Ribbons and Homogeneous Symmetric Functions

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The Symmetric Functions

\[ \Lambda = \mathbb{Q}[h_1, h_2, h_3, \ldots] \]

The space of symmetric functions is generated algebraically by the simple homogeneous symmetric functions. This may be taken as a definition.

The Schur Functions

\[ s_\lambda = \text{det}|h_{\lambda_i+j-\lambda_j}| \]

The definition of the Schur polynomials is well known and they are a fundamental basis of the symmetric functions. Schur functions will be identified here with the Young diagrams for the partition.
Rule 1: 
A Straightening Rule for Schur Functions

A column of size \( m \) & a column of \( n \) =
\[- \text{ a col. of size } n - 1 \text{ & a col. of size } m + 1 \]

Note: a column of size \( m \) on a column of \( m + 1 \)
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\begin{array}{c}
\text{= -}
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An example of the straightening rule:

Example 1:

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\begin{align*}
\text{before} & = \text{after} \\
\end{align*}
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Example 2:

= before = after

= = - = - = - = -

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Rule 2: The Littlewood-Richardson Rule

A combinatorial rule for expanding skew Schur functions in terms of Schur functions indexed by partitions.

Definition: skew-Schur function

for $\lambda/\mu$ skew partition

$$s_{\lambda/\mu} = det|h_{\lambda_i-\mu_j+i-j}|$$

The LR-rule:

$$s_{\lambda/\mu} = \sum_{\nu} c_{\nu \mu}^\lambda s_\nu$$

where the coefficients $c_{\nu \mu}^\lambda$ are the number of ways of filling a Young diagram of shape $\lambda/\mu$ with $\nu_1$ 1’s, $\nu_2$ 2’s, $\nu_3$ 3’s, etc. such that the filling increases weakly in the rows, strictly in the columns AND the for each $k$, the first $k$ entries of the reverse reading word has partition content.
Example 1: In the case when the inner partition consists of only one square the result is equivalent to removing each of the corner cells of the outer partition:

Example 2: In the case that the inner partition is a single row, the result is equivalent to removing all horizontal strips of the same size from the border of the outer partition.
Example 3: Something a little more complicated
Ribbon Operators

Ribbon operators use a combination of the operation of straightening columns followed by the Littlewood-Richardson rule.

Example 1:

Now reduce this with the Littlewood-Richardson rule.
A dual Pieri rule:

The sum of all ribbon operators of size $m$ adds a column on the homogeneous symmetric functions.

- adds a column of size 1 on a homogeneous symmetric function with at most 1 part
- adds a column of size 2 on a homogeneous symmetric function with at most 2 parts
- adds a column of size 3 on a homogeneous symmetric function with at most 3 parts
- adds a column of size 4... etc.

Example 1: adding a column of size 3 on the empty Schur function yields $h$.
Example 2: Add a column of size 3 on $h$ yields $h$

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Open question:

Combinatorially prove the positivity of a composition of these operators (they yield the homogeneous symmetric functions, of course they are Schur positive). Does this give a new combinatorial interpretation of the homogeneous symmetric functions?

Generalizations:

There exist $q$ (a dual Morris recurrence) and $q, t$ (a Macdonald-Morris recurrence) analogs of the ribbon rule. Can these generalized operators be used to show positivity of the Hall-Littlewood and Macdonald symmetric functions?