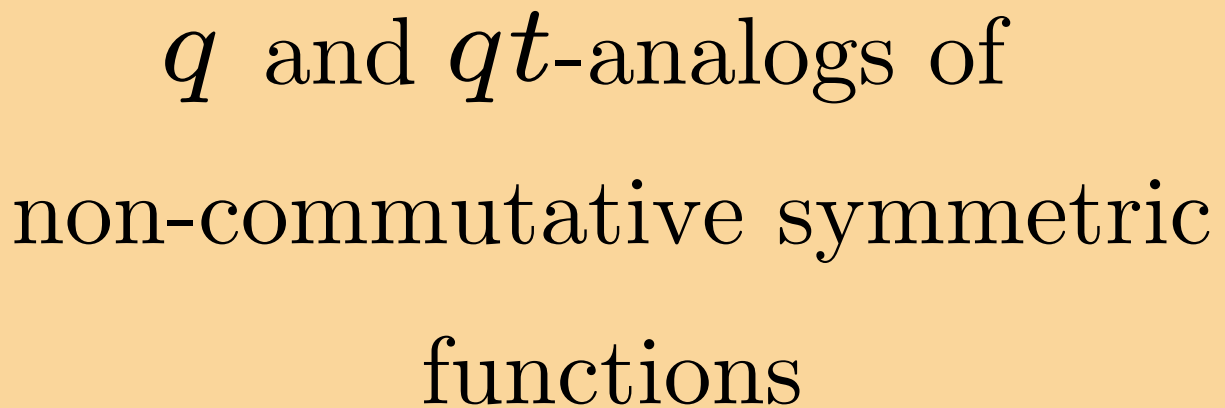


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q and qt -analogs of
non-commutative symmetric
functions

The Symmetric Functions

$$\Lambda = \mathbb{Q}[h_1, h_2, h_3, \dots]$$
$$\deg(h_k) = k$$

The space of symmetric functions is generated algebraically by the simple homogeneous and power symmetric functions.

The Schur Functions

$$s_\lambda = \det |h_{\lambda_i + i - j}|$$

Example:

$$s_{(2,2,1)} = \begin{vmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} = h_2^2 h_1 - h_2 h_3 - h_3 h_1^2 + h_4 h_1$$

More symmetric functions

$$e_k := e_{k-1}h_1 - e_{k-2}h_2 + \cdots + (-1)^k h_k$$

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)}}$$

$$s_\lambda = \det |h_{\lambda_i + i - j}|$$

more generally

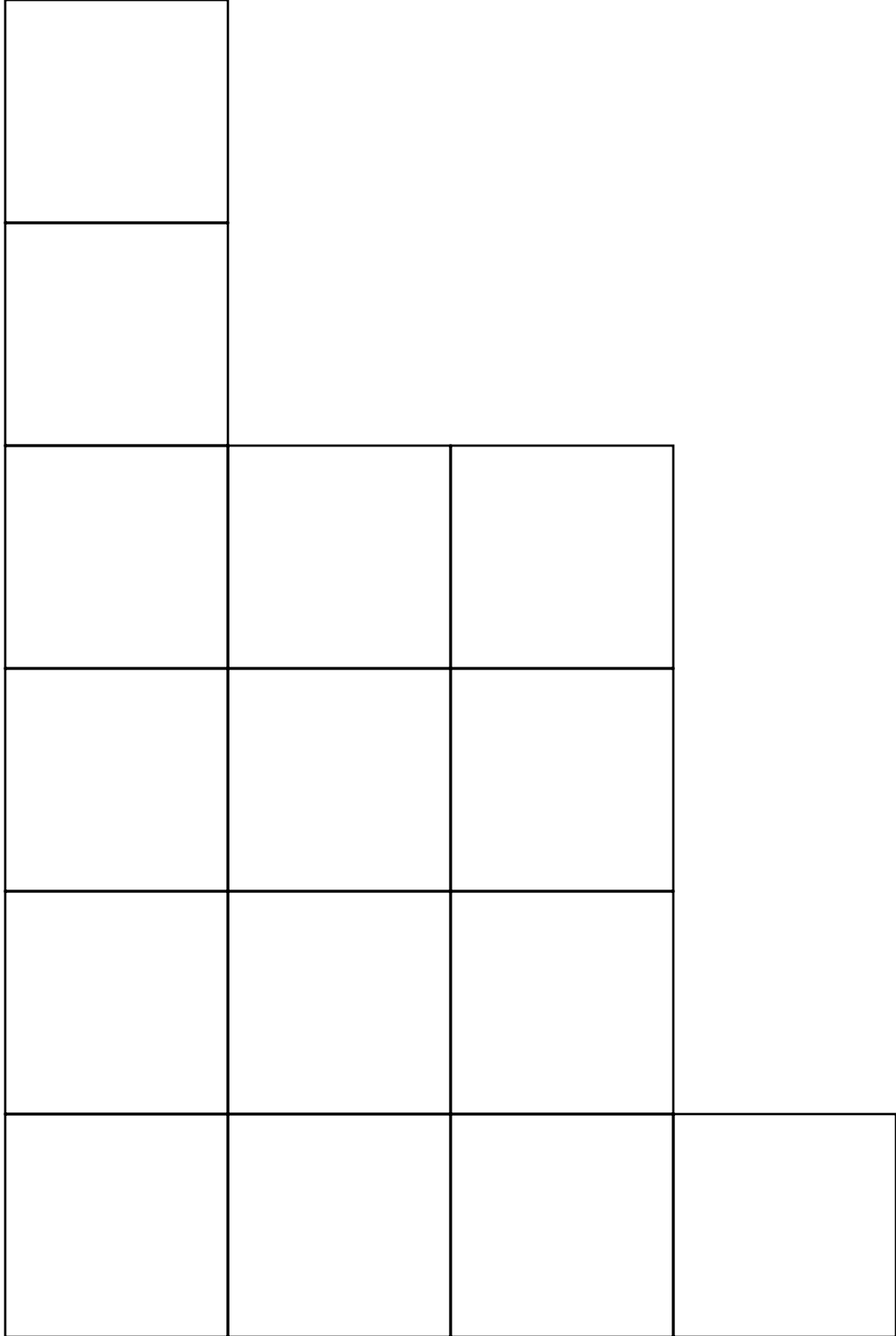
$$s_{\lambda/\mu} = \det |h_{\lambda_i - \mu_j + i - j}|$$

$$\{s_\lambda\}_{\lambda \vdash n} \quad \{h_\lambda\}_{\lambda \vdash n} \quad \{e_\lambda\}_{\lambda \vdash n}$$

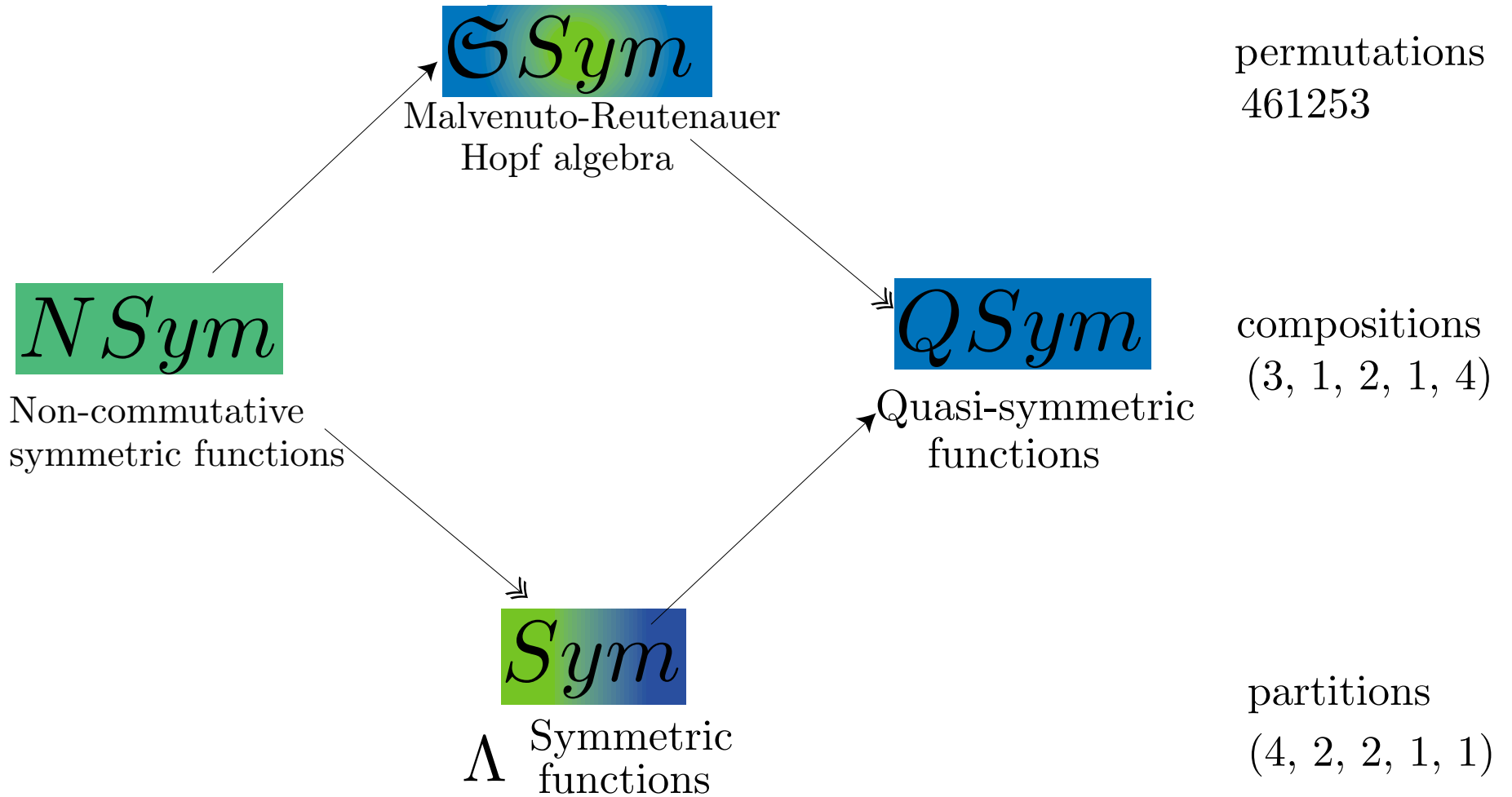
bases for the symmetric functions of degree n

$$\omega(h_\lambda) = e_\lambda \quad \omega(s_\lambda) = s_{\lambda'}$$

$$Y_1 = (0, 4, 4, 1)$$



$$\lambda = (4, 3, 3, 3, 1, 1)$$



Non-commutative symmetric functions

$$\begin{aligned}
 NSym &= \mathbb{Q}\langle \mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3, \dots \rangle \\
 \deg(\mathbf{h}_k) &= k \\
 \mathbf{h}_i \mathbf{h}_j &\neq \mathbf{h}_j \mathbf{h}_i
 \end{aligned}$$

The space of non-commutative symmetric functions is the free algebra with one generator in each degree

The NCSchur Functions

$$\mathbf{s}_\alpha = \sum_{\beta \geq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} \mathbf{h}_\beta$$

Example:

$$\mathbf{s}_{221} = \mathbf{h}_{221} - \mathbf{h}_{23} - \mathbf{h}_{41} + \mathbf{h}_5$$

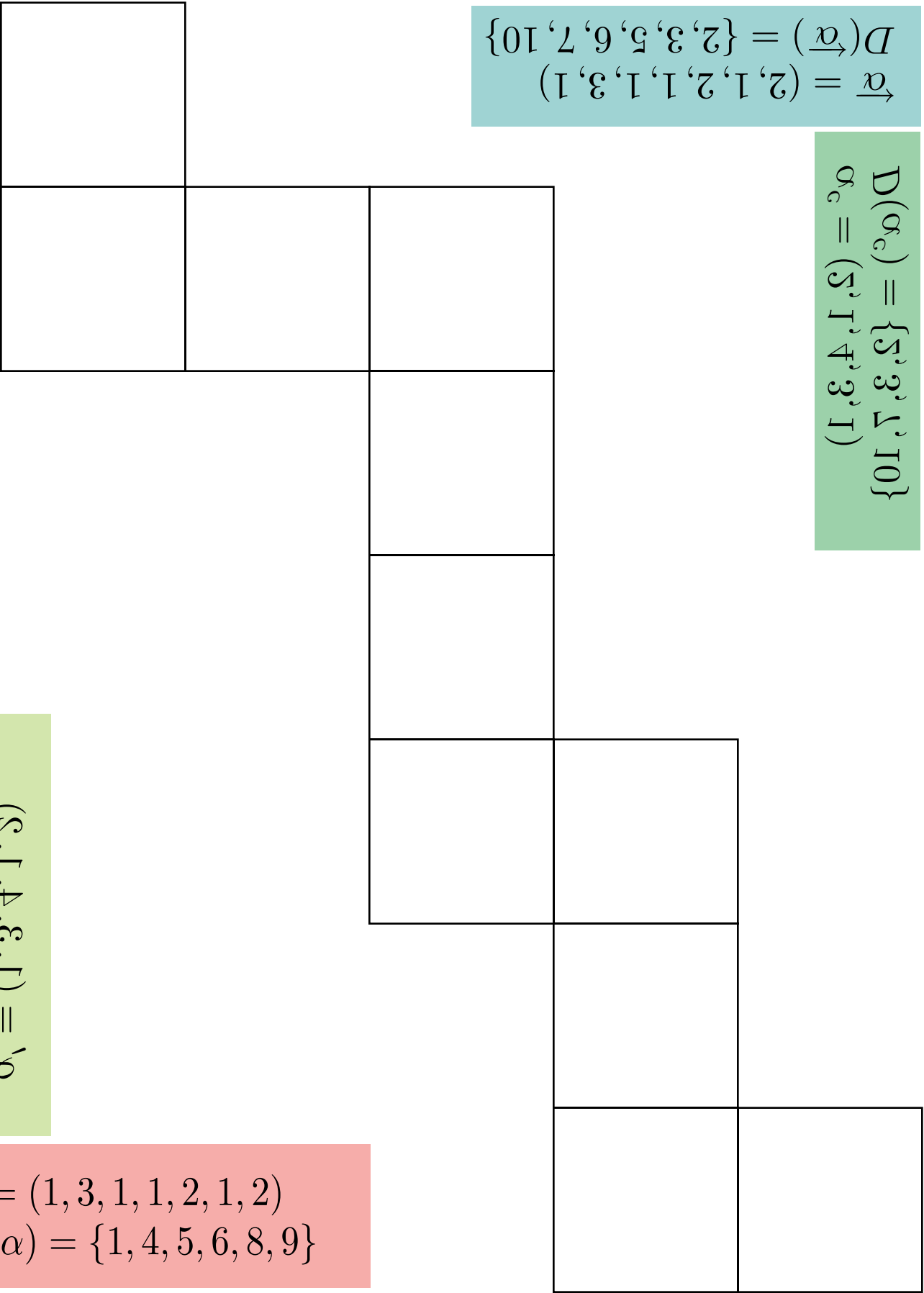
$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline & \square \\ \hline \end{array} = \mathbf{h}_{\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline & \square \\ \hline \end{array}} - \mathbf{h}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline & \square & \square \\ \hline & & \square \\ \hline \end{array}} - \mathbf{h}_{\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline & & \square \\ \hline \end{array}} + \mathbf{h}_{\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline & & & \square \\ \hline & & & \square \\ \hline & & & \square \\ \hline \end{array}}$$

$$D(\alpha_c) = \{5, 3, 5, 10\}$$

$$\alpha_c = (5, 1, 4, 3, 1)$$

$$\vec{\alpha} = (2, 1, 2, 1, 1, 3, 1)$$

$$D(\vec{\alpha}) = \{2, 3, 5, 6, 7, 10\}$$



$$D(\alpha_1) = \{1, 4, 8, 9\}$$

$$\alpha_1 = (1, 3, 4, 1, 5)$$

$$\alpha = (1, 3, 1, 1, 2, 1, 2)$$

$$D(\alpha) = \{1, 4, 5, 6, 8, 9\}$$

More non-commutative symmetric functions

$$\begin{aligned}
 e_k &= e_{k-1}h_1 - e_{k-2}h_2 + \cdots + (-1)^k h_k \\
 &= \sum_{\alpha \models k} (-1)^{k+\ell(\alpha)} h_\alpha
 \end{aligned}$$

$$e_\alpha = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_{\ell(\alpha)}}$$

$$s_\alpha = \sum_{\beta \geq \alpha} (-1)^{\ell(\alpha) - \ell(\beta)} h_\beta$$

$$\{s_\alpha\}_{\alpha \models n} \quad \{h_\alpha\}_{\alpha \models n} \quad \{e_\alpha\}_{\alpha \models n}$$

are bases for NCSF of degree n

$$\begin{array}{lll}
 \omega'(s_\alpha) = s_{\alpha'} & \omega^c(s_\alpha) = s_{\alpha^c} & \overleftarrow{\omega}(s_\alpha) = s_{\overleftarrow{\alpha}} \\
 \omega'(h_\alpha) = e_{\overleftarrow{\alpha}} & \omega^c(h_\alpha) = e_\alpha & \overleftarrow{\omega}(h_\alpha) = h_{\overleftarrow{\alpha}} \\
 \omega'(e_\alpha) = h_{\overleftarrow{\alpha}} & \omega^c(e_\alpha) = h_\alpha & \overleftarrow{\omega}(e_\alpha) = e_{\overleftarrow{\alpha}}
 \end{array}$$

For the map $\chi : NSym \rightarrow \Lambda$ by

$$\chi(\mathbf{h}_\alpha) = h_{\alpha_1} \cdots h_{\alpha_{\ell(\alpha)}}$$

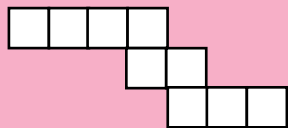
$$\chi(\mathbf{e}_\alpha) = e_{\alpha_1} \cdots e_{\alpha_{\ell(\alpha)}}$$

$$\chi(\mathbf{s}_{(1^a, b)}) = s_{(b, 1^a)}$$

in general

$$\chi(\mathbf{s}_\alpha) = s_\alpha \text{ as a skew Schur function}$$

Example:



$$\alpha = (4, 2, 3)$$

$$\chi(\mathbf{s}_\alpha) = s_{(7, 5, 4)/(4, 3)}$$

$$s_{\lambda/\mu} = \det |h_{\lambda_i - \mu_j + i - j}|$$

Hall-Littlewood symmetric functions

$$\begin{aligned}\mathbf{S}_m(s_\lambda) &= s_{(m,\lambda)} \\ \mathbf{S}_{\lambda_1} \mathbf{S}_{\lambda_2} \cdots \mathbf{S}_{\lambda_{\ell(\lambda)}} 1 &= s_\lambda\end{aligned}$$

For $V \in \text{Hom}(\Lambda, \Lambda)$

$$\overline{V} = \mu \circ \text{id} \otimes (V \circ S) \circ \Delta$$

$$R^q(f) = q^{\text{deg}(f)} f$$

for $f \in \Lambda$ of homogeneous degree

$$\widetilde{V}^q = \overline{\overline{V} R^q}$$

$$\mathbf{H}_m = \widetilde{\mathbf{S}}_m^q$$

$$H_\lambda^q = \mathbf{H}_{\lambda_1} \mathbf{H}_{\lambda_2} \cdots \mathbf{H}_{\lambda_{\ell(\lambda)}} 1$$

Theorem (Jing)

$$H_\mu^q = \sum_\lambda K_{\lambda\mu}(q) s_\lambda$$

Non-commutative Hall-Littlewood symmetric functions

$$\mathbb{S}_m(\mathfrak{s}_\alpha) = \mathfrak{s}_{(\alpha, m)}$$

$$\mathbb{S}_{\alpha_{\ell(\alpha)}} \mathbb{S}_{\alpha_{\ell(\alpha)-1}} \cdots \mathbb{S}_{\alpha_1} 1 = \mathfrak{s}_\alpha$$

For $V \in \text{Hom}(NCA, NCA)$

$$\overline{V} = \mu \circ \text{id} \otimes (V \circ S) \circ \Delta$$

$$R^q(f) = q^{\text{deg}(f)} f$$

for $f \in NCA$ of homogeneous degree

$$\widetilde{V}^q = \overline{\overline{V}} \overline{R^q}$$

$$\mathbb{H}_m = \widetilde{\mathbb{S}}_m^q$$

$$\mathbb{H}_\alpha^q := \mathbb{H}_{\alpha_{\ell(\alpha)}} \mathbb{H}_{\alpha_{\ell(\alpha)-1}} \cdots \mathbb{H}_{\alpha_1} 1$$

Example

$$\mathbb{H}_{\begin{array}{|c|} \hline \square \\ \hline \square \end{array}}^q = \begin{array}{|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + q^2 \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}$$

Theorem (Z-Bergeron)

$$\mathbf{H}_\alpha^q = \sum_{\beta \geq \alpha} q^{c(\alpha, \beta^c)} \mathbf{s}_\beta$$

where

$$c(\alpha, \beta) = \sum_{i \in D(\alpha) \cap D(\beta)} i$$

moreover

$$\chi(\mathbf{H}_{(1^a, b)}^q) = H_{(b, 1^a)}^q$$

$$\mathbf{H}_{(113)}^q = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} + q \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + q^2 \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} + q^3 \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}$$

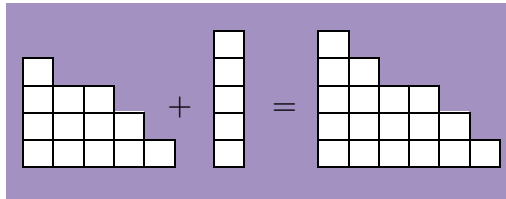
$$\chi(\mathbf{H}_{(113)}^q) = H_{(311)}^q = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} + q \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + (q + q^2) \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} + q^3 \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}$$

$$\mathbf{H}_\alpha^q \Big|_{q=0} = \mathbf{s}_\alpha$$

$$\mathbf{H}_\alpha^q \Big|_{q=1} = \mathbf{h}_\alpha$$

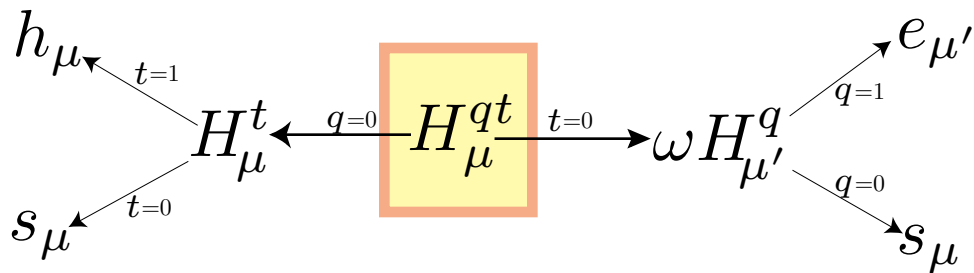
Macdonald symmetric functions

Define $\mathbf{H}_{1^m}^t(H_\lambda^t) = H_{\lambda+1^m}^t$



set $\mathbf{H}_{1^m}^{qt} = \widetilde{\mathbf{H}}_{1^m}^t{}^q$

$$\begin{aligned}
 H_{\mu'}^{qt} &= \mathbf{H}_{1^{\mu_1}}^{qt} \mathbf{H}_{1^{\mu_2}}^{qt} \cdots \mathbf{H}_{1^{\mu_{\ell(\mu)}}}^{qt} 1 \\
 &= \sum_{\lambda} K_{\lambda\mu'}(q, t) s_{\lambda}
 \end{aligned}$$



$$\omega H_{\mu}^{qt} = q^{n(\mu')} t^{n(\mu)} H_{\mu}^{\frac{1}{q} \frac{1}{t}}$$

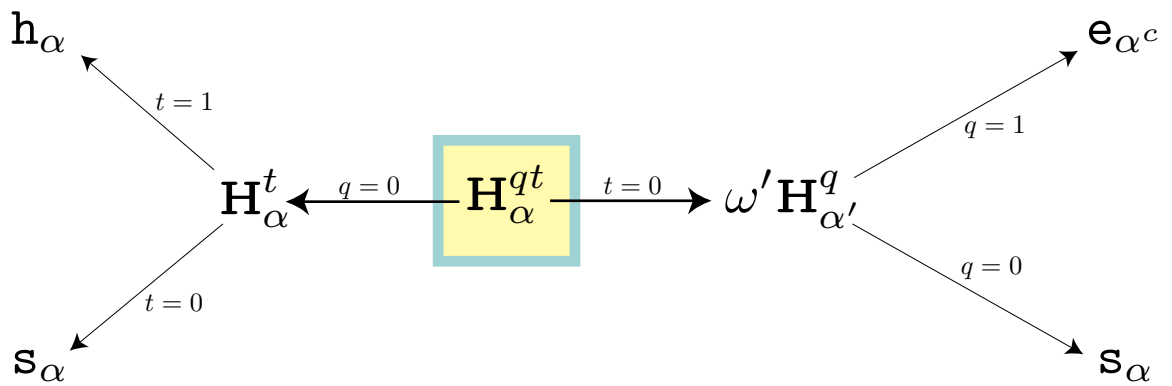
$$\omega H_{\mu}^{tq} = H_{\mu'}^{qt}$$

Example

$$\mathbf{H}_{\begin{smallmatrix} \square \\ \square \end{smallmatrix}}^{qt} = t \begin{smallmatrix} \square & \square & \square \\ \square & & \end{smallmatrix} + (qt + 1) \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} + q \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}$$

A non-commutative Macdonald symmetric function

$$\mathbf{H}_\alpha^{qt} = \sum_{\beta} q^{c(\alpha', \overleftarrow{\beta})} t^{c(\alpha, \beta^c)} \mathbf{s}_\beta$$



$$\omega^c \mathbf{H}_\alpha^{qt} = q^{n(\alpha')} t^{n(\alpha)} \mathbf{H}_\alpha^{\frac{1}{q} \frac{1}{t}}$$

$$\omega' \mathbf{H}_\alpha^{qt} = \mathbf{H}_{\alpha'}^{tq}$$

$$\chi(\mathbf{H}_\alpha^{qt}) = H_{(b1^a)}^{qt}$$

if $\alpha = (1^a b)$

Example

$$\mathbf{H}_{\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}}^{qt} = t \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} + qt \begin{array}{|c|} \hline \square & \square \\ \hline \square \\ \hline \end{array} + q \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$$