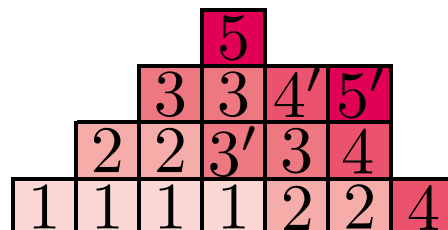
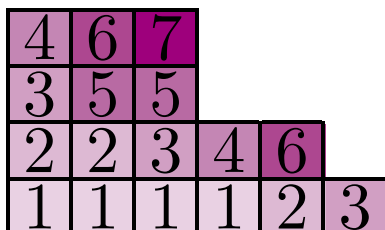


# Hall-Littlewood analogs in the $Q$ -function algebra

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# The Symmetric Functions

$$\begin{aligned}\Lambda &= \mathbb{Q}[p_1, p_2, p_3, \dots] \\ &= \mathbb{Q}[h_1, h_2, h_3, \dots] \\ \deg(h_k) &= k\end{aligned}$$

The space of symmetric functions is generated algebraically by the simple homogeneous and power symmetric functions.

# The Schur Functions

$$s_\lambda = \det |h_{\lambda_i + i - j}|$$

Example:

$$s_{(2,2,1)} = \begin{vmatrix} h_2 & h_3 & h_4 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} = h_2^2 h_1 - h_2 h_3 - h_3 h_1^2 + h_4 h_1$$

Bernstein Schur row adding operator

$$\mathbf{S}_m(s_\mu) = s_{(m,\mu)}$$

Jing's Hall-Littlewood row adding operator

$$\mathbf{H}_m(H_\mu^q) = H_{(m,\mu)}^q$$

Hopf  $q$ -ification of linear operators

$$\Delta \text{ coproduct} \quad \Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$$

$$\mu \text{ multiplication} \quad \mu(f \otimes g) = fg$$

$$S \text{ antipode} \quad S(p_k) = -p_k$$

$$R^q(f) = q^{\deg(f)} f$$

Define for  $V \in \text{Hom}(\Lambda, \Lambda)$

$$\overline{V} := \mu \circ \text{id} \otimes (VS) \circ \Delta$$

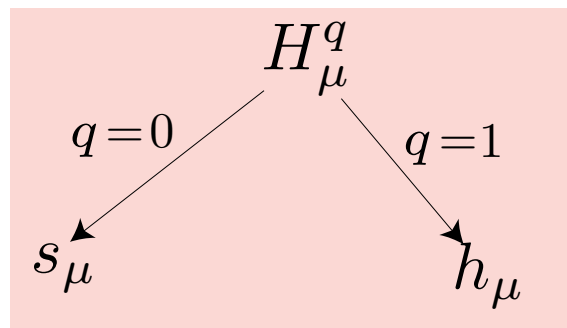
$$\widetilde{V}^q := \overline{V} \overline{R^q}$$

$$\mathbf{H}_m = \widetilde{\mathbf{S}}_m^q$$

# Hall-Littlewood symmetric functions

## Kostka-Foulkes coefficients

$$H_{\mu}^q = \sum_{\lambda} K_{\lambda\mu}(q) s_{\lambda}$$



$$K_{\lambda\mu}(0) = 1 \text{ if } \lambda = \mu \text{ and} \\ 0 \text{ otherwise}$$

$$K_{\lambda\mu}(1) = h_{\mu} \Big|_{s_{\lambda}}$$

Example:

$$H_{(2,2,2)}^q = q^6 s_6 + q^4 (q+1) s_{5,1} + q^2 (q^2 + q + 1) s_{4,2} \\ + q^3 s_{4,1,1} + q^3 s_{3,3} + q (q+1) s_{3,2,1} + s_{2,2,2}$$

# Column strict tableaux (CST)

Young diagram of a partition with entries that increase weakly in the rows and strictly in the columns

Example:

4	6	7			
3	5	5			
2	2	3	4	6	
1	1	1	1	2	3

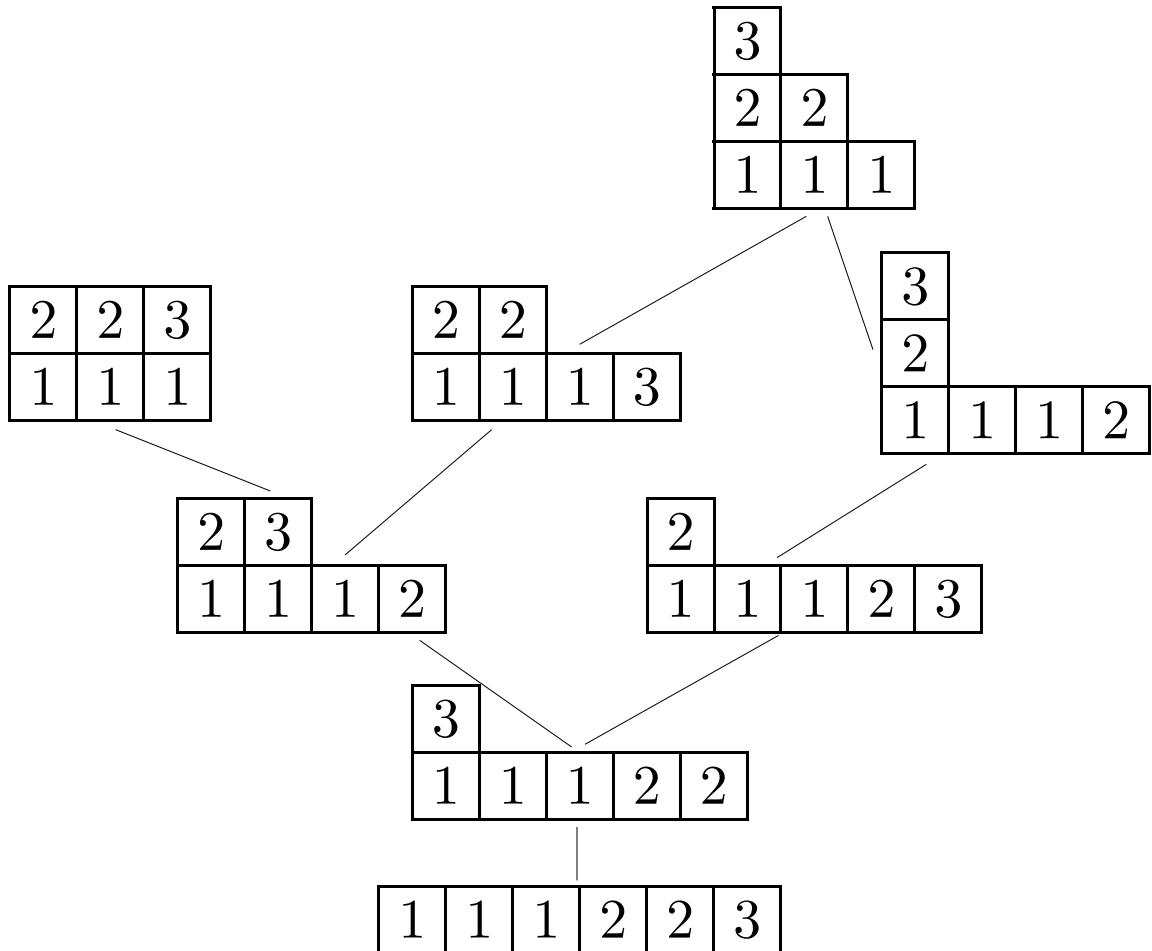
$\lambda(T)$  = the shape of a tableau  $T$

$\mu(T)$  = the content of a tableau  $T$

$$h_{\mu} = \sum_{\substack{T \in CST \\ \mu(T) = \mu}} s_{\lambda(T)}$$

$$s_{\lambda} = \sum_{\substack{T \in CST \\ \lambda(T) = \lambda}} m_{\mu(T)}$$

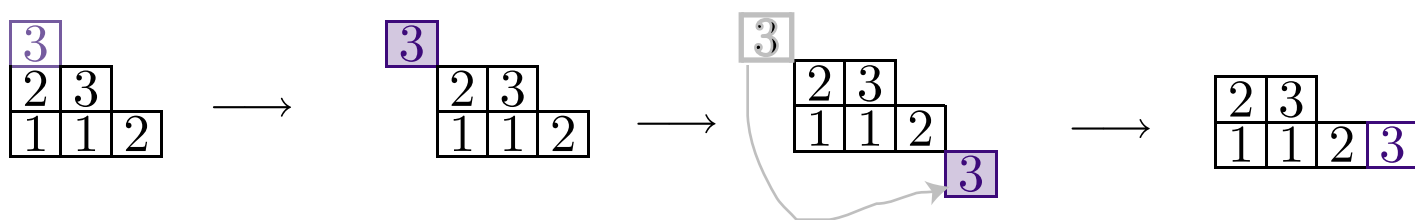
Poset structure of the column strict tableaux of content  $(3, 2, 1)$



$$H_{(3,2,1)}^q = q^4 s_6 + q^2 (q + 1) s_{5,1} + q (q + 1) s_{4,2} + q s_{4,1,1} + q s_{3,3} + s_{3,2,1}$$

# Covering relation - cyclage

1. Pick a corner
2. column evacuate
3. row insert



If  $\mu \geq \nu$ , then there is an injection from  $CST^\mu$  to  $CST^\nu$  which preserves the statistic

This implies

$$q^{n(\mu)} H_\nu^q - q^{n(\nu)} H_\mu^q$$

is Schur positive

where  $n(\mu) := \sum_i (i - 1)\mu_i$

# Schur's $Q$ -functions

$$\Gamma = \mathbb{Q}[p_1, p_3, p_5, \dots] \subset \Lambda$$

$$\theta : \Lambda \longrightarrow \Gamma$$

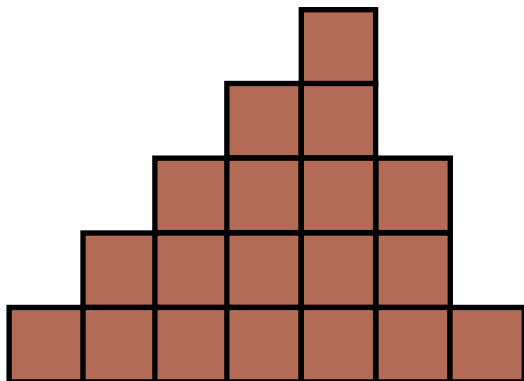
$$\theta(p_k) = (1 - (-1)^k)p_k$$

$$\theta(h_\mu) = q_\mu$$

$$\text{Definition: } Q_\mu = \theta(H_\mu^q) \Big|_{q=-1}$$

Basis indexed by strict partitions

$(7, 5, 4, 2, 1)$



# Pieri rule for the product $q_m Q_\lambda$

$$q_m Q_\mu = \sum_{\lambda/\mu \in \mathcal{H}_m} 2^{a(\lambda/\mu) - \ell(\lambda) + \ell(\mu)} Q_\lambda$$

$a(\lambda/\mu) = 1 +$  the number of  $1 < j \leq \ell(\lambda)$   
such that  $\lambda_j > \mu_j$  and  $\mu_{j-1} > \lambda_j$

$$\begin{aligned}
 q_3 \begin{array}{cccc} & & \blacksquare & \\ & \blacksquare & \blacksquare & \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array} &= 2^2 \begin{array}{ccccc} & & \blacksquare & \blacksquare & \\ & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array} + 2^1 \begin{array}{ccccc} & & \blacksquare & & \\ & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array} \\
 &+ 2^2 \begin{array}{ccccc} & & \blacksquare & & \\ & \blacksquare & \blacksquare & \blacksquare & \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array} + 2^1 \begin{array}{ccccccc} & & \blacksquare & & & & \\ & \blacksquare & \blacksquare & & & & \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array} \\
 &+ 2^0 \begin{array}{cccc} & & & \blacksquare \\ & & \blacksquare & \blacksquare \\ & \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{array}
 \end{aligned}$$

$q_\lambda \longrightarrow$  Schur's  $Q$ -functions

$$q_4 q_3 q_2$$

$$q_4 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 1 \\ \hline \end{array}$$

$$= Q_4$$

$$q_3 q_4 = \begin{array}{|c|c|c|c|} \hline & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} + 2 \begin{array}{|c|c|c|c|} \hline & 2 & 2 & \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} 2^* + 2 \begin{array}{|c|c|c|c|} \hline & 2 & & \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} 2^* 2 + 2 \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2^* & 2 & 2 \\ \hline \end{array}$$

$$= 2Q_7 + 2Q_{6,1} + 2Q_{5,2} + Q_{4,3}$$

$$q_2 q_3 q_4 = \begin{array}{|c|c|c|} \hline & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array} + 2 \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array} 3^* + 2 \begin{array}{|c|c|c|c|} \hline & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \begin{array}{|c|} \hline 3^* \\ \hline \end{array} + 2 \begin{array}{|c|c|c|c|} \hline 2 & 2 & 2 & \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} \begin{array}{|c|} \hline 3^* \\ \hline \end{array} 3$$

$$+ 4 \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 2 & 2 & 3^* \\ \hline 1 & 1 & 1 \\ \hline \end{array} 2^* + 4 \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 2 & 2 & \\ \hline 1 & 1 & 1 \\ \hline \end{array} 2^* 3^* + 4 \begin{array}{|c|c|c|c|} \hline & 2 & 2 & 3^* \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} 2^*$$

$$+ 8 \begin{array}{|c|c|c|c|} \hline & 2 & 2 & 3^* \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} 2^* 3^* + 4 \begin{array}{|c|c|c|} \hline & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array} 2^* 3^* 3 + 2 \begin{array}{|c|c|} \hline & 3 \\ \hline 2 & 3' \\ \hline 1 & 1 \\ \hline \end{array} 2^* 2$$

$$+ 4 \begin{array}{|c|c|c|c|} \hline & 2 & 3^* & 3 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array} 2^* 2 + 8 \begin{array}{|c|c|c|} \hline & 2 & 3^* \\ \hline 1 & 1 & 1 \\ \hline \end{array} 2^* 2 3^* + 4 \begin{array}{|c|c|} \hline & 2 \\ \hline 1 & 1 \\ \hline \end{array} 2^* 2 3^* 3$$

$$+ 2 \begin{array}{|c|c|c|} \hline & 3 & 3 \\ \hline 1 & 1 & 1 \\ \hline \end{array} 2^* 2 2 + 4 \begin{array}{|c|c|c|} \hline & 3 & \\ \hline 1 & 1 & 1 \\ \hline \end{array} 2^* 2 2 3^* + 4 \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2^* & 2 & 2 \\ \hline \end{array} \begin{array}{|c|} \hline 3^* \\ \hline \end{array} 3$$

$$= 4Q_9 + 8Q_{8,1} + 14Q_{7,2} + 14Q_{6,3} + 6Q_{6,2,1} + 6Q_{5,4} + 6Q_{5,3,1} + Q_{4,3,2}$$

# Marked Shifted Tableaux (MST)

Shifted Young diagram with entries weakly increasing in the rows and columns in the alphabet

$$1' < 1 < 2' < 2 < 3' < 3 < \dots$$

with the configurations  $\begin{array}{|c|} \hline k \\ \hline k \\ \hline \end{array}$  and  $\begin{array}{|c|c|} \hline k' & k' \\ \hline \end{array}$  excluded

and diagonal entries are not marked (MST')

Example:

			5			
		3	3	4'	5'	
	2	2	3'	3	4	
1	1	1	1	2	2	4

$\lambda(T)$  = the shape of a tableau  $T$

$\mu(T)$  = the content of a tableau  $T$

$$q_\mu = \sum_{\substack{T \in MST' \\ \mu(T) = \mu}} Q_{\lambda(T)}$$

$$Q_\lambda = 2^{\ell(\lambda)} \sum_{\substack{T \in MST' \\ \lambda(T) = \lambda}} m_{\mu(T)}$$

Jing

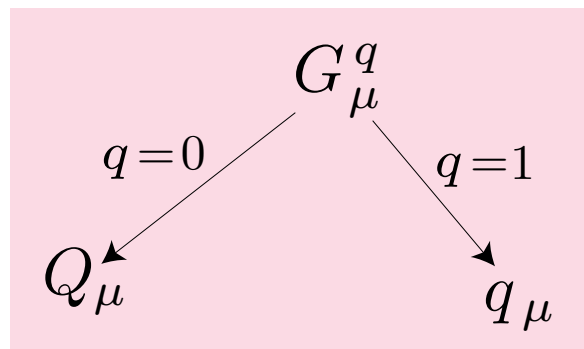
$$\mathbf{Q}_m = \theta(\mathbf{H}_m^{q=-1})$$

$$\mathbf{Q}_m(Q_\lambda) = Q_{(m,\lambda)} \quad \text{for } m > \lambda_1$$

$$\mathbf{G}_m := \widetilde{\mathbf{Q}_m}^q$$

$$G_\mu^q := G_{\mu_1} G_{\mu_2} \cdots G_{\mu_k} 1$$

$$G_\mu^q = \sum_{\lambda} L_{\lambda\mu}(q) Q_\lambda$$



Example :

$$G_{(4,3,1)}^q = 4q^5 Q_8 + (2q^3 + 4q^4) Q_{7,1} + (4q^3 + 4q^2) Q_{6,2} \\ + (4q^2 + 2q) Q_{5,3} + 2q Q_{5,2,1} + Q_{4,3,1}$$

# Some Results

Theorem (Tudose & Zabrocki)

$$L_{\alpha, (n, \mu)}(q) = \sum_{s=1}^{t: \alpha_t \geq n} (-1)^{s-1} q^{\alpha_s - n} \sum_{\lambda: \lambda/\alpha^{(s)} \in \mathcal{H}_{(\alpha_s - n)}} 2^{a(\lambda/\alpha^{(s)})} L_{\lambda\mu}(q)$$

where  $n > \mu_1$  and  $\alpha^{(s)}$  is  $\alpha$  with part  $\alpha_s$  removed.

Proposition

$$\deg_q L_{\lambda\mu}(q) = q^{n(\mu) - n(\lambda)}$$

Proposition

$$2^{\ell(\mu) - \ell(\lambda)} \text{ divides } L_{\lambda\mu}(q)$$

# Conjectures

Conjecture

$$G_{\mu}^q = \sum_{T \in MST^{\mu}} q^{d(T)} Q_{\lambda(T)}$$

In particular,  
 $L_{\lambda\mu}(q)$  is a polynomial in  $q$  with  
non-negative integer coefficients

Conjecture

If  $\mu > \lambda$  are strict partitions then

$$2^{\ell(\mu)} q^{n(\mu)} G_{\lambda}^q - 2^{\ell(\lambda)} q^{n(\lambda)} G_{\mu}^q$$

is  $Q$ -Schur positive