=0}^{a}\sum_{s=0}^{b}\sum_{u=0}^{n-m}\sum_{i=0}^{n-m} e_i[X]e_{a-r}[1/M]h_{b-s}[1/M](-1)^{n-r-s+u+i} e_u\left[\frac{X}{1-t}\right] t^{u+m-1} \\
&\qquad \times \sum_{\gamma\vdash r+s-1} \frac{1}{w_\gamma} e_{r-1}[B_\gamma] e_{n-m-u-i}[XD_\gamma/M] e_{u+m+i-1}\left[D_\gamma\right]
\end{aligned}
\tag{21}
$$

$$
+\chi(m=1)\sum_{r=0}^{a}\sum_{s=0}^{b} e_{a-r}[1/M]h_{b-s}[1/M] \sum_{\gamma\vdash r+s-1} \frac{1}{w_\gamma}(-1)^{n-r-s} e_r\left[B_\gamma\right] e_{n-1}[XD_\gamma/M]
\tag{22}
$$

Notice now that (22) does not quite align with (9), but as it turns out it is only off by something that is equal to 0. Consider equation (22) and break off the break off the $s = 0$ term and change the sum so that $s \to s + 1$.

$$
\sum_{r=0}^{a}\sum_{s=0}^{b} e_{a-r}[1/M]h_{b-s}[1/M] \sum_{\gamma\vdash r+s-1} (-1)^{n-r-s}\frac{e_r\left[B_\gamma\right]}{w_\gamma} e_{n-1}[XD_\gamma/M]
\tag{23}
$$

$$
= \sum_{r=0}^{a}\sum_{s=0}^{b-1} e_{a-r}[1/M]h_{b-s-1}[1/M] \sum_{\gamma\vdash r+s} (-1)^{n-r-s-1}\frac{e_r\left[B_\gamma\right]}{w_\gamma} e_{n-1}[XD_\gamma/M]
\tag{24}
$$

$$
+ \sum_{r=0}^{a} e_{a-r}[1/M]h_b[1/M] \sum_{\gamma\vdash r-1} (-1)^{n-r}\frac{e_r\left[B_\gamma\right]}{w_\gamma} e_{n-1}[XD_\gamma/M]
\tag{25}
$$

$$
= \nabla h_a^* e_{b-1}^* e_{n-a-b}^*
\tag{26}
$$

This is because (25) is 0 because $e_r[B_\gamma] = 0$ when $\gamma \vdash r - 1$. Therefore, what we have shown (21)-(22) may be expressed by moving the $\chi(m = 1)$ terms on the left hand side of the equation and shifting the sum of $0 \le i \le n - m$ and $0 \le u \le n - m$ by noticing that $u \le n - m - i$ and then it is possible to shift $u \to u - i$.

$$
\begin{aligned}
&\mathbb{C}_m^* \nabla h_a^* e_b^* e_{n-a-b}^* - \chi(m=1)(\nabla h_a^* e_b^* e_{n-1-a-b}^* + \nabla h_a^* e_{b-1}^* e_{n-a-b}^*)\\
&\quad = \sum_{r=0}^{a}\sum_{s=0}^{b}\sum_{i=0}^{n-m}\sum_{u=i}^{n-m} e_i[X]e_{a-r}[1/M]h_{b-s}[1/M](-1)^{n-r-s+u-1}e_{u-i}\left[\frac{X}{1-t}\right]t^{u-i+m-1}\\
&\qquad\quad \times \sum_{\gamma\vdash r+s-1}\frac{1}{w_\gamma}e_r\big[B_\gamma\big]e_{n-m-u}[XD_\gamma/M]e_{u+m-2}\big[D_\gamma\big]\\
&\quad + \sum_{r=0}^{a}\sum_{s=0}^{b}\sum_{i=0}^{n-m}\sum_{u=i}^{n-m} e_i[X]e_{a-r}[1/M]h_{b-s}[1/M](-1)^{n-r-s+u}e_{u-i}\left[\frac{X}{1-t}\right]t^{u-i+m-1}\\
&\qquad\quad \times \sum_{\gamma\vdash r+s-1}\frac{1}{w_\gamma}e_{r-1}[B_\gamma]e_{n-m-u}[XD_\gamma/M]e_{u+m-1}\big[D_\gamma\big]
\end{aligned}
$$

The sums with $0 \leq i \leq n-m$ and $i \leq u \leq n-m$ implies that $0 \leq u \leq n-m$ and $0 \leq i \leq u$ and hence the part of the expression with $\sum_{i=0}^{u} e_i[X]e_{u-i}[tX/(1-t)] = e_u[X/(1-t)]$.

$$= t^{m-1}\sum_{r=0}^{a}\sum_{s=0}^{b}\sum_{u=0}^{n-m} e_{a-r}[1/M]h_{b-s}[1/M](-1)^{n-r-s+u-1}e_u\left[\frac{X}{1-t}\right] \tag{27}$$

$$\times \sum_{\gamma\vdash r+s-1}\frac{1}{w_\gamma}e_r\big[B_\gamma\big]e_{n-m-u}[XD_\gamma/M]e_{u+m-2}\big[D_\gamma\big] \tag{28}$$

$$+ t^{m-1}\sum_{r=0}^{a}\sum_{s=0}^{b}\sum_{u=0}^{n-m} e_{a-r}[1/M]h_{b-s}[1/M](-1)^{n-r-s+u}e_u\left[\frac{X}{1-t}\right] \tag{29}$$

$$\times \sum_{\gamma\vdash r+s-1}\frac{1}{w_\gamma}e_{r-1}[B_\gamma]e_{n-m-u}[XD_\gamma/M]e_{u+m-1}\big[D_\gamma\big] \tag{30}$$

Now in equation (27)-(28) the $s=0$ term is 0 because $e_r[B_\gamma] = 0$ for all $\gamma \vdash r-1$ and in (29)-(30) the $r=0$ term is 0 since $e_{r-1} = 0$. These are dropped from the sum and the corresponding sum on r and s are shifted to align with the expression in Proposition 7.

$$
\begin{aligned}
&= t^{m-1}\sum_{r=0}^{a}\sum_{s=0}^{b-1}\sum_{u=0}^{n-m} e_{a-r}[1/M]h_{b-s-1}[1/M](-1)^{n-r-s+u}e_u\left[\frac{X}{1-t}\right]\\
&\qquad\times\sum_{\gamma\vdash r+s}\frac{1}{w_\gamma}e_r\left[B_\gamma\right]e_{n-m-u}[XD_\gamma/M]e_{u+m-2}\left[D_\gamma\right]\\
&+ t^{m-1}\sum_{r=0}^{a-1}\sum_{s=0}^{b}\sum_{u=0}^{n-m} e_{a-r-1}[1/M]h_{b-s}[1/M](-1)^{n-r-s+u+1}e_u\left[\frac{X}{1-t}\right]\\
&\qquad\times\sum_{\gamma\vdash r+s}\frac{1}{w_\gamma}e_r[B_\gamma]e_{n-m-u}[XD_\gamma/M]e_{u+m-1}\left[D_\gamma\right]\\
&= \mathbb{B}^*_{m-2}\nabla h^*_a e^*_{b-1}e^*_{n-a-b-1} \;+\; \mathbb{B}^*_{m-1}\nabla h^*_{a-1}e^*_b e^*_{n-a-b}
\end{aligned}
$$

□

3. The combinatorial recursion

In the remainder of the paper we will establish that the recurrence from Corollary 5 implies Theorem 1.

Remark 8. The colored paths that we use for our combinatorial interpretation are actually just a different way of viewing parking functions. A parking function is a Dyck path of length n along with a labeling of the vertical edges of the path with the numbers $\{1, 2, \ldots, n\}$ such that the labels increase along the consecutive vertical edges from the bottom of the path to the top.

The reading word of a parking function is a reading of the labels along the diagonals starting with the diagonal farthest away from the main diagonal and moving towards the $x = y$ diagonal. The reading along each diagonal starts at the top of the diagram. For instance in the example below the reading word of the parking function is 327415896.

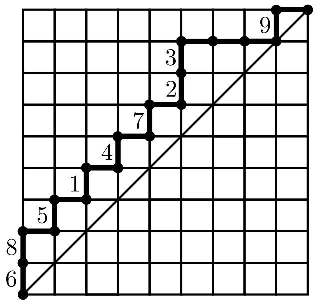

Figure 4. A typical parking function with a reading word that is a shuffle of 321, 456 and 789.

The paths we consider for our combinatorial interpretation are simply parking functions whose reading word is a shuffle of the words $(a(a-1)\cdots 21)$, $((a+1)(a+2)\cdots(a+b))$

and $((a+b+1)(a+b+2)\cdots(a+b+c))$. To see the bijection we replace the edges with labels between $a+1$ and $a+b$ by diagonal red edges and the labels between $a+b+1$ and $a+b+c$ with diagonal blue edges and such that the slopes of those edges are equal to 2 iff a red edge is on top of a blue edge.

In our example above, the reading word is a shuffle of 321, 456 and 789 and by replacing the edges labeled with 456 by red and those labled 789 by blue we obtain precisely the colored path that appears in the introduction.

Proof. (of Theorem 1) For integers a, b, c and a composition $\alpha \models a+b+c$ we let $\Gamma_{\alpha}^{(a,b,c)}$ be the set of paths with a black vertical steps, and b red diagonal steps and c blue diagonal steps of the type in figure 1 and with touch composition equal to α.

Assume by induction that

$$\langle \nabla C_{\beta}[X;q], e_r h_s h_t \rangle = \sum_{P \in \Gamma_{\beta}^{(r,s,t)}} q^{dinv(P)} t^{area(P)}$$

is true for all compositions $\beta \models r+s+t < a+b+c$.

We consider $(m,\alpha) \models a+b+c$ for two cases, $m>1$ and $m=1$.

Case 1: $m > 1$.

By Corollary 5 we have for $m>1$,

$$\begin{aligned}
&\left\langle \nabla C_{(m,\alpha)}[X;q], e_a h_b h_c \right\rangle \\
&= t^{m-1} q^{\ell(\alpha)} \left(\sum_{\beta \models m-1} \left\langle \nabla C_{(\alpha,\beta)}[X;q], e_{a-1} h_b h_c \right\rangle + \sum_{\beta \models m-2} \left\langle \nabla C_{(\alpha,\beta)}[X;q], e_a h_{b-1} h_{c-1} \right\rangle \right) \\
&= \sum_{P'} q^{dinv(P')+\ell(\alpha)} t^{area(P')+m-1}
\end{aligned}$$

where the sum is over all P' that are in the set

$$\Gamma' := \bigcup_{\beta \models m-1} \Gamma_{(\alpha,\beta)}^{(a-1,b,c)} \cup \bigcup_{\beta \models m-2} \Gamma_{(\alpha,\beta)}^{(a,b-1,c-1)}.$$

In this case there is a bijection between Γ' and $\Gamma_{(m,\alpha)}^{(a,b,c)}$. We describe the correspondence starting with a P in $\Gamma_{(m,\alpha)}^{(a,b,c)}$. In this case either the first step of P is a black step N or it begins with a red/blue NNE step. We refer to the part of the path from $(0,0)$ to (m,m) as the 'first part' and the part from (m,m) to (n,n) as the 'second part.'

If P begins with a black step N then we let P' be the path formed by removing the first vertical step and the last horizontal step from the first part and moving this remaining piece to the end of the path. We remark that $P' \in \bigcup_{\beta \models m-1} \Gamma_{(\alpha,\beta)}^{(a-1,b,c)}$ and it is easy to see that $area(P) = area(P') + m - 1$. The first step in P will contribute one for each time that the second part of the path touches the diagonal. If pairs that are not in the first row contribute towards the $dinv(P)$, then they are either both in the first part, or both in the second part then they are pairs that also contribute to $dinv(P')$. If one is in the first part and the other is in the second part then those pairs which contribute to $dinv_1(P)$ then contribute to $dinv_2(P')$, and those that contribute to $dinv_2(P)$ then contribute to $dinv_1(P')$. In summary, we have that $dinv(P) = dinv(P') + \ell(\alpha)$.

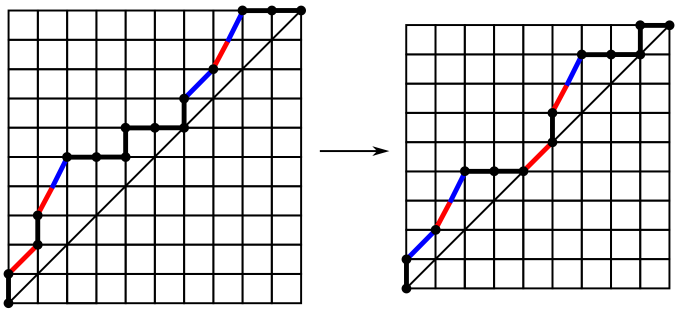

FIGURE 5. A transformation from P to P' when the first step is vertical. In this case $dinv(P) = |\{(1,7),(2,8),(3,6),(3,8),(3,9),(4,10),(6,8),(6,9)\} \cup \{(2,7),(3,7),(4,6),(6,7)\}| = 12$, $area(P) = 12$ and $dinv(P') = |\{(1,5),(1,6),(1,9),(6,9)\} \cup \{(2,5),(2,6),(2,9),(3,6),(3,9),(4,7),(7,9)\}| = 11$, $area(P') = 7$.

If P begins with a red/blue NNE step, then set P' as the path where the NNE step as well as the last horizontal step before the path touches the diagonal are deleted and the remainder is placed at the end of the second part of the path. In this case $P' \in \bigcup_{\beta \models m-2} \Gamma^{(a,b-1,c-1)}_{(\alpha,\beta)}$ since we have deleted both a blue and a red step and the first part of the path now may touch anywhere. The same reasoning in the previous case shows that $area(P) = area(P') + m - 1$ and $dinv(P) = dinv(P') + \ell(\alpha)$.

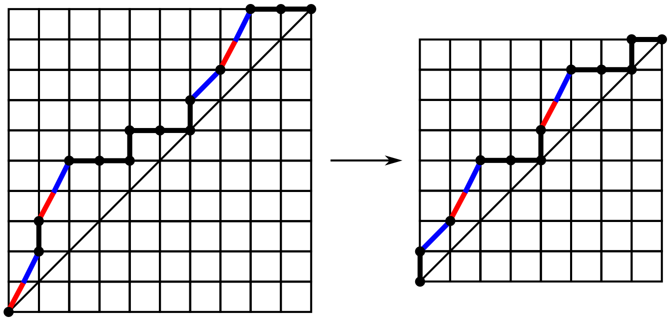

FIGURE 6. A transformation from P to P' when the first step is a NNE red/blue step. In in this case, $dinv(P) = |\{(3,6),(3,8),(3,9),(4,10),(6,8),(6,9)\} \cup \{(2,7),(3,7),(4,6),(6,7)\}| = 10$, $area(P) = 12$ and $dinv(P') = |\{(1,5),(1,8),(5,8)\} \cup \{(2,5),(2,8),(3,5),(3,8),(4,6),(6,8)\}| = 9$, $area(P') = 7$.

The procedure taking P to P' that we have described is a bijection from $\Gamma^{(a,b,c)}_{(m,\alpha)}$ to Γ' and hence we conclude

$$\sum_{P'\in\Gamma'} q^{dinv(P')+\ell(\alpha)} t^{area(P')+m-1} = \sum_{P\in\Gamma^{(a,b,c)}_{(m,\alpha)}} q^{dinv(P)} t^{area(P)} \,.$$

Case 1: $m=1$.

By Corollary 5 we have an expression for $\left\langle C_{(1,\alpha)}, e_a h_b h_c \right\rangle$.

$$\begin{aligned}
&\left\langle \nabla C_{(1,\alpha)}[X;q], e_a h_b h_c \right\rangle\\
&\quad = q^{\ell(\alpha)} \left\langle \nabla C_\alpha[X;q], e_{a-1} h_b h_c \right\rangle + \left\langle \nabla C_\alpha[X;q], e_a(h_{b-1}h_c + h_b h_{c-1}) \right\rangle\\
&\qquad + (q-1) \sum_{i:\alpha_i=1} q^{i-1} \left\langle \nabla C_{\hat\alpha^{(i)}}[X;q], e_a h_{b-1} h_{c-1} \right\rangle
\end{aligned}$$

$$= \sum_{P'\in\Gamma^{(a-1,b,c)}_{\alpha}} q^{dinv(P')+\ell(\alpha)} t^{area(P')} + \sum_{P'\in\Gamma^{(a,b,c-1)}_{\alpha}} q^{dinv(P')} t^{area(P')} \tag{31}$$

$$+ \sum_{P'\in\Gamma^{(a,b-1,c)}_{\alpha}} q^{dinv(P')} t^{area(P')} + (q-1) \sum_{i:\alpha_i=1} \sum_{P'\in\Gamma^{(a,b-1,c-1)}_{\hat\alpha^{(i)}}} q^{dinv(P')+i-1} t^{area(P')} \,. \tag{32}$$

Now consider a path in $\Gamma^{(a,b,c)}_{(1,\alpha)}$ that must begin with either a black vertical followed by a horizontal, a red diagonal or a blue diagonal. Let P' be the path formed by deleting the first step from P. If P begins with a black vertical step, then $dinv(P) = dinv(P') + \ell(\alpha)$. If P begins with a red step then $dinv(P) = dinv(P') +$ the number of blue steps on the diagonal of P (which may only occur in positions where $\alpha_i = 1$). If P begins with a blue step then $dinv(P) = dinv(P')$.

Say that there are r blue steps in the diagonal of P' in positions $i_1, i_2, \ldots, i_r$ and let $P^{i_1}, P^{i_2}, \ldots, P^{i_r}$ be the paths with the blue step at position i_d deleted from P'. The blue step contributes to the $dinv$ in P' for each step in the diagonal which is not blue so $dinv(P^{i_d}) = dinv(P') - (i_d - d)$. We note that

$$\begin{aligned}
&q^{dinv(P')} + (q-1)q^{dinv(P^{i_1})+i_1-1} + (q-1)q^{dinv(P^{i_2})+i_2-1} + \cdots + (q-1)q^{dinv(P^{i_r})+i_r-1}\\
&\quad = q^{dinv(P')} + (q-1)q^{dinv(P')} + (q-1)q^{dinv(P')+1} + \cdots + (q-1)q^{dinv(P')+r-1}\\
&\quad = q^{dinv(P')+r} = q^{dinv(P)}
\end{aligned}$$

We can conclude the two sums which appear in (32) reduces to

$$\sum_{P'\in\Gamma^{(a,b-1,c)}_{\alpha}} q^{dinv(P')+r(P')} t^{area(P')}$$

where $r(P')$ is the number of blue steps which occur in the diagonal of P'. Therefore (31)-(32) sum up to

$$\sum_{P\in\Gamma^{(a,b,c)}_{(1,\alpha)}} q^{dinv(P)} t^{area(P)}$$

By induction it therefore follows that

$$\langle \nabla(C_\alpha[X;q]), e_a h_b h_c \rangle = \sum_{P \in \Gamma_\alpha^{(a,b,c)}} q^{dinv(P)} t^{area(P)}$$

for all $a + b + c \geq 0$. □

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