Ribbons and Column Strict Tableaux

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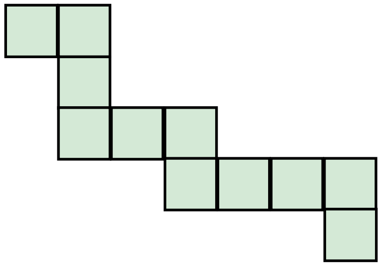
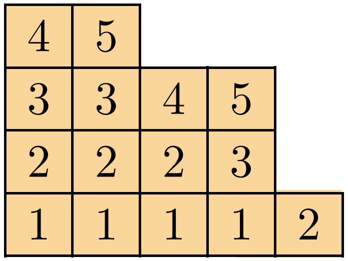




4	5			
3	3	4	5	
2	2	2	3	
1	1	1	1	2

The Symmetric Functions

$$\boxed{\begin{gathered}\Lambda = \mathbb{Q}[h_1, h_2, h_3, \ldots] \\ deg(h_k) = k\end{gathered}}$$

The space of symmetric functions is generated algebraically by the simple homogeneous symmetric functions.

The Schur Functions

$$\boxed{s_\lambda = det\left|h_{\lambda_i+i-j}\right|}$$

Example:

$$s_{(2,2,1)} = \begin{vmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} = {h_2}^2 h_1 - h_2 h_3 - h_3 {h_1}^2 + h_4\, h_1$$

$$h_3^2 h_7 h_2 h_1^3 h_4$$

degree $2 \cdot 3 + 7 + 2 + 3 \cdot 1 + 4 = 22$

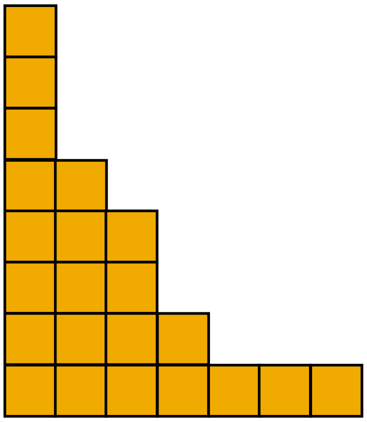
$$h_7 h_4 h_3^2 h_2 h_1^3$$

$$h_{(7,4,3,3,2,1,1,1)}$$

definition: a *partition* of n

sequence of non-negative integers $(\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$
such that $\lambda_1 + \lambda_2 + \cdots + \lambda_{\ell(\lambda)} = n$
and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0.$

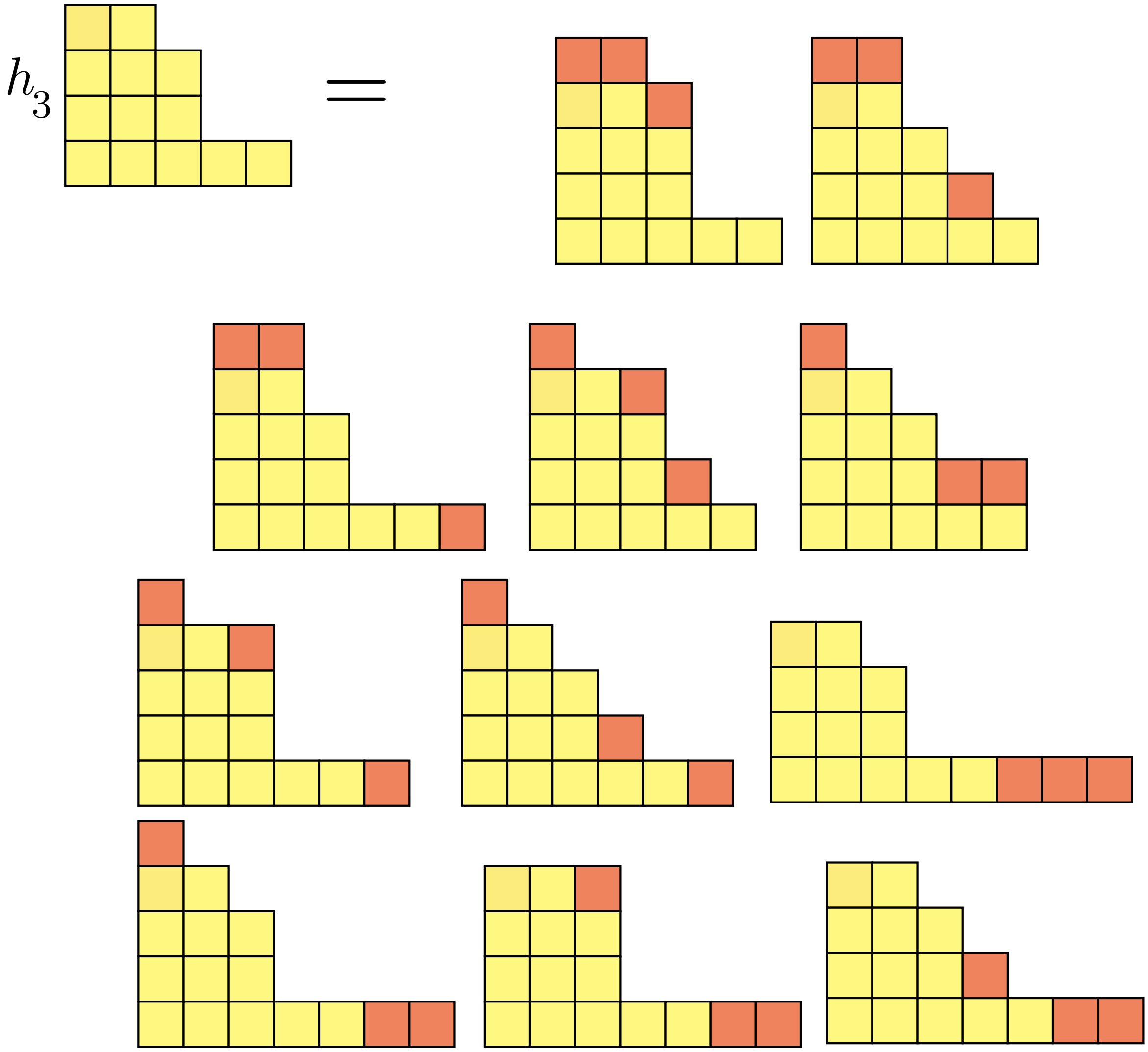
Young diagram of a partition



The Pieri Rule

$$h_m s_\lambda = \sum_{\mu} s_\mu$$

μ contains λ and the difference has at most one cell per column



Homogeneous ⟶ Schur

$$h_{\begin{array}{l}\boxed{}\\ \boxed{}\boxed{}\boxed{}\boxed{}\\ \boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\end{array}} = h_5 h_4 h_1$$

$$h_5 = \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1} = s_5$$

$$\begin{aligned}h_4 h_5 = {} & \begin{array}{l}\boxed{2}\boxed{2}\boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\end{array} + \begin{array}{l}\boxed{2}\boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\end{array} + \begin{array}{l}\boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\end{array}\\ & + \begin{array}{l}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{2}\end{array} + \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{2}\boxed{2}\\ & = s_9 + s_{8,1} + s_{7,2} + s_{6,3} + s_{5,4}\end{aligned}$$

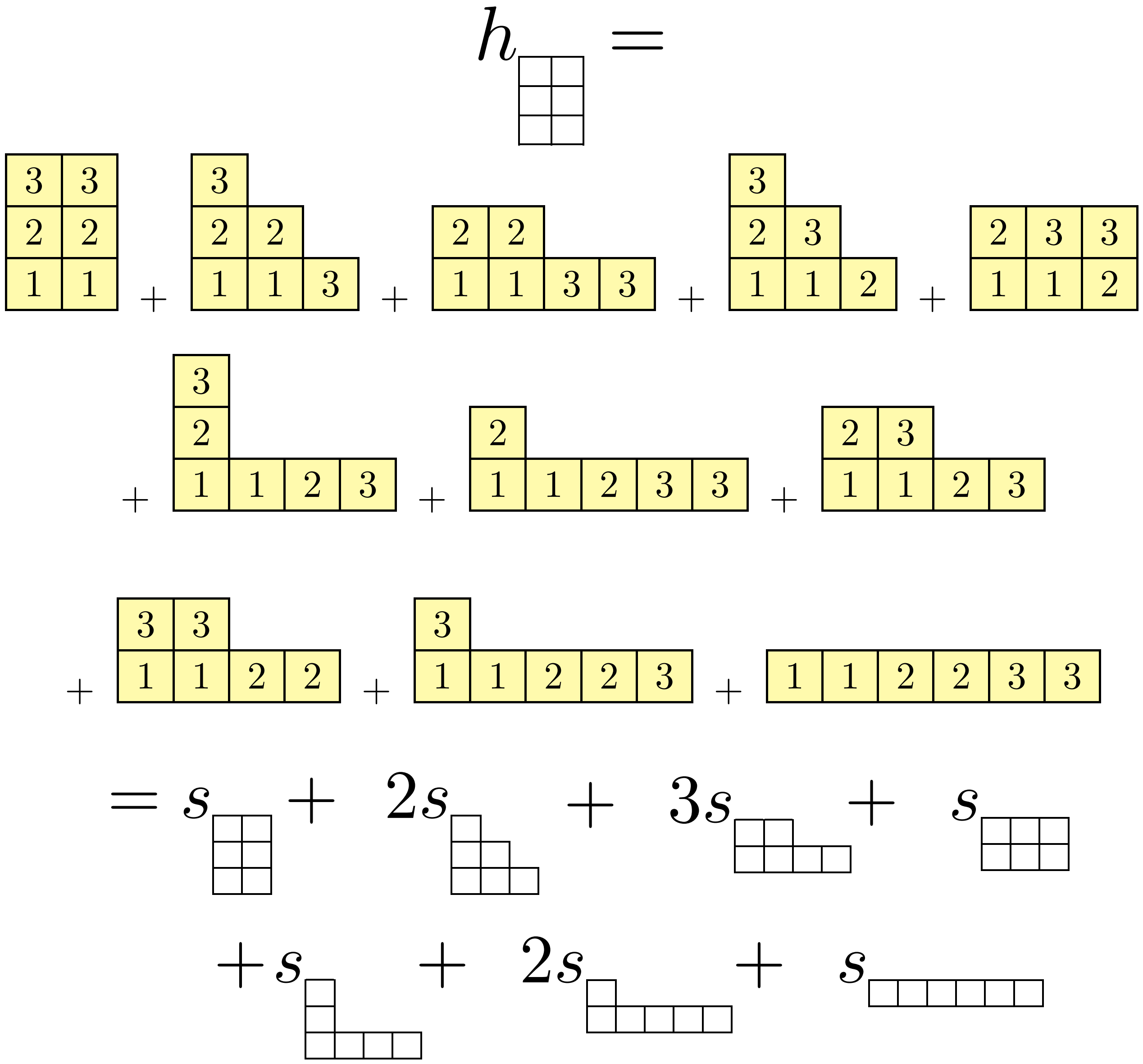
$$\begin{aligned}h_1 h_4 h_5 = {} & \begin{array}{l}\boxed{3}\\ \boxed{2}\boxed{2}\boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\end{array} + \begin{array}{l}\boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{3}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\end{array} + \begin{array}{l}\boxed{2}\boxed{2}\boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{3}\end{array} +\\ & + \begin{array}{l}\boxed{3}\\ \boxed{2}\boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\end{array} + \begin{array}{l}\boxed{2}\boxed{2}\boxed{2}\boxed{3}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\end{array} + \begin{array}{l}\boxed{2}\boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{3}\end{array}\\ & + \begin{array}{l}\boxed{3}\\ \boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\end{array} + \begin{array}{l}\boxed{2}\boxed{2}\boxed{3}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\end{array} + \begin{array}{l}\boxed{2}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{3}\end{array}\\ & + \begin{array}{l}\boxed{3}\\ \boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{2}\end{array} + \begin{array}{l}\boxed{2}\boxed{3}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{2}\end{array} + \begin{array}{l}\boxed{2}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{2}\boxed{3}\end{array}\\ & + \begin{array}{l}\boxed{3}\\ \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{2}\boxed{2}\end{array} + \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{3}\end{aligned}$$

$$\begin{aligned}& = s_{10} + 2\,s_{9,1} + 2\,s_{8,2} + s_{8,1,1} + 2\,s_{7,3} + s_{7,2,1}\\ & + 2\,s_{6,4} + s_{6,3,1} + s_{5,5} + s_{5,4,1}\end{aligned}$$

The Homogeneous functions

$$\boxed{h_\lambda = \sum_T s_{\lambda(T)}}$$

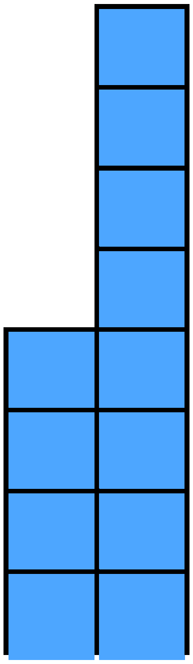
Example:



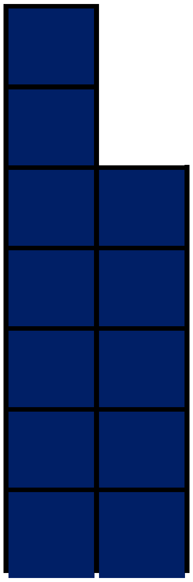
For each column strict tableau of content λ there is exactly one Schur function that appears in the sum.

Rule 1:
A Straightening Rule for Schur Functions

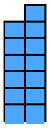
A column of size m & a column of n =
− a col. of size $n-1$ & a col. of size $m+1$



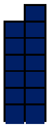
$= -$



Note: a column of size m on a column of $m+1$



$= -$

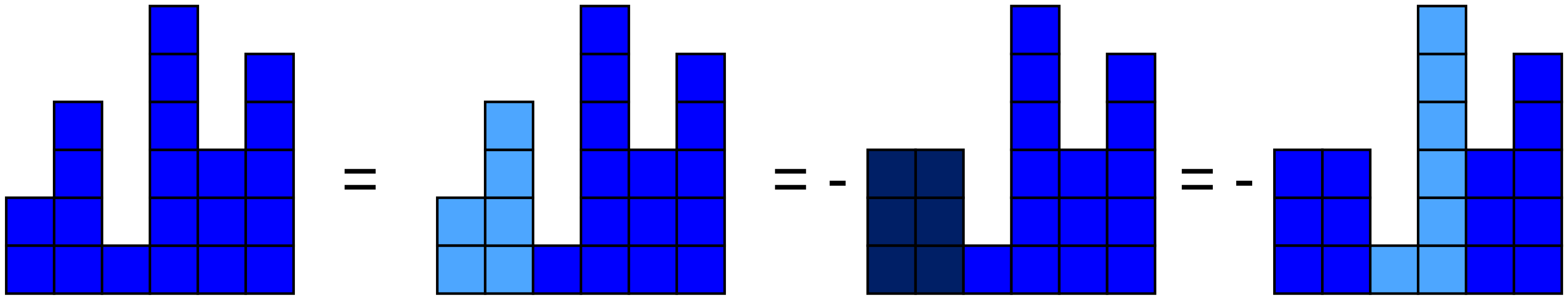


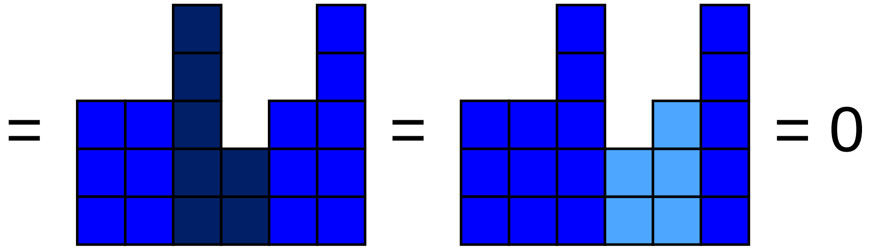
$= 0$

An example of the straightening rule:

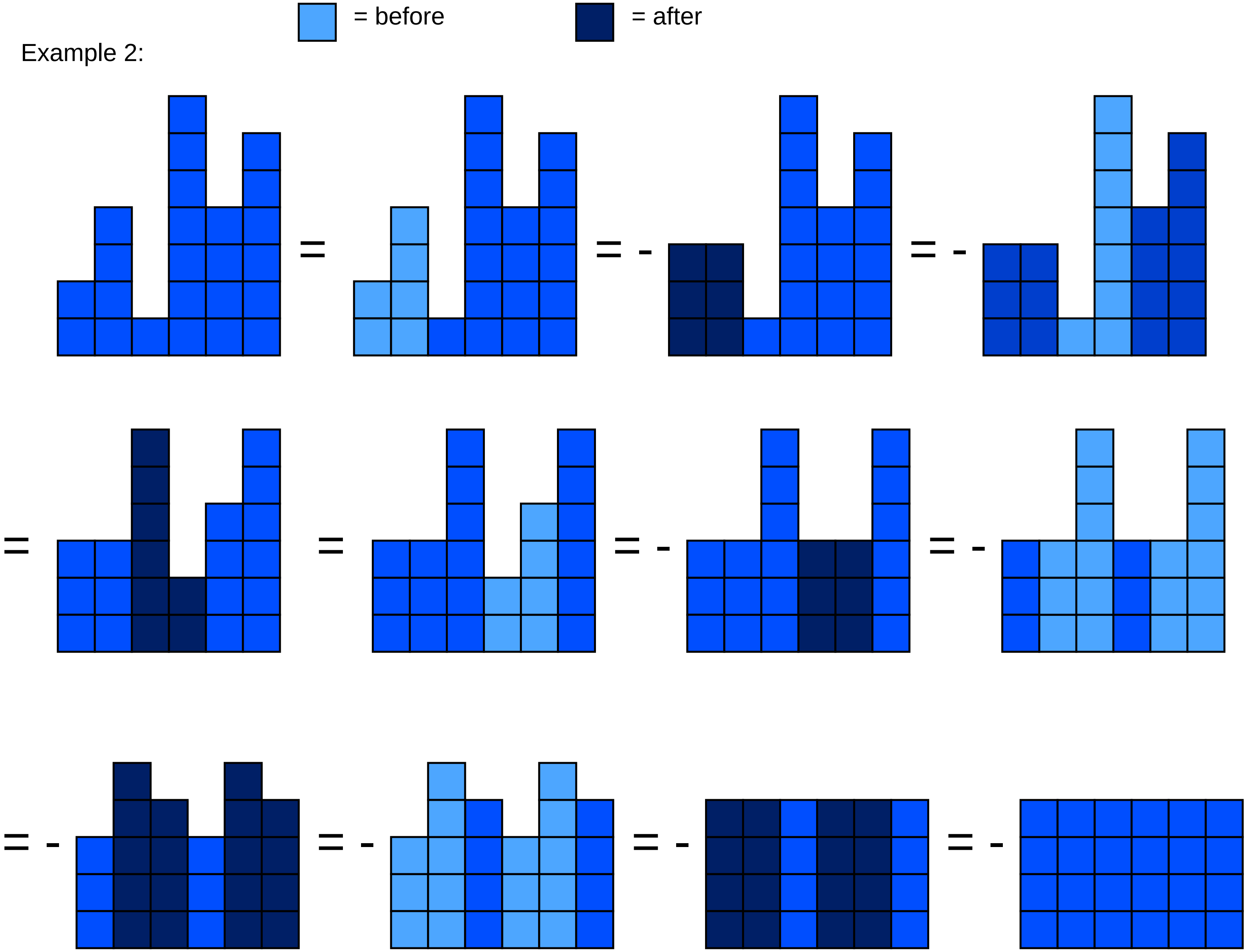
🟦 = before ■ = after

Example 1:





Example 2:



Rule 2: The Littlewood-Richardson Rule

A combinatorial rule for expanding skew Schur functions in terms of Schur functions indexed by partitions.

Definition: skew-Schur function
for λ/μ skew partition

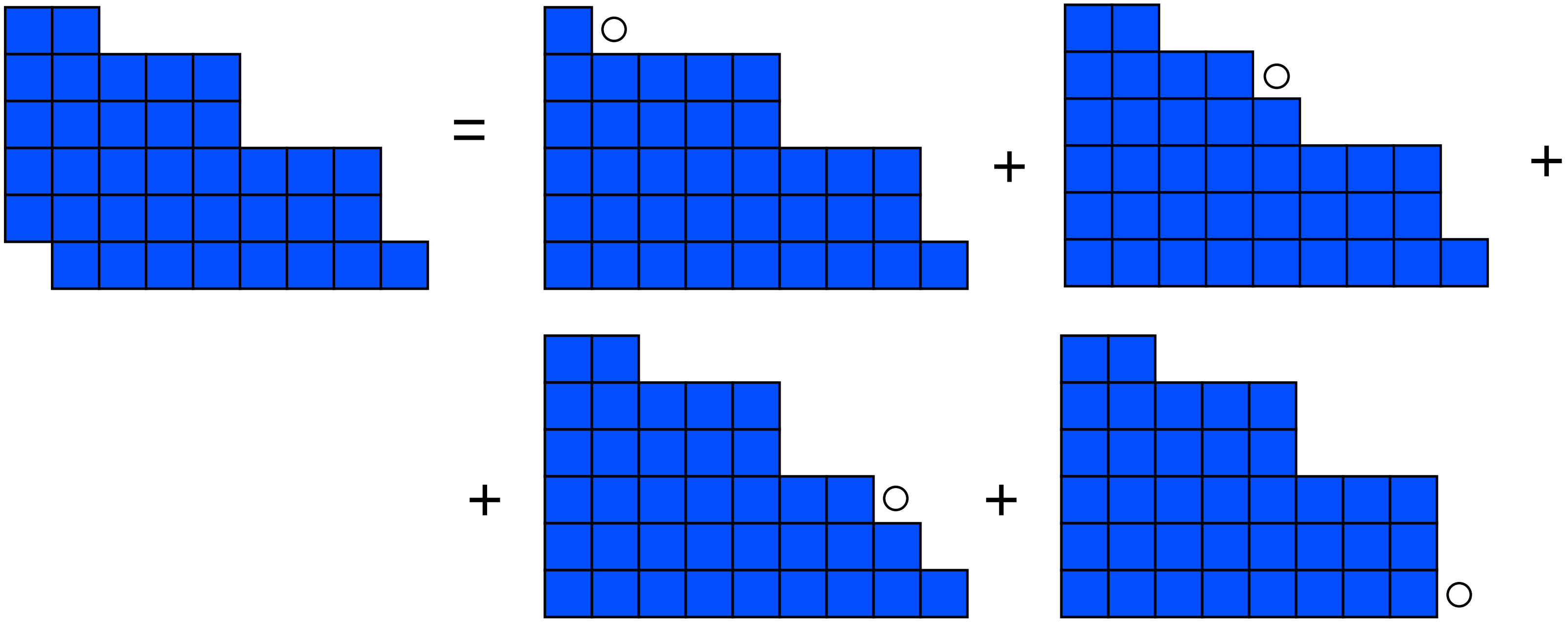
$$s_{\lambda/\mu} = det|h_{\lambda_i-\mu_j+i-j}|$$

The LR-rule:

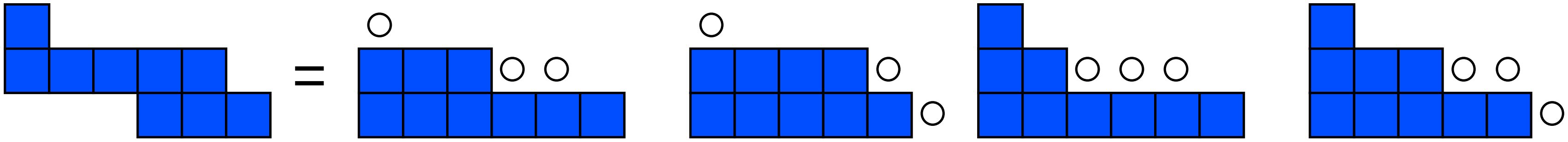
$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\nu\mu} s_{\nu}$$

where the coefficients $c^{\lambda}_{\nu\mu}$ are the number of ways of filling a Young diagram of shape λ/μ with ν_1 1's, ν_2 2's, ν_3 3's, etc. such that the filling increases weakly in the rows, strictly in the columns AND the for each k, the first k entries of the reverse reading word has partition content.

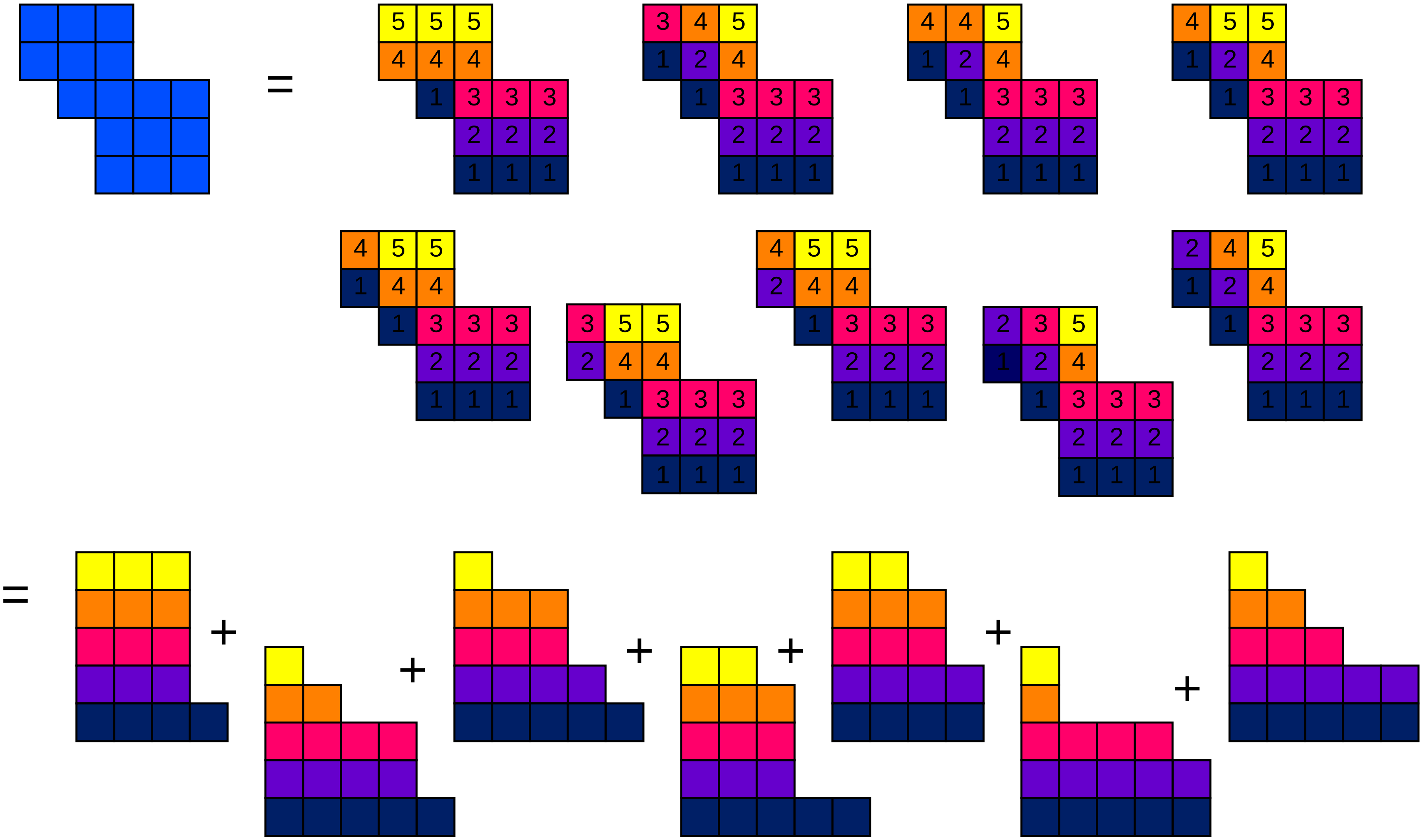
Example 1: In the case when the inner partition consists of only one square the result is equivalent to removing each of the corner cells of the outer partition:



Example 2: In the case that the inner partition is a is a single row, the result is equivalent to removing all horizontal strips of the same size from the border of the outer partition.

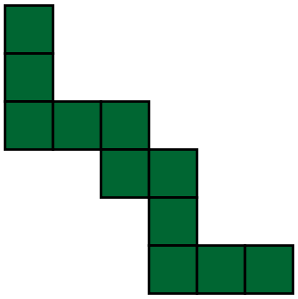


Example 3: Something a little more complicated

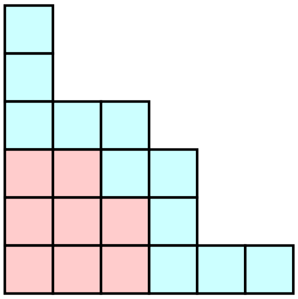


Ribbon Operators

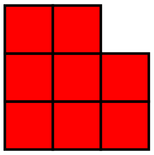
Ribbon operators use a combination of the operation of straightening columns followed by the Littlewood-Richardson rule.



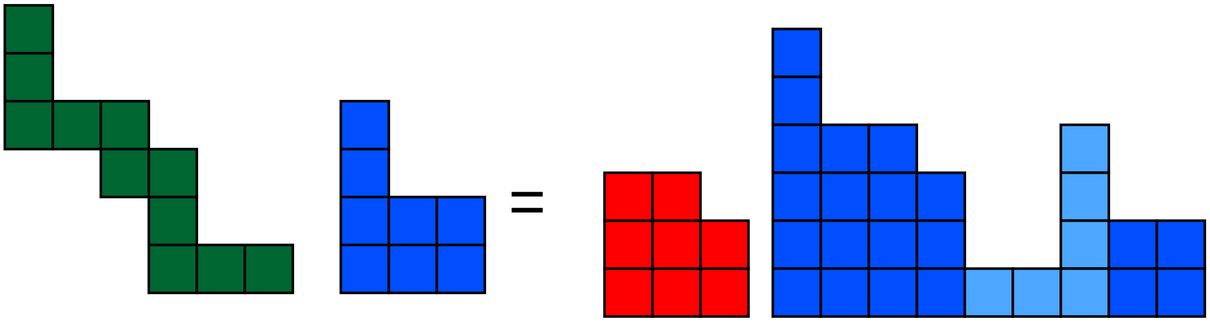
means first add these columns on the left



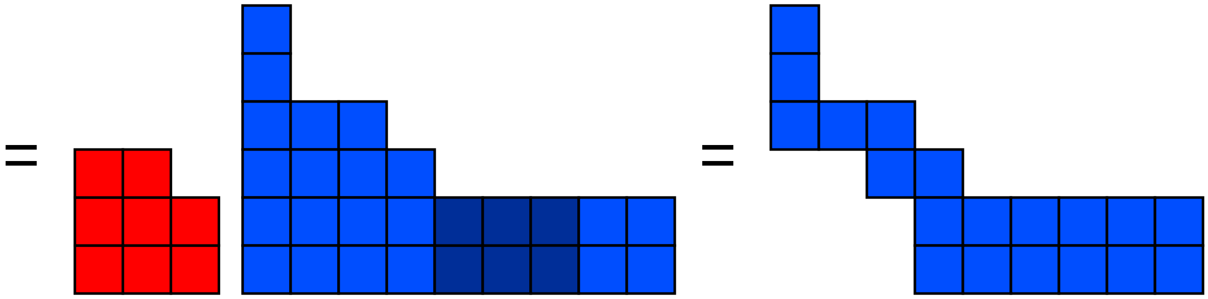
then remove the shape



Example 1:



remove



remove

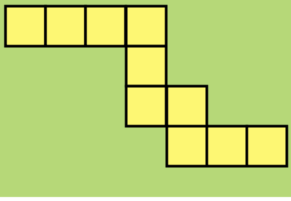
Now reduce this with the Littlewood-Richardson rule.

A ribbon may be represented by a seqence of rows.
$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$ with $\alpha_i > 0$ for $1 \leq i \leq \ell(\alpha)$

$$D(\alpha) = \text{descent set of } \alpha$$
$$= \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots \alpha_{\ell(\alpha)-1}\}$$

Example

$\alpha = (4, 1, 2, 3)$



$D(\alpha) = \{4, 5, 7\}$

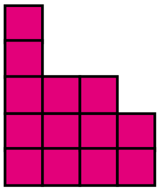
We will identify a ribbon operator with the decscent set
If $D \subseteq \{1, 2, \ldots, m-1\}$, then S^D is the ribbon operator
whose associated ribbon α has $D(\alpha) = D$

Theorem (Zabrocki):

$$H_{1^m} := \sum_{D \subseteq \{1,2,\ldots m-1\}} S^D$$

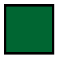
$$H_{1^m}(h_\lambda) = h_{\lambda + (1^m)}$$

Example

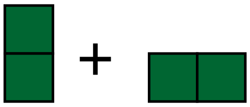
$H_{1^5} H_{1^3} H_{1^3} H_{1^2} 1 = h_{(1,1,3,4,4)} = h$ 

Theorem: a dual Pieri rule (Zabrocki)

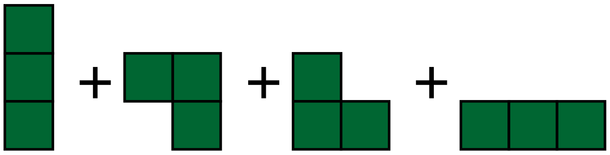
The sum of all ribbon operators of size m adds a column on the homogeneous symmetric functions.



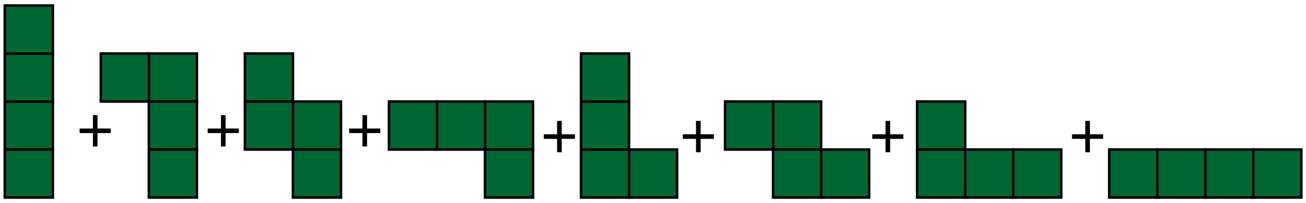
adds a column of size 1 on a homogeneous symmetric function with at most 1 part



adds a column of size 2 on a homogeneous symmetric function with at most 2 parts

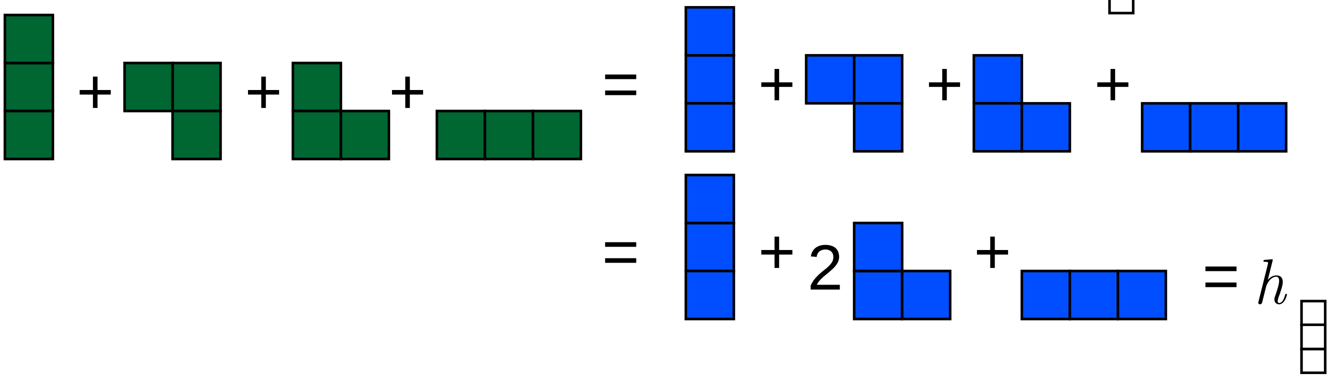


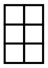
adds a column of size 3 on a homogeneous symmetric function with at most 3 parts

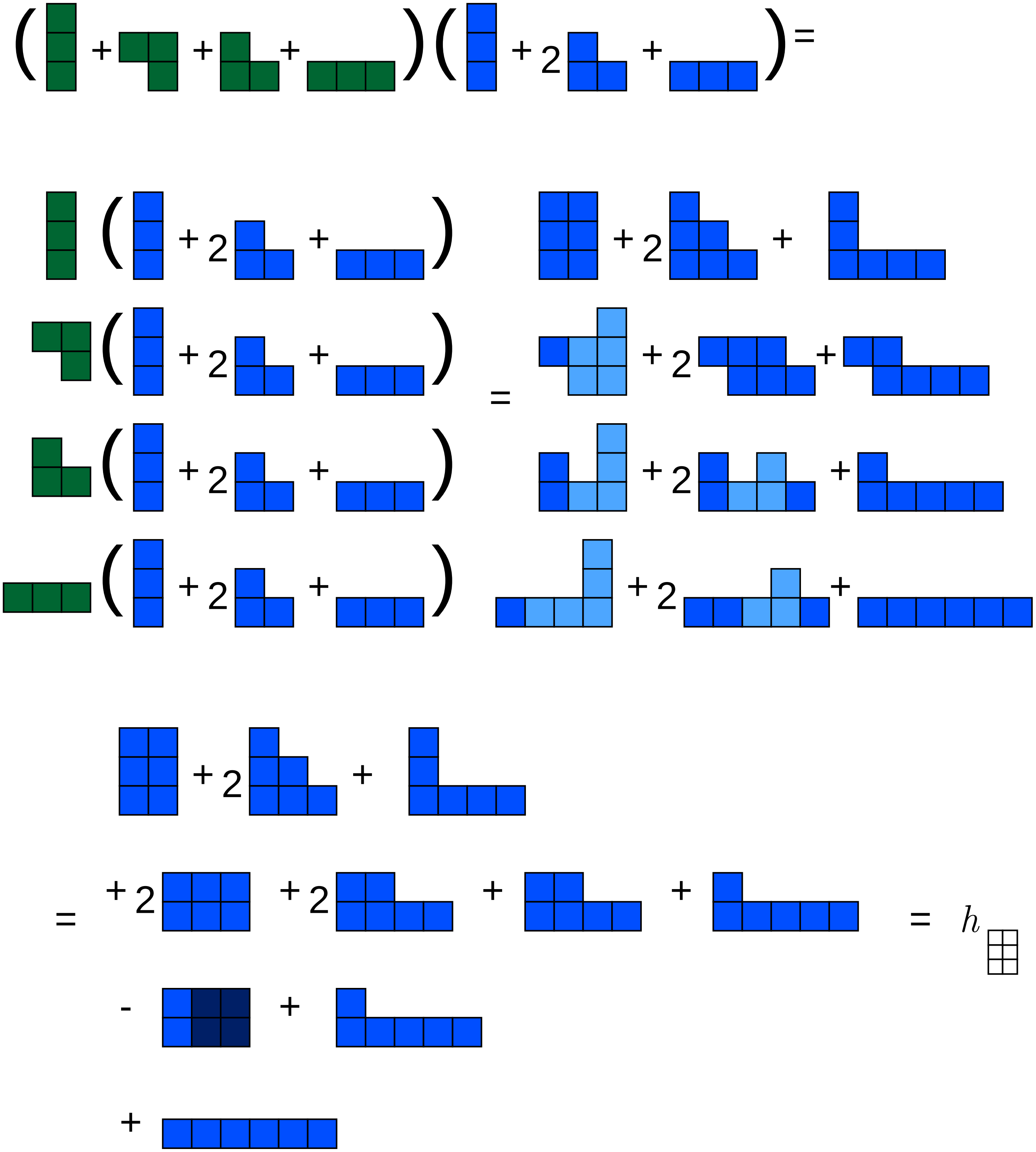


adds a column of size 4... etc.

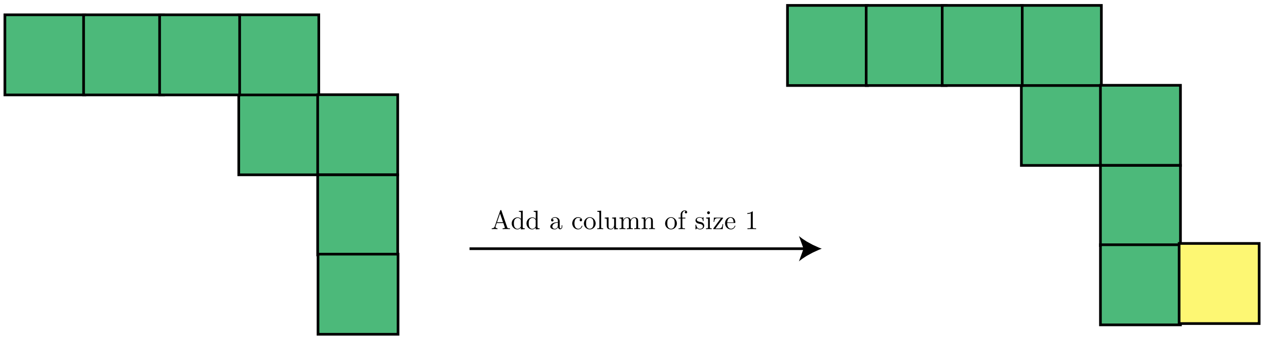
Example 1: adding a column of size 3 on the empty Schur function yields :$h_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$

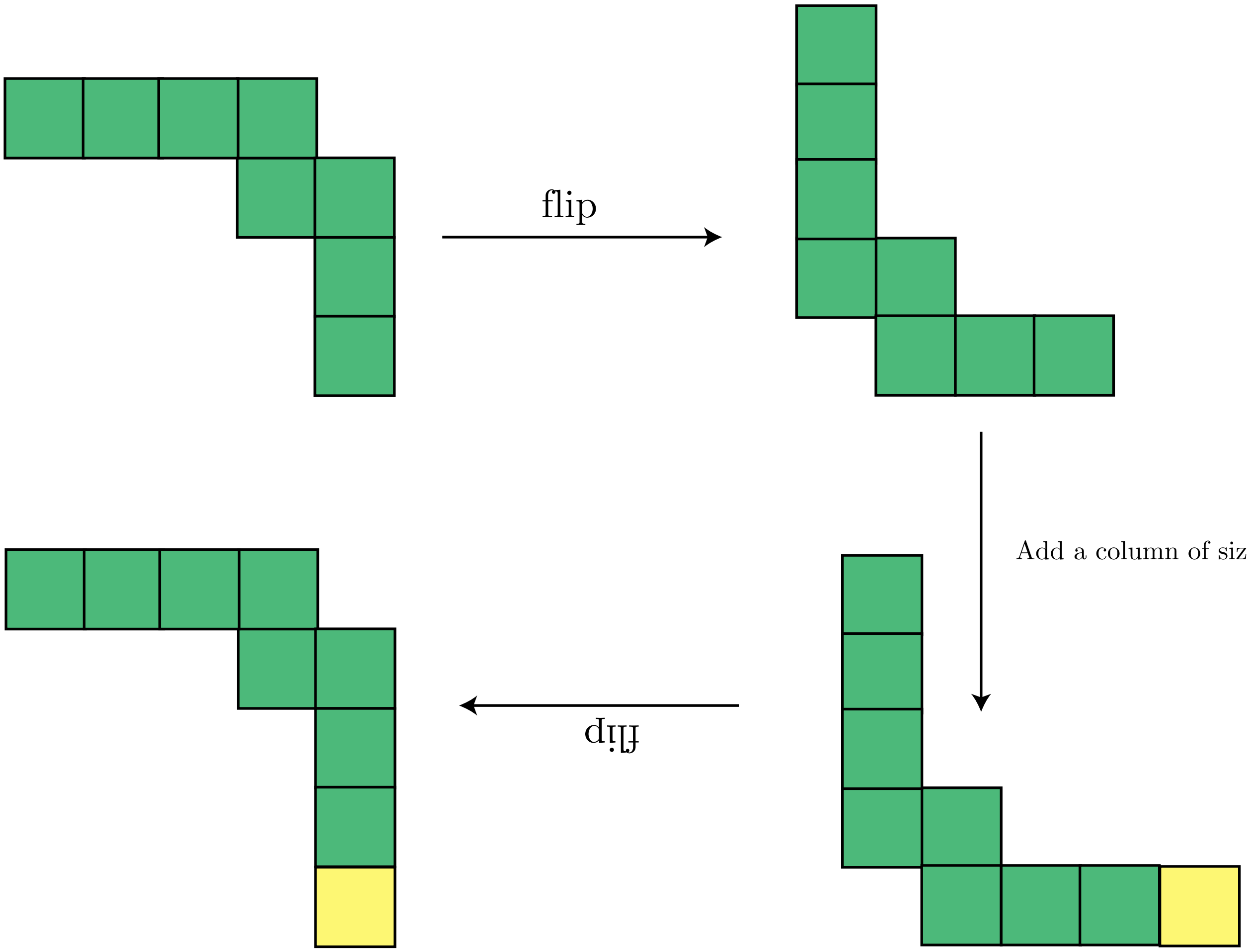


Example 2: Add a column of size 3 on h  yields h 



Building ribbon operators





A q-refinement

Theorem:

$$H^q_{1^m} := \sum_{D \subseteq \{1,2,\ldots,m-1\}} q^{\sum_{i\in D} i} S^D$$

$$H^q_{1^m} H_\mu[X;q] = H_{\mu+(1^m)}[X;q]$$

$$\begin{aligned} H_\mu[X;q]\Big|_{s_\lambda} &= q \text{ count column strict tableaux shape } \lambda \text{ content } \mu \\ &= \text{Kostka Foulkes polynomial} \end{aligned}$$

$$\begin{aligned} &H_{(2,2,2)}[X;q] = \\ &\quad s_6 + q(1+q)\,s_{5,1} + q^2\left(q^2+q+1\right)s_{4,2} + q^3 s_{4,1,1} + q^3 s_{3,3} + q^4(1+q)\,s_{3,2,1} + q^6 s_{2,2,2} \end{aligned}$$

A qt-refinement

Theorem:

$$H^{qt}_{1^m} := \sum_{D \subseteq \{1,2,\ldots,m-1\}} q^{\sum_{i\in D} i} (S^D)^t$$

$$H^{qt}_{1^m} H_\mu[X;q,t] = H_{\mu+(1^m)}[X;q,t]$$

$$\begin{aligned} H_\mu[X;q,t]\Big|_{s_\lambda} &= q,t \text{ count column strict tableaux shape } \lambda \text{ content } (1^{|\mu|}) \\ &= \text{Macdonald-Kostka polynomial} \end{aligned}$$

Some examples