

Ribbons and Homogeneous Symmetric Functions

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The Symmetric Functions

$$\Lambda = \mathbb{Q}[h_1, h_2, h_3, \dots]$$

The space of symmetric functions is generated algebraically by the simple homogeneous symmetric functions. This may be taken as a definition.

The Schur Functions

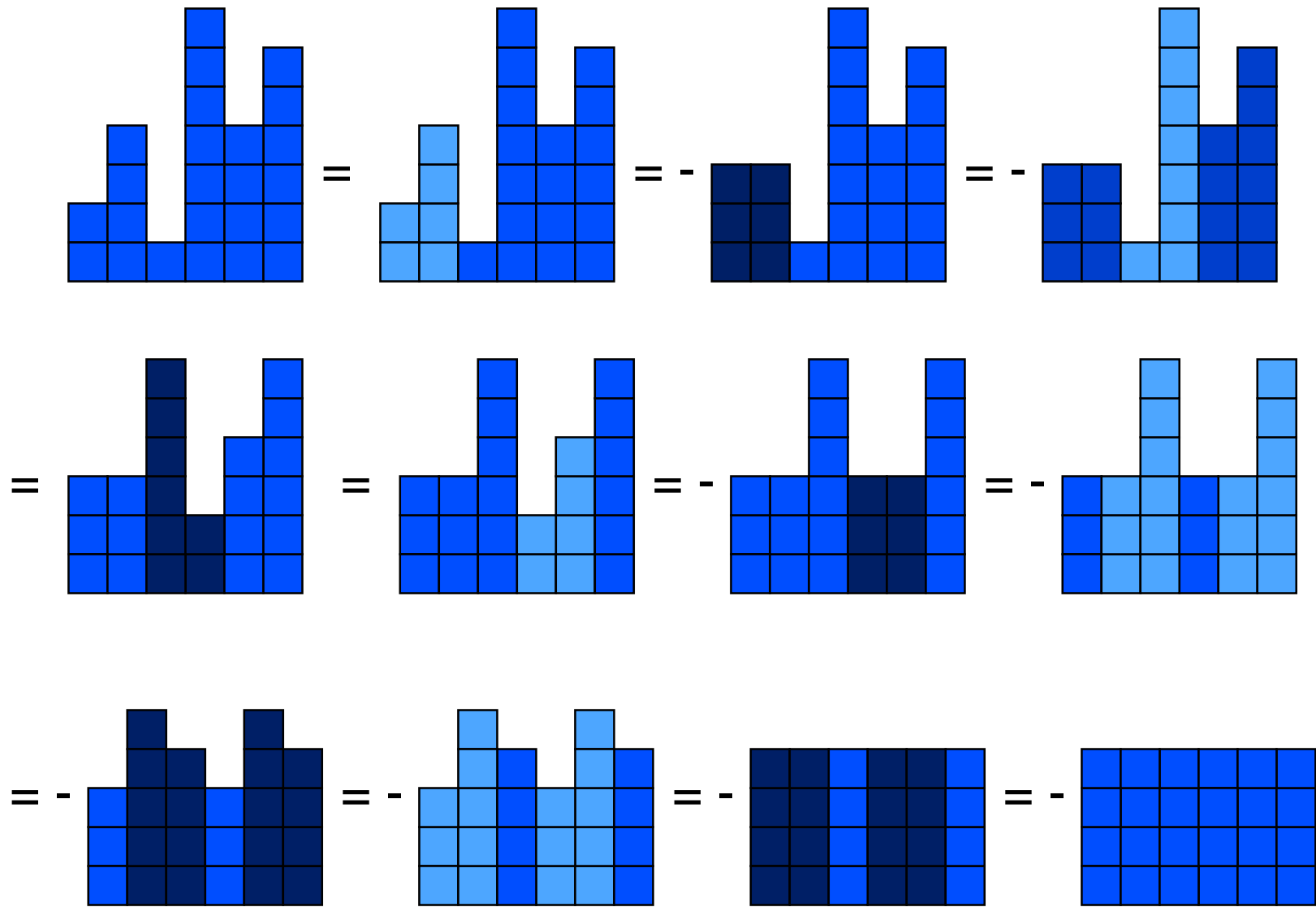
$$s_\lambda = \det |h_{\lambda_i + i - j}|$$

The definition of the Schur polynomials is well known and they are a fundamental basis of the symmetric functions.

Schur functions will be identified here with the Young diagrams for the partition.

Example 2:

 = before  = after



Rule 2: The Littlewood-Richardson Rule

A combinatorial rule for expanding skew Schur functions in terms of Schur functions indexed by partitions.

Definition: skew-Schur function
for λ/μ skew partition

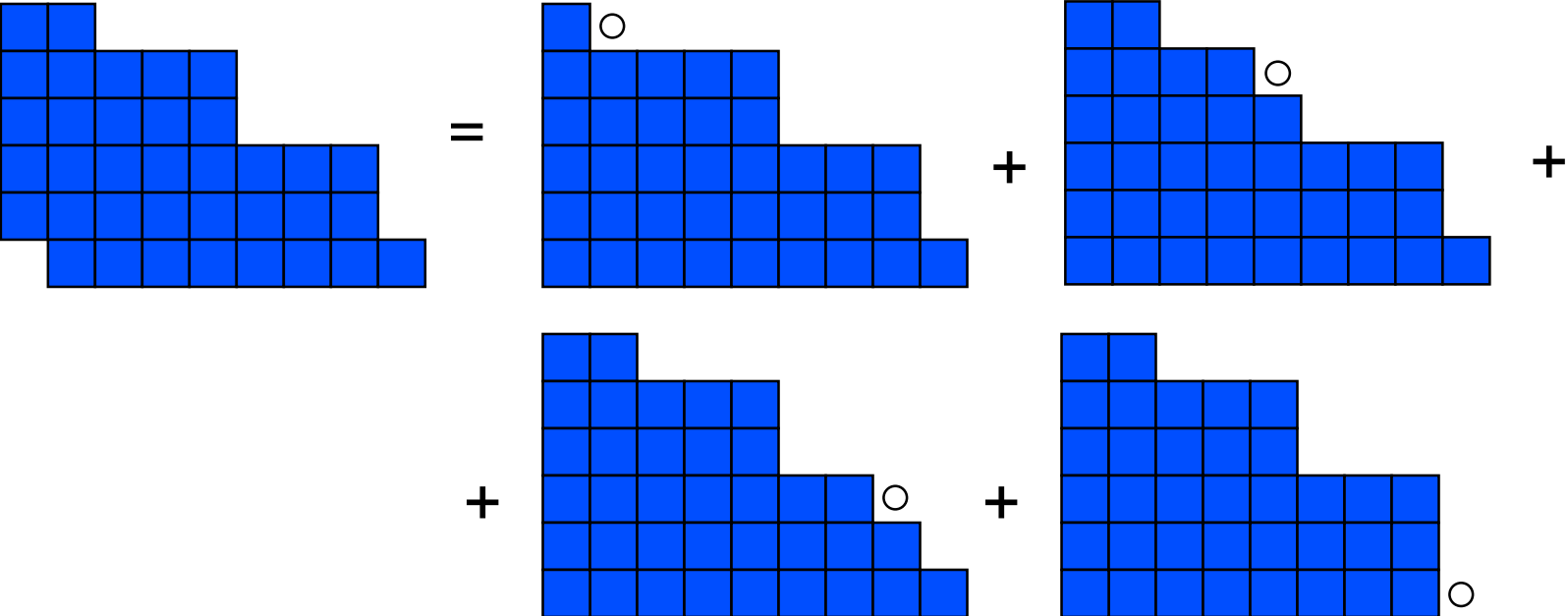
$$s_{\lambda/\mu} = \det |h_{\lambda_i - \mu_j + i - j}|$$

The LR-rule:

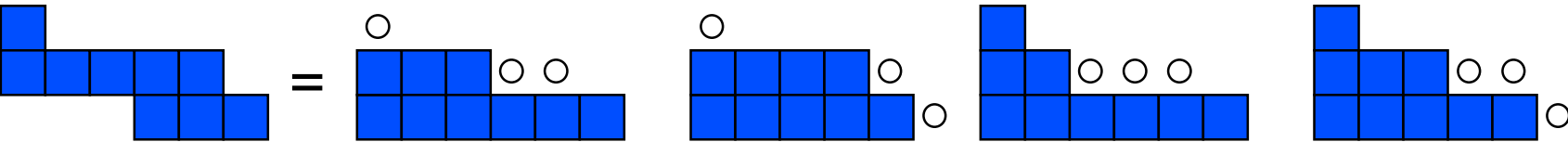
$$s_{\lambda/\mu} = \sum_{\nu} c_{\nu\mu}^{\lambda} s_{\nu}$$

where the coefficients $c_{\nu\mu}^{\lambda}$ are the number of ways of filling a Young diagram of shape λ/μ with ν_1 1's, ν_2 2's, ν_3 3's, etc. such that the filling increases weakly in the rows, strictly in the columns AND the for each k , the first k entries of the reverse reading word has partition content.

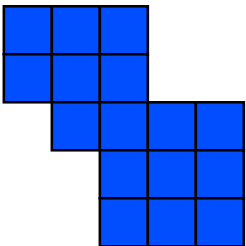
Example 1: In the case when the inner partition consists of only one square the result is equivalent to removing each of the corner cells of the outer partition:



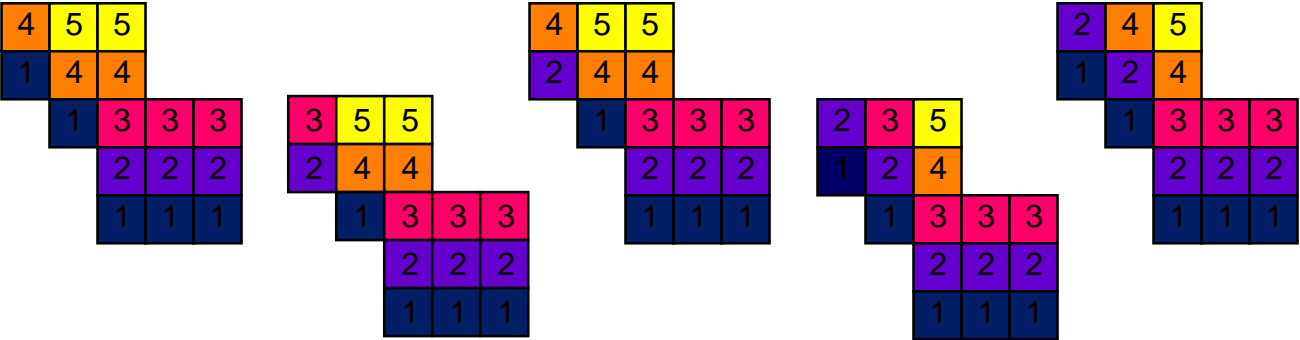
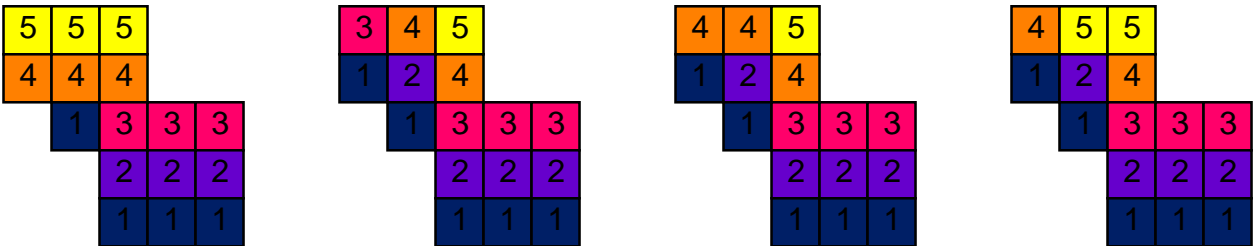
Example 2: In the case that the inner partition is a single row, the result is equivalent to removing all horizontal strips of the same size from the border of the outer partition.



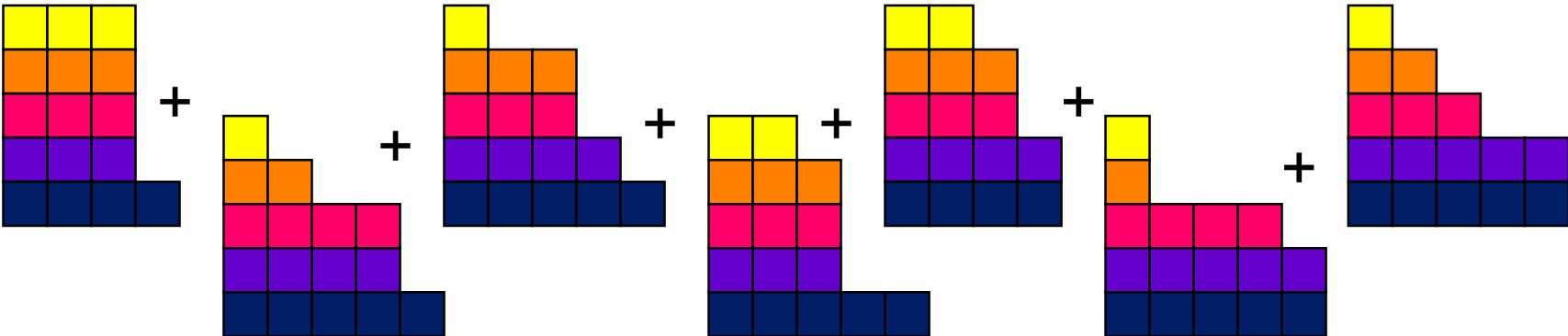
Example 3: Something a little more complicated



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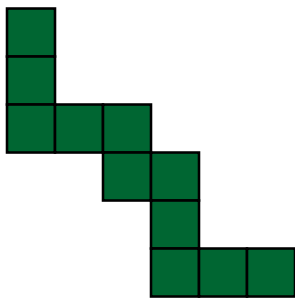


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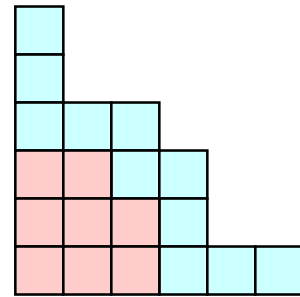


Ribbon Operators

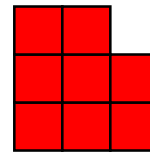
Ribbon operators use a combination of the operation of straightening columns followed by the Littlewood-Richardson rule.



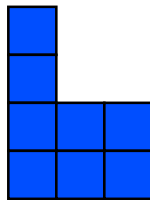
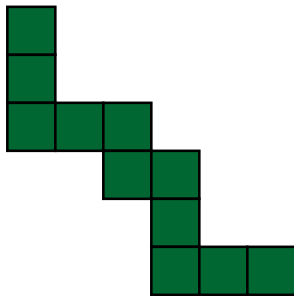
means first add these columns
on the left



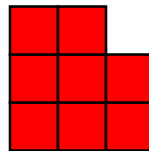
then remove the shape



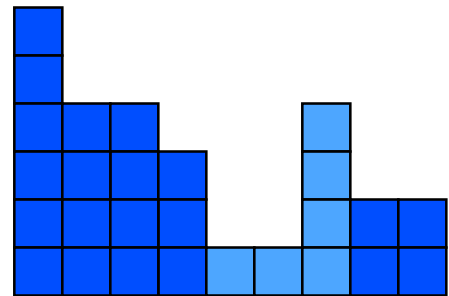
Example 1:



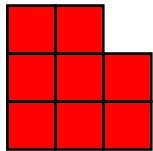
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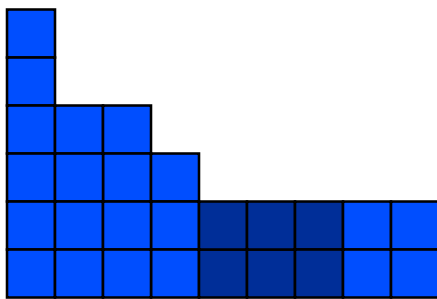
remove



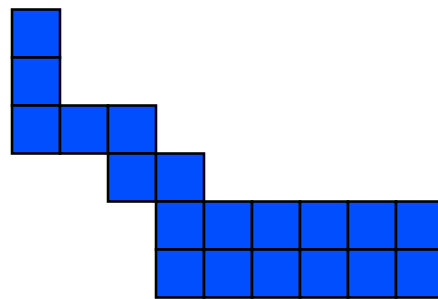
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remove




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Now reduce this with the Littlewood-Richardson rule.



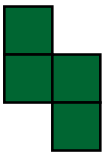
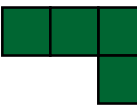
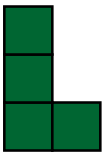
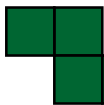
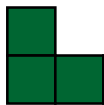

A dual Pieri rule :

The sum of all ribbon operators of size m adds a column on the homogeneous symmetric functions.

 adds a column of size 1 on a homogeneous symmetric function with at most 1 part

 +  adds a column of size 2 on a homogeneous symmetric function with at most 2 parts

 +  +  +  adds a column of size 3 on a homogeneous symmetric function with at most 3 parts

 +  +  +  +  +  +  + 

adds a column of size 4... etc.

Example 1: adding a column of size 3 on the empty Schur function yields h

$$\begin{aligned}
 & \left(\begin{array}{c} \square \\ \square \\ \square \end{array} \right) + \left(\begin{array}{cc} \square & \square \\ & \square \end{array} \right) + \left(\begin{array}{c} \square \\ \square & \square \end{array} \right) + \left(\begin{array}{ccc} \square & \square & \square \end{array} \right) \\
 &= \left(\begin{array}{c} \square \\ \square \\ \square \end{array} \right) + \left(\begin{array}{cc} \square & \square \\ & \square \end{array} \right) + \left(\begin{array}{c} \square \\ \square & \square \end{array} \right) + \left(\begin{array}{ccc} \square & \square & \square \end{array} \right) \\
 &= \left(\begin{array}{c} \square \\ \square \\ \square \end{array} \right) + 2 \left(\begin{array}{c} \square \\ \square & \square \end{array} \right) + \left(\begin{array}{ccc} \square & \square & \square \end{array} \right) = h_{\left(\begin{array}{c} \square \\ \square \\ \square \end{array} \right)}
 \end{aligned}$$

Open question:

Combinatorially prove the positivity of a composition of these operators (they yield the homogeneous symmetric functions, of course they are Schur positive). Does this give a new combinatorial interpretation of the homogeneous symmetric functions?

Generalizations:

There exist q (a dual Morris recurrence) and q, t (a Macdonald-Morris recurrence) analogs of the ribbon rule. Can these generalized operators be used to show positivity of the Hall-Littlewood and Macdonald symmetric functions?