This worksheet is an implementation of the product rules that are in “Products of characters of the symmetric group” (arXiv:1709.08098) by Rosa Orellana and Mike Zabrocki.

**Abstract:** In a recent paper (arXiv:1605.06672), the authors introduced a new basis of the ring of symmetric functions which evaluate to the irreducible characters of the symmetric group at roots of unity. The structure coefficients for this new basis are the stable Kronecker coefficients. In this paper we give combinatorial descriptions for several products that have as consequences several versions of the Pieri rule for this new basis of symmetric functions. In addition, we give several applications of the products studied in this paper.

In this worksheet we will compute a few examples of the three main theorems of the paper. These theorems provide a combinatorial rule for the product of characters of the symmetric group as symmetric functions in terms of certain types of multiset tableaux. The coefficients represent the multiplicity of the irreducibles in a tensor product of symmetric group modules.

This worksheet is available in both pdf (http://garsia.math.yorku.ca/~zabrocki/papers/stbasisPieriTableauxStructure.pdf) and jupyter notebook (http://garsia.math.yorku.ca/~zabrocki/papers/stbasisPieriTableauxStructure.ipynb) form.

Note that the majority of the tableaux and multiset partition code that makes these functions run has not been integrated into Sage, but for the time being I am making these programs available on my website (subject to update and change):


Programs for calculating with the symmetric functions $s_\gamma$ and $h_\gamma$ is part of SymmetricFunctions code as a part of Sage.

In [1]: load("http://garsia.math.yorku.ca/~zabrocki/mpy")
   load("http://garsia.math.yorku.ca/~zabrocki/mst.py")

Loading MultiSetPartition, SetPartitionsOfMultiSet, MultiSetPartitions and SetPartitionsOfMultiSet
BiSetPartition, BiSetPartitions, MCPartition, MCPartitions (v 5.0 date: Jan 7, 2018)
grlex_msp, lex_msp, gr_rev_lex_msp, rev_lex_msp, rev_gr_revlex_msp
Loading MultiSetTableau, MultiSetTableaux, SetTableau, SetTableaux,
BiSetTableau and BiSetTableaux, MCTableau, MCTableaux (v 4.0 date: Jan 7, 2018)

In [2]: SymmetricFunctions(QQ).inject_shorthands('all')

Defining e as shorthand for Symmetric Functions over Rational Field
   in the elementary basis
Defining f as shorthand for Symmetric Functions over Rational Field
   in the forgotten basis
Defining h as shorthand for Symmetric Functions over Rational Field
   in the homogeneous basis
Defining ht as shorthand for Symmetric Functions over Rational Field
   in the induced trivial character basis
Defining m as shorthand for Symmetric Functions over Rational Field
   in the monomial basis
Defining o as shorthand for Symmetric Functions over Rational Field
   in the orthogonal basis
Defining p as shorthand for Symmetric Functions over Rational Field
   in the powersum basis
Defining s as shorthand for Symmetric Functions over Rational Field
   in the Schur basis
Defining sp as shorthand for Symmetric Functions over Rational Field
   in the symplectic basis
Defining st as shorthand for Symmetric Functions over Rational Field
   in the irreducible symmetric group character basis
Defining w as shorthand for Symmetric Functions over Rational Field
   in the Witt basis

In [ ]:

**Definition 1.** Let $\gamma$ be a partition, $\alpha$ and $\beta$ be compositions. Then the set $\mathcal{MCT}_\gamma(\alpha, \beta)$ contains tableaux $T$ that are column strict with respect to the reverse lex order, have shape $((\gamma_1, \ldots, \gamma_r))$, and content $\{1^{\gamma_1}, 2^{\gamma_2}, \ldots, r^{\gamma_r}, 1^\beta, 2^\beta, \ldots, k^\beta\}$ with at most one barred entry in each cell. Further, we require that cells in the first row cannot be filled with multisets containing only barred entries.
In [3]: MCTableaux([3,2],[3,3],[2,1,1],rlex_bsp)
Out[3]: multiset common tableaux with shape \{x\}+[3, 2] and content vectors {3, 3} and \{2, 1, 1\}

In [4]: #optional: At least some of the tableaux options are followed by this class (including 'convention')
    Tableaux.options(convention='French')

In [5]: MCTableaux([3,2],[3,3],[2,1,1],rlex_bsp).cardinality()
Out[5]: 2453

In [6]: MCTableaux([3,2],[3,3],[2,1,1],rlex_bsp).first().pp()
   2 2
   -1 -1 -2
   . . 1 1 1 2

In [7]: # Here is an example of a set of tableaux that can be computed for this class
    for T in MCTableaux([1,1],[2],[1,1],rlex_bsp):
      T.pp()
-2
-1
. 1 1
-2
-1
. 11
-11
-2
. 1
1
-2
. -11
-111
-2
.
-21
-1
. 1
1
-1
. -21
-211
-1
.
-21
-11
.

In [ ]:

In [ ]:

**Definition 3.** If \( T \in \mathcal{MCT}_3(\lambda, \mu) \) for some partitions \( \lambda, \mu \) and \( \gamma \), then let \( S_1 < S_2 < \ldots < S_4 \) be the multisets of the unbarred entries that appear in \( T \) (ignoring the barred entries), then we say that \( T \) is a **lattice tableau** if the word

\[
\text{read}(T)_{S_1}\text{read}(T)_{S_2}\text{read}(T)_{S_3}\ldots\text{read}(T)_{S_4}
\]

is lattice.

In [8]: Tlist = MCTableaux([3,2],[2],[2,2],rlex_bsp).list()
In [9]: len(Tlist)
Out[9]: 10
In [10]:

    # some examples of the negative reading word is NOT lattice
    for T in Tlist:
        if not T.neg_reading_word_is_lattice():
            T.pp()
            print "reading word = ", T.neg_reading_word()

-2 1
-1 -1 -2
     1
reading word = [-2, -1, -1, -2]
-2 1
-1 -1 -2
     1
reading word = [-2, -1, -1, -2]
 1  -1
-1 -2 -2
     1
reading word = [-2, -1, -1, -1]
-2 -1
-1 -2 -1
     1
reading word = [-2, -1, -2, -1]
-2 1
-1 -2 -1
     1
reading word = [-2, -1, -2, -1]
 1  -2
-1 -1 -2
     1
reading word = [-2, -1, -1, -2]
 1  -2
-1 -1 -2
     1
reading word = [-2, -1, -1, -2]
In 

    for T in Tlist:
        if T.neg_reading_word_is_lattice():
            T.pp()
            print "reading word = ", T.neg_reading_word()
    
    reading word = [-1, -1, -2, -2]
    -2 -2
    -1 -1 1
    . . 1
    reading word = [-1, -1, -2, -2]
    -2 -2
    -1 -1 11
    . .
    reading word = [-1, -1, -2, -2]
    -2 -21
    -1 -1 1
    . .
    reading word = [-1, -1, -2, -2]
    -2 1
    -1 -1 -21
    . .
    reading word = [-1, -1, -2, -2]

In 

Theorem 13. Let \( \lambda \) and \( \gamma \) be partitions and \( \alpha \) a composition, then the coefficient of \( \hat{s}_\gamma \) in \( h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_d} \) is equal to the number of \( T \in MCT_{\gamma}(\lambda, \alpha) \) such that \( T \) is a lattice tableau.

In 

    # here is a single coefficient in the product from Theorem 13
    (st[2,1]*h[3,2,1]).coefficient([3,2])

Out[12]: 3776

In 

    # and here it is computed combinatorially
    len([T for T in MCTableaux([3,2],[3,2,1],[2,1],rlex_bsp) if T.neg_reading_word_is_lattice()])

Out[13]: 3776

In 

    # Consider the following symmetric function expression
    st[1,1]*h[2,1]


In 

    # and this may be computed combinatorially with the following sum
    sum(len([T for T in MCTableaux([la],[2,1],[1,1],rlex_bsp) if T.neg_reading_word_is_lattice()]) for la in Partitions(d))


In 

Theorem 14. Let \( \lambda \) and \( \gamma \) be partitions and \( \alpha \) a composition, then the coefficient of \( \hat{s}_\gamma \) in \( h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_d} \) is equal to the number of \( T \in MCT_{\gamma}(\lambda, \alpha) \) such that the entries of the tableaux are sets (no repeated entries) and \( T \) is a lattice tableau.

In 

    # Here is a demonstration of the calculation of a single coefficient from Theorem 14
    (st[2,1]*ht[3]*ht[2]*ht[1]).coefficient([3,2])

Out[16]: 1730
In [17]: 
# and here it is computed combinatorially

len([T for T in MCTableaux([3,2],[3,2,1],[2,1],rlex_bsp) if T.neg_reading_word_is_lattice() and T.has_set_entries()])

Out[17]: 1730

In [18]: 
st[1,1]*ht[2]*ht[1]

Out[18]: 

In [19]: 
# and this may be computed combinatorially with the following sum

sum(len([T for T in MCTableaux([la, [2,1],[1,1],rlex_bsp) if T.neg_reading_word_is_lattice() and T.has_set_entries()]*st[la] for d in range(6) for la in Partitions(d))

Out[19]: 

Out[20]: 757

In [20]: 
# Here is a demonstration of the calculation of a single coefficient f from Theorem 15

(st[2]*st[3]*st[2]*st[1]).coefficient([3,2])

Out[20]: 757

In [21]: 
# and here it is computed combinatorially

len([T for T in MCTableaux([3,2],[3,2,1],[2,1],rlex_bsp) if T.neg_reading_word_is_lattice() and T.has_set_entries() and T.has_first_row_entries_size_greater_than_one()])

Out[21]: 757

In [22]: 
st[1,1]*st[2]*st[1]

Out[22]: 

Out[23]: 

Out[23]: 757

Out[23]: 
coefficient([3,2])

Out[23]: 
coefficient([3,2])

Out[23]: 757

Out[23]: 
coefficient([3,2])

Out[23]: 757

In [ ]:

In [ ]:

Theorem 15. Let λ and γ be partitions and α a composition, then the coefficient of s_α in δ_{α_1}δ_{α_2}⋯δ_{α_d}s_λ is equal to the number of T ∈ MCT_γ(λ,α) such that the entries of the tableau are sets (no repeated entries), T is a lattice tableau, and only labels of sets of size greater than 1 are allowed in the first row.