Mike Zabrocki

York University

A quantization of non-commutative symmetric functions

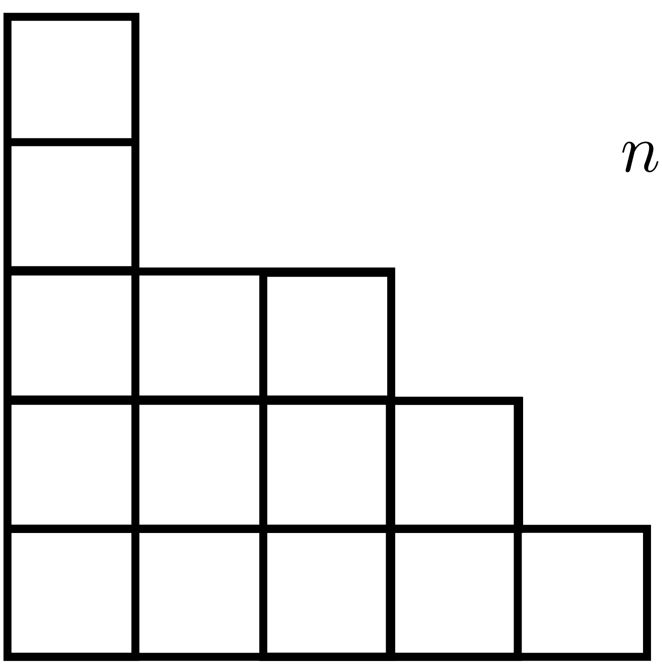
joint work with Nantel Bergeron

Symmetric functions

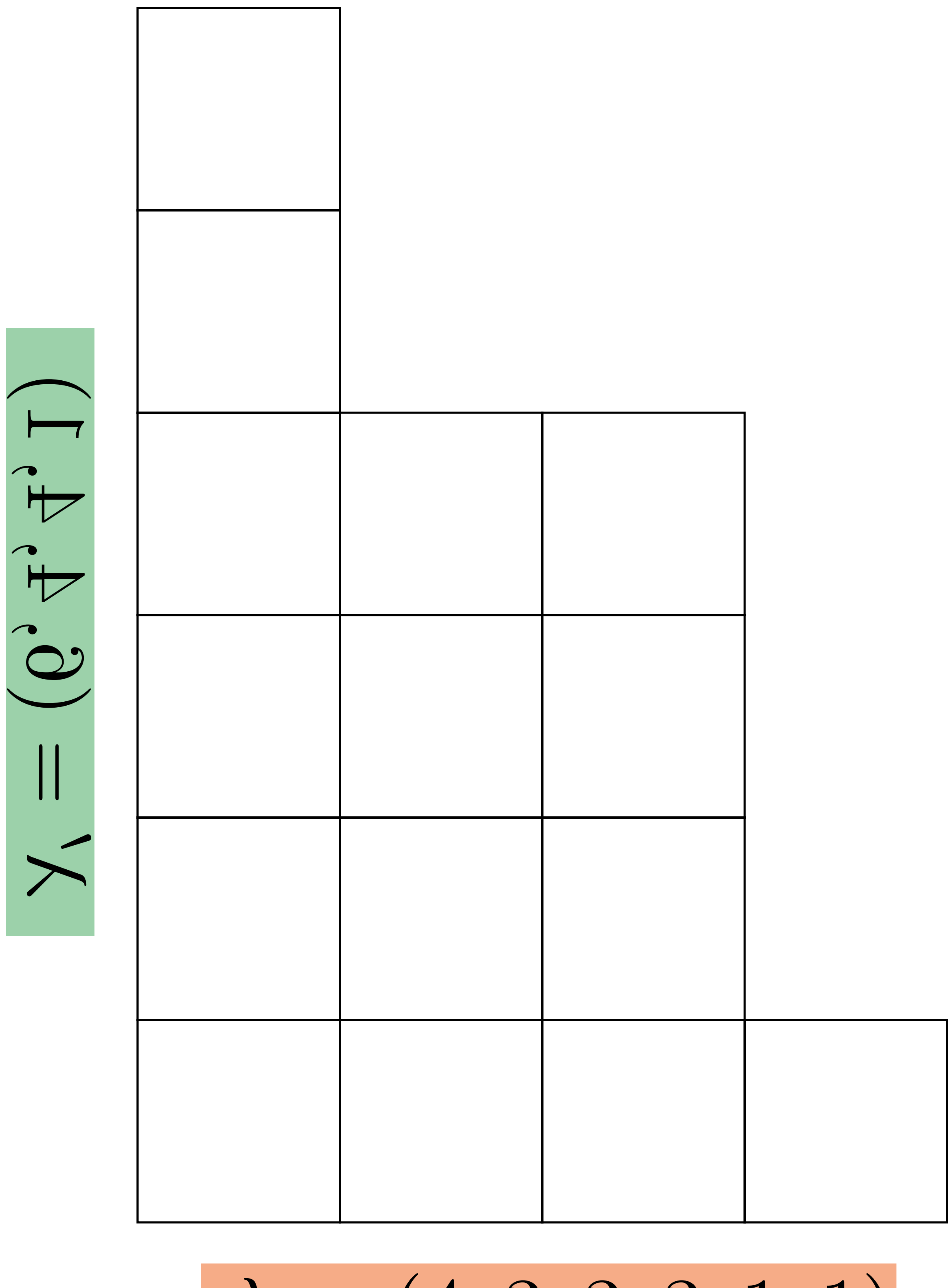
$$\Lambda = \mathbb{Q}[h_1, h_2, h_3, \dots]$$
$$deg(h_k) = k$$

Basis:

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}}$$
$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)} > 0)$$
$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_{\ell(\lambda)}$$
$$\ell(\lambda) = \#\text{ of non-zero parts}$$



$$n(\lambda) = \sum_i (i-1)\lambda_i$$


λ' = (6, 4, 4, 1)
λ = (4, 3, 3, 3, 1, 1)

More symmetric functions

$$e_k := e_{k-1}h_1 - e_{k-2}h_2 + \cdots + (-1)^k h_k$$
$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)}}$$

$$s_\lambda = det|h_{\lambda_i+i-j}|$$

$$s_{\lambda/\mu} = det|h_{\lambda_i-\mu_j+i-j}|$$

$$\{s_\lambda\}_{\lambda\vdash n} \qquad \{h_\lambda\}_{\lambda\vdash n} \qquad \{e_\lambda\}_{\lambda\vdash n}$$

bases for the symmetric functions of degree n

$$\omega(h_\lambda) = e_\lambda \qquad \omega(s_\lambda) = s_{\lambda'}$$

$$\Lambda = \mathbb{Q}[h_1, h_2, h_3, \ldots]$$
$$deg(h_k) = k \qquad h_i h_j = h_j h_i$$

Littlewood-Richardson coefficients

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$$

Pieri Rule

$$h_k s_\lambda = \sum_{\mu = \lambda + \text{horiz strip}} s_\mu$$

Kostka coefficients

$$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$$

Λ is a graded Hopf algebra

multiplication:	$\mu(f \otimes g) = fg$
co-multiplication:	$\Delta(h_k) = \sum_{i=0}^k h_i \otimes h_{k-i}$
anitpode:	$S(h_\lambda) = (-1)^{\lvert\lambda\rvert} e_\lambda$

Non-commutative symmetric functions

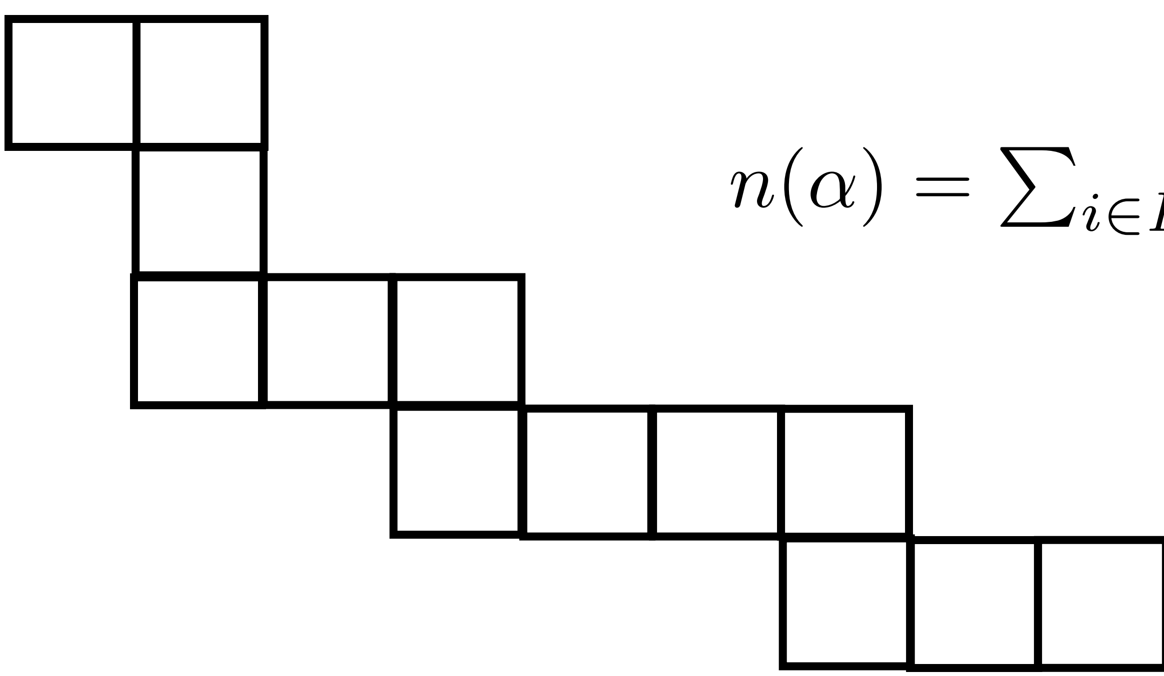
$$NC\Lambda = \mathbb{Q}\left\langle \mathsf{h}_1, \mathsf{h}_2, \mathsf{h}_3, \ldots \right\rangle$$
$$deg(\mathsf{h}_k) = k \qquad \mathsf{h}_i\mathsf{h}_j \neq \mathsf{h}_j\mathsf{h}_i \text{ if } i \neq j$$

Basis:

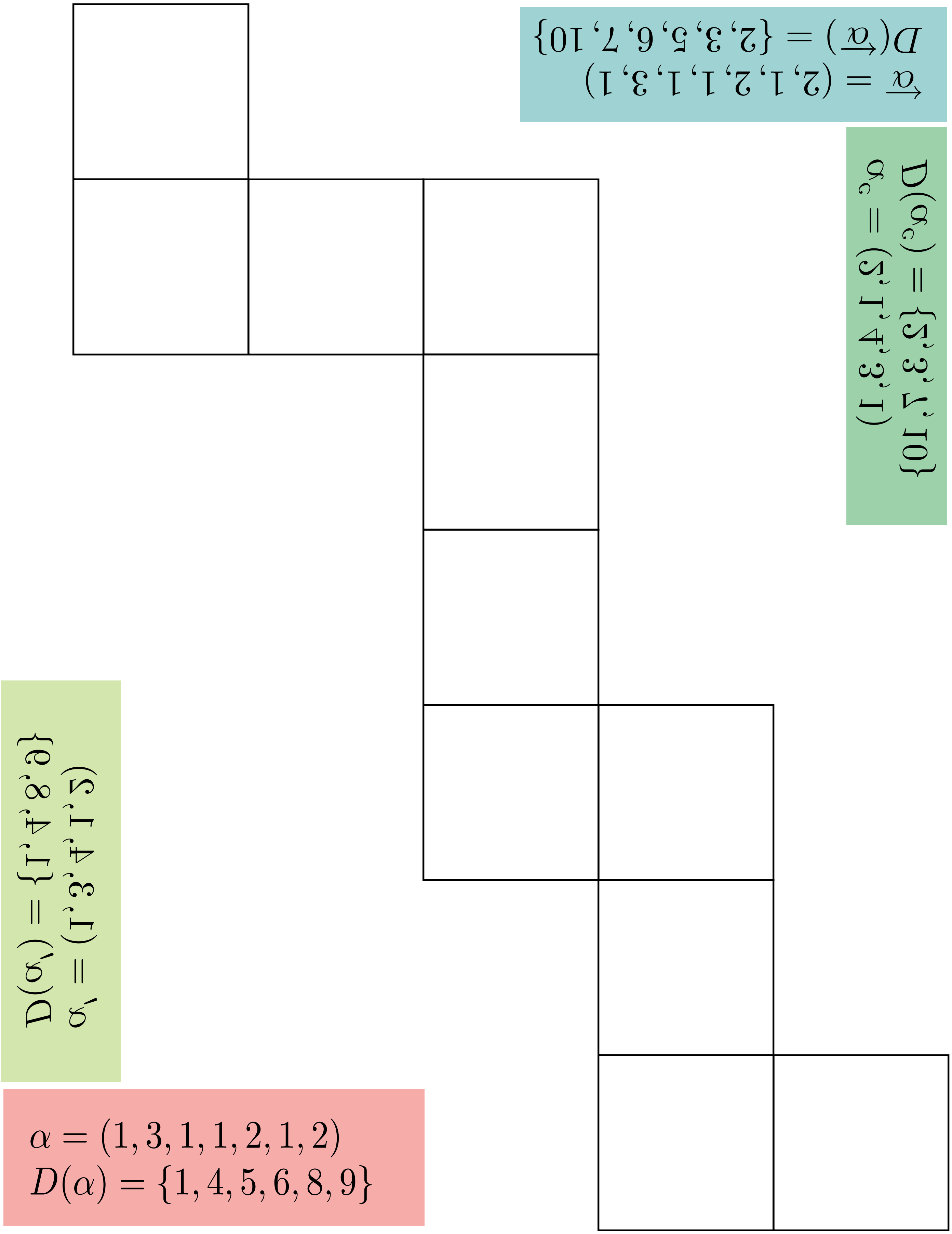
$$\mathsf{h}_\alpha = \mathsf{h}_{\alpha_1}\mathsf{h}_{\alpha_2}\cdots\mathsf{h}_{\alpha_{\ell(\alpha)}}$$
$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)}) \qquad \alpha_i > 0$$

$\ell(\alpha) =$ number of non-zero parts of α

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell(\alpha)}$$
$$D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{\ell(\alpha)}\}$$



$$n(\alpha) = \sum_{i \in D(\alpha)} i$$



$\overleftarrow{\alpha} = (2, 1, 2, 1, 1, 3, 1)$
$D(\overleftarrow{\alpha}) = \{2, 3, 5, 6, 7, 10\}$
$D(\alpha^c) = \{2, 3, 7, 10\}$
$\alpha^c = (2, 1, 4, 3, 1)$
$D(\alpha') = \{1, 4, 8, 9\}$
$\alpha' = (1, 3, 4, 1, 5)$
$\alpha = (1, 3, 1, 1, 2, 1, 2)$
$D(\alpha) = \{1, 4, 5, 6, 8, 9\}$

More non-commutative symmetric functions

$$\mathsf{e}_k = \mathsf{e}_{k-1}\mathsf{h}_1 - \mathsf{e}_{k-2}\mathsf{h}_2 + \cdots + (-1)^k \mathsf{h}_k$$
$$= \sum_{\alpha \models k} (-1)^{k+\ell(\alpha)} \mathsf{h}_\alpha$$
$$\mathsf{e}_\alpha = \mathsf{e}_{\alpha_1}\mathsf{e}_{\alpha_2} \cdots \mathsf{e}_{\alpha_{\ell(\alpha)}}$$

$$\mathsf{s}_\alpha = \sum_{\beta \geq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \mathsf{h}_\beta$$

For the map $\chi : NC\Lambda \to \Lambda$ by

$$\chi(\mathsf{h}_\alpha) = h_{\alpha_1} \cdots h_{\alpha_{\ell(\alpha)}}$$
$$\chi(\mathsf{e}_\alpha) = e_{\alpha_1} \cdots e_{\alpha_{\ell(\alpha)}}$$
$$\chi(\mathsf{s}_{(1^a,b)}) = s_{(b,1^a)}$$

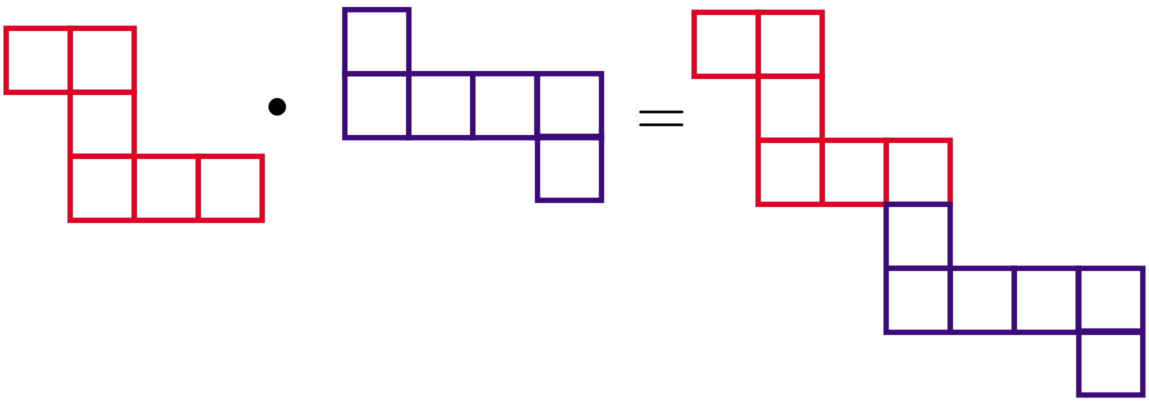
$$\{\mathsf{s}_\alpha\}_{\alpha \models n} \qquad \{\mathsf{h}_\alpha\}_{\alpha \models n} \qquad \{\mathsf{e}_\alpha\}_{\alpha \models n}$$

are bases for NCSF of degree n

$$\omega'(\mathsf{s}_\alpha) = \mathsf{s}_{\alpha'} \qquad \omega^c(\mathsf{s}_\alpha) = \mathsf{s}_{\alpha^c} \qquad \overleftarrow{\omega}(\mathsf{s}_\alpha) = \mathsf{s}_{\overleftarrow{\alpha}}$$

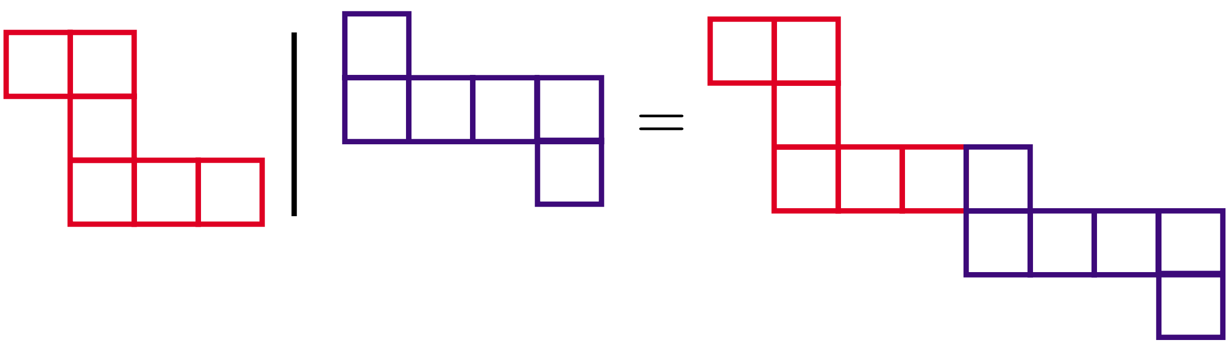
concatenate

$\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_{\ell(\alpha)}, \beta_1, \ldots, \beta_{\ell(\beta)})$

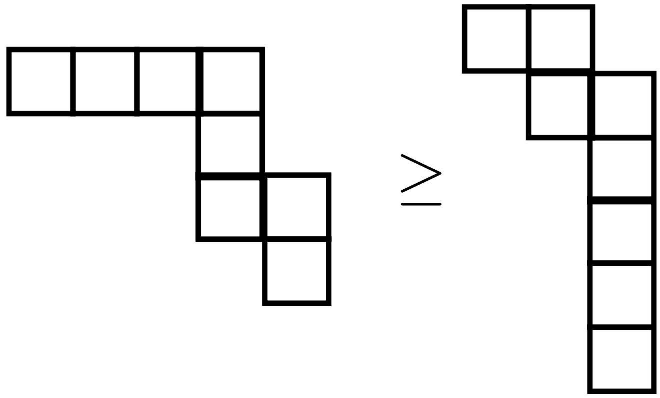


attach

$\alpha | \beta = (\alpha_1, \ldots, \alpha_{\ell(\alpha)-1}, \alpha_{\ell(\alpha)} + \beta_1, \beta_2, \ldots, \beta_{\ell(\beta)})$



$\alpha \geq \beta$ if and only if $D(\alpha) \subseteq D(\beta)$



$$NC\Lambda = \mathbb{Q}\langle \mathsf{h}_1, \mathsf{h}_2, \mathsf{h}_3, \ldots \rangle$$

$$deg(\mathsf{h}_k) = k \qquad \mathsf{h}_i\mathsf{h}_j \neq \mathsf{h}_j\mathsf{h}_i \text{ if } i \neq j$$

$$\mathsf{s}_\alpha \mathsf{s}_\beta = \mathsf{s}_{\alpha\cdot\beta} + \mathsf{s}_{\alpha|\beta}$$

$$\mathsf{s}_\alpha \mathsf{h}_k = \mathsf{s}_{\alpha\cdot(k)} + \mathsf{s}_{\alpha|(k)}$$

$$\mathsf{h}_\alpha = \sum_{\beta \geq \alpha} \mathsf{s}_\beta$$

$NC\Lambda$ is a graded Hopf algebra

multiplication: $\mu(f \otimes g) = fg$

co-multiplication: $\Delta(\mathsf{h}_k) = \sum_{i=0}^{k} \mathsf{h}_{k-i} \otimes \mathsf{h}_i$

anitpode: $S(\mathsf{h}_\alpha) = (-1)^{\ell(|\alpha|)} \mathsf{e}_{\overleftarrow{\alpha}}$

Hall-Littlewood symmetric functions

$$\mathbf{S}_m(s_\lambda) = s_{(m,\lambda)}$$
$$\mathbf{S}_{\lambda_1}\mathbf{S}_{\lambda_2}\cdots\mathbf{S}_{\lambda_{\ell(\lambda)}}1 = s_\lambda$$

For $V \in Hom(\Lambda, \Lambda)$
$$\overline{V} = \mu \circ id \otimes (V \circ S) \circ \Delta$$
$$R^q(f) = q^{deg(f)} f$$
for $f \in \Lambda$ of homogeneous degree
$$\widetilde{V}^q = \overline{\overline{V}\ \overline{R^q}}$$

$$\mathbf{H}_m = \widetilde{\mathbf{S}_m}^q$$
$$H^q_\lambda = \mathbf{H}_{\lambda_1}\mathbf{H}_{\lambda_2}\cdots\mathbf{H}_{\lambda_{\ell(\lambda)}}1$$

Theorem (Jing)
$$H^q_\mu = \sum_\lambda K_{\lambda\mu}(q) s_\lambda$$

Non-commutative Hall-Littlewood symmetric functions

$$\mathbb{S}_m(\mathsf{s}_\alpha) = \mathsf{s}_{(\alpha,m)}$$
$$\mathbb{S}_{\alpha_{\ell(\alpha)}}\mathbb{S}_{\alpha_{\ell(\alpha)-1}}\cdots\mathbb{S}_{\alpha_1}1 = \mathsf{s}_\alpha$$

For $V \in Hom(NC\Lambda, NC\Lambda)$
$\overline{V} = \mu \circ id \otimes (V \circ S) \circ \Delta$
$R^q(f) = q^{deg(f)} f$
for $f \in NC\Lambda$ of homogeneous degree
$\widetilde{V}^q = \overline{\overline{V}\ \overline{R^q}}$

$$\mathbb{H}_m = \widetilde{\mathbb{S}_m}^q$$
$$\mathsf{H}^q_\alpha := \mathbb{H}_{\alpha_{\ell(\alpha)}}\mathbb{H}_{\alpha_{\ell(\alpha)-1}}\cdots\mathbb{H}_{\alpha_1}1$$

Theorem (Z-Bergeron)

$$\mathsf{H}^q_\alpha = \sum_{\beta\geq\alpha} q^{c(\alpha,\beta^c)}\mathsf{s}_\beta$$
$$\chi(\mathsf{H}^q_{(1^a,b)}) = H^q_{(b,1^a)}$$