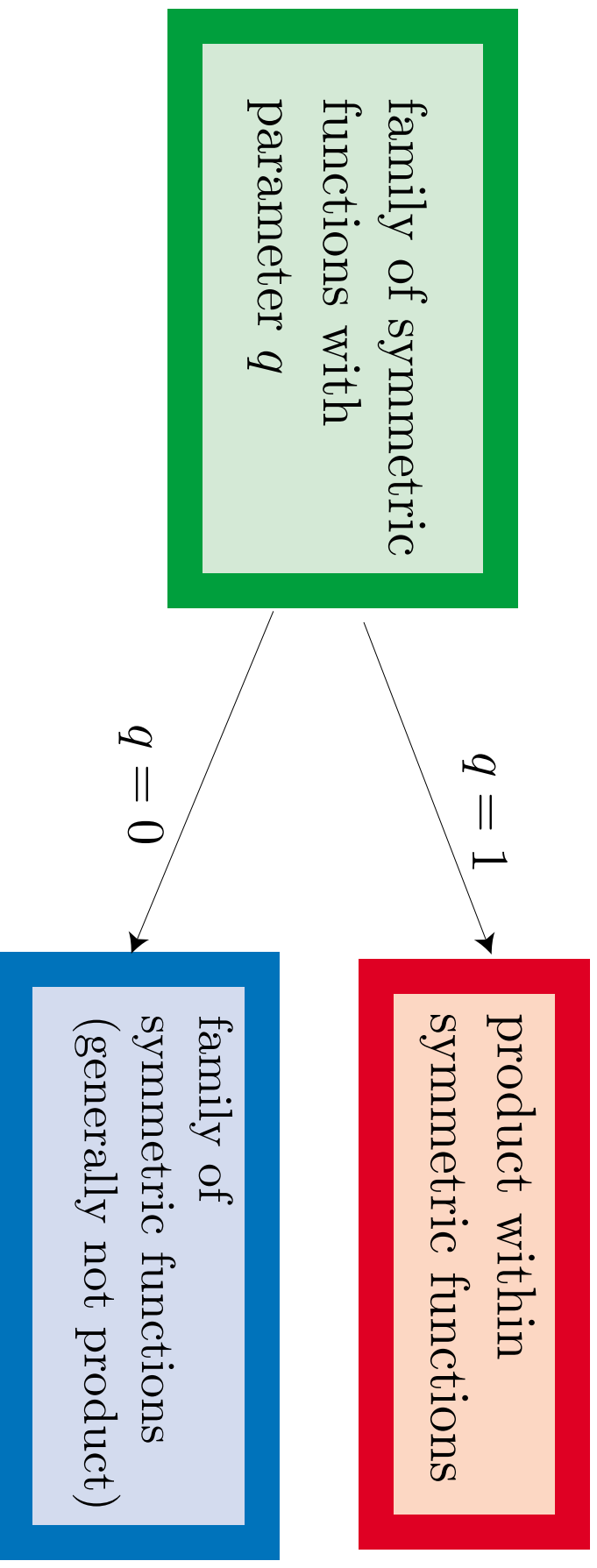




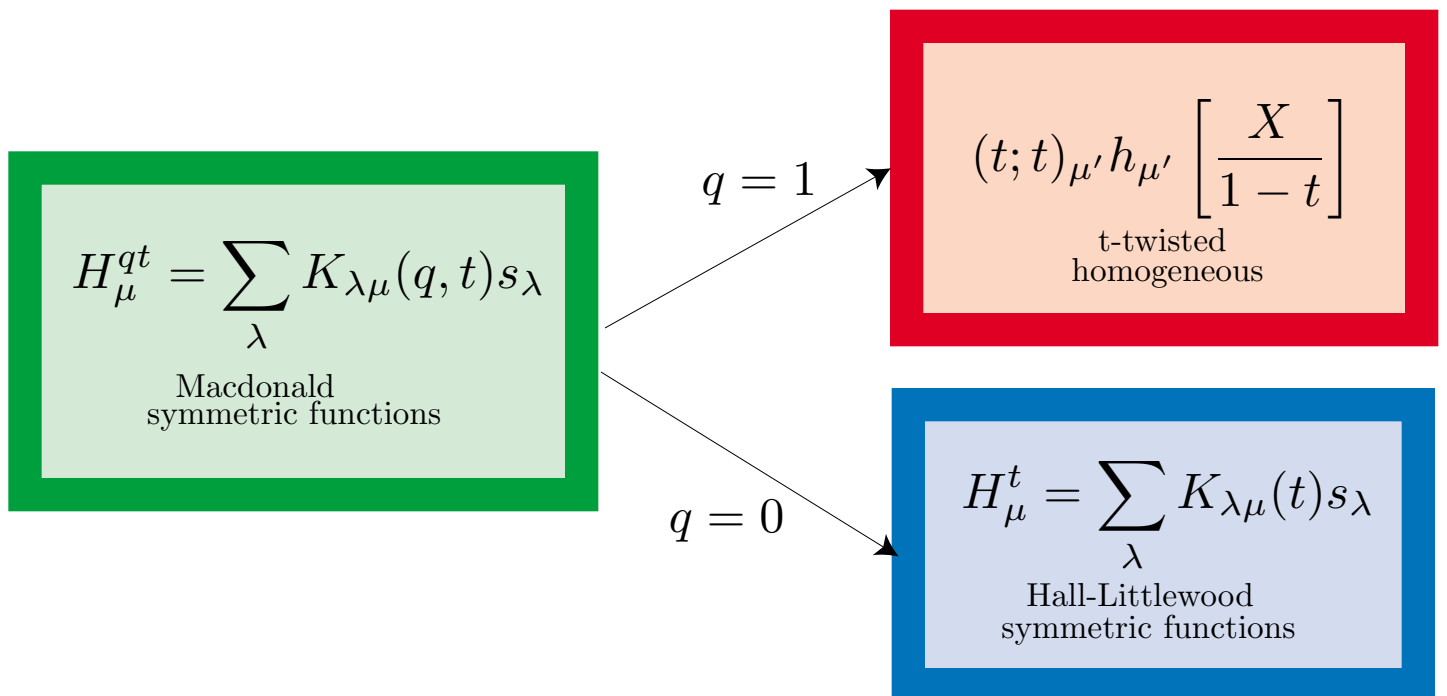
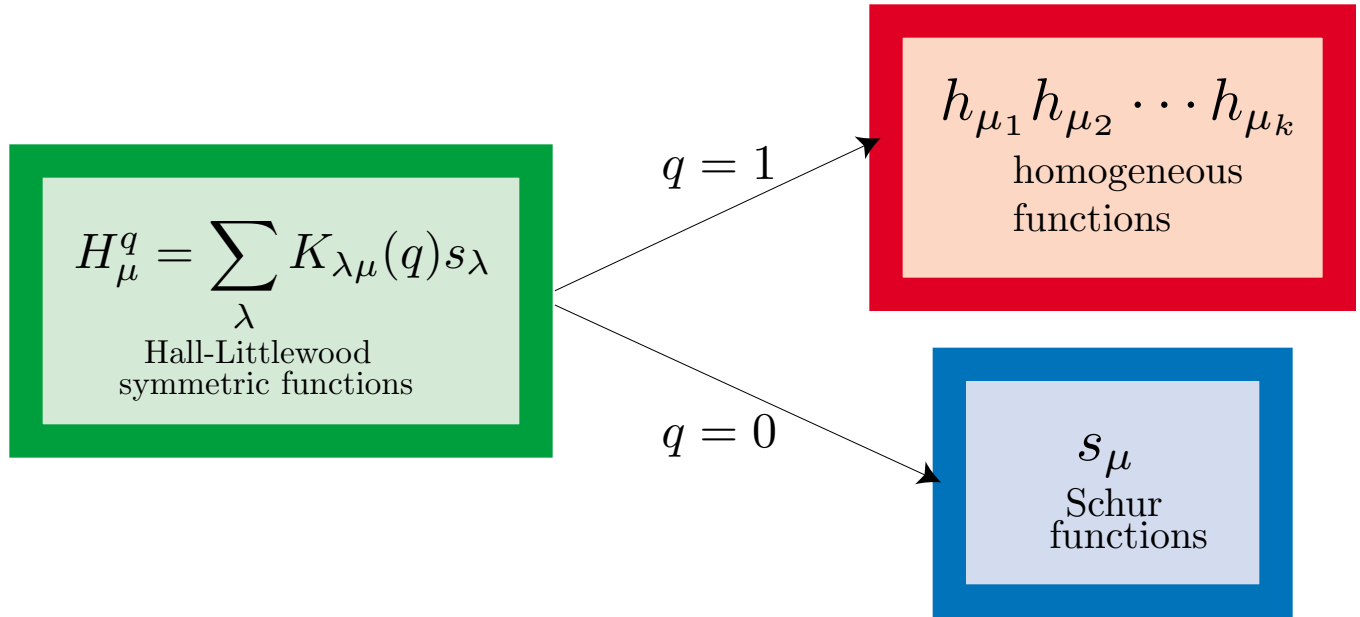
q-Analogs of Hopf Algebra Structures

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q-Analogues in Symmetric Functions



Examples:



Why look at q -SF's like this?

1. Derive algebraic relations and recurrences

2. Create general means for defining q -analog

Identify other natural q -analog families.

3. Identify when this is occurring.

Can we classify q -analog 'types?'

4. Derive properties on abstract level.

When will coefficients be positive?

q -Analogues in graded Hopf Algebras

generating function
for coefficients
with a parameter q

$q = 1$

product within
Hopf algebra

$q = 0$

expression that
is generally not a
product

Examples:

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$$

$q = 1$

$$(1 + x)^n$$

$q = 0$

$$1 + x + \cdots + x^n$$

Hopf Algebra Notation/Definitions

Algebra

A - module over a ring R

μ - multiplication

η - unit

$$\mu : A \otimes A \rightarrow A$$

$$\eta : R \rightarrow A$$

satisfying:

$$\mu \circ (id \otimes \mu) = \mu \circ (\mu \otimes id) \quad \text{associativity}$$

$$\mu \circ (\eta \otimes id) = id = \mu \circ (id \otimes \eta)$$

Let $\tau : A \otimes A \rightarrow A \otimes A$ by $\tau(f \otimes g) = g \otimes f$
 A is commutative if $\mu \circ \tau = \mu$.

Co-algebra

C - module over a ring R

Δ - co-multiplication

ε - co-unit

$$\Delta : C \rightarrow C \otimes C$$

$$\varepsilon : C \rightarrow R$$

satisfying:

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \quad \text{associativity}$$

$$(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta$$

C is co-commutative if $\tau \circ \Delta = \Delta$

Hopf Algebra

A bi-algebra is an algebra and a co-algebra at the same time such that Δ is a homomorphism with respect to μ .

H is a Hopf algebra if it is a bi-algebra with a S - antipode $S : H \rightarrow H$

satisfying

$$\mu \circ id \otimes S \circ \Delta = \mu \circ S \otimes id \circ \Delta = \eta \circ \varepsilon$$

If H is commutative or co-commutative then S is an involution.

product	$\mu : H \otimes H \rightarrow H$
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unit	$\eta : R \rightarrow H$
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co-product	$\Delta : H \rightarrow H \otimes H$
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co-unit	$\varepsilon : H \rightarrow R$
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antipode	$S : H \rightarrow H$
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The q -Analog for a graded Hopf algebra

Convolution $U, V \in \text{Hom}(H, H)$

$$U * V = \mu \circ U \otimes V \circ \Delta$$

this operation is associative and satisfies

$$V * (\eta\varepsilon) = (\eta\varepsilon) * V = V$$

Bar operation

$$V \in \text{Hom}(H, H)$$

$$\bar{V} = id * (VS)$$

if H is co-commutative then

$$\bar{\bar{V}} = V$$

q -Analog

H is graded, then there exists an operator $R^q \in \text{Hom}(H, H)$

For $f \in H$ and f of degree d $R^q(f) = q^d f$ Let $F^q = \bar{R^q}$

$$\tilde{V}^q = \overline{\bar{V} R^q} = \overline{\bar{V} F^q}$$

Example 1: Binomial coefficients

Hopf algebra: polynomials ring in x $\mathbb{Q}[x]$

product: $\mu(x^i \otimes x^j) = x^{i+j}$

co-product: $\Delta(x^n) = \sum_{k=0}^n \binom{n}{k} x^{n-k} \otimes x^k$

antipode: $S(x^n) = (-1)^n x^n$

grading: $R^q(x^n) = q^n x^n$

Linear operators

$$B(1) = 1 + x$$

$$B(x^n) = x^{n+1} \text{ for } n > 0$$

$$\tilde{B}^q(\tilde{B}^q(\dots \tilde{B}^q(1)\dots)) = \sum_{k=0}^n \left[\begin{matrix} n \\ k \end{matrix} \right]_q x^k$$

$q = 0$

$$B(B(\dots B(1)\dots)) = 1 + x + \dots + x^n$$

$q = 1$

$$\begin{aligned} B(1)B(1)\dots B(1) &= (1+x)^n \\ &= \sum_{k=0}^n \binom{n}{k} x^k \end{aligned}$$

Example 2: Partitions with distinct parts

Hopf algebra: polynomials ring in x $\mathbb{Q}[x]$

product: $\mu(x^i \otimes x^j) = x^{i+j}$

co-product: $\Delta(x^n) = \sum_{k=0}^n \binom{n}{k} x^{n-k} \otimes x^k$

antipode: $S(x^n) = (-1)^n x^n$

grading: $R^q(x^n) = q^n x^n$

Linear operators

$$B_k(1) = 1 + x^k$$

$$B_k(x^n) = x^n \quad \text{for } n > 0$$

$$\tilde{B}_k^q(\cdots \tilde{B}_2^q(\tilde{B}_1^q(1)) \cdots)$$

$$= \sum_{m \geq 0} \sum_{\lambda} q^{n(\lambda)} x^m$$

$n(\lambda) = \sum_{i \geq 1} \lambda_i(i-1)$

the sum is over all partitions λ of m with distinct parts of size $\leq k$.

$q = 0$

$$B_k(\cdots B_2(B_1(1)) \cdots)$$

$$= 1 + x + \cdots + x^k$$

$q = 1$

$$B_1(1) B_2(1) \cdots B_k(1)$$

$$= \prod_{i=1}^k (1 + x^i) = \sum_{m \geq 0} \sum_{\lambda} x^m$$

Example 3: Falling factorial

Hopf algebra: polynomials ring in x $\mathbb{Q}[x]$

product: $\mu(x^i \otimes x^j) = \binom{i+j}{j} x^{i+j}$

co-product: $\Delta(x^n) = \sum_{i=0}^n x^{n-i} \otimes x^i$

antipode: $S(x^n) = (-1)^n x^n$

grading: $R^q(x^n) = q^n x^n$

Linear operators

$$B(1) = 1 + x$$

$$B(x^n) = x^{n+1} \text{ for } n > 0$$

$$\tilde{B}^q(\tilde{B}^q(\dots \tilde{B}^q(1)\dots)) = \sum_{i=0}^n [n]_q [n-1]_q \dots [n-i+1]_q x^i$$

$q = 0$

$q = 1$

$$B(B(\dots B(1)\dots)) = 1 + x + \dots + x^n$$

$$B(1) \bullet B(1) \bullet \dots \bullet B(1) = \sum_{i=0}^n n(n-1)\dots(n-i+1)x^i$$

Example 4: q -Koskta coefficients

Hopf algebra: Symmetric functions $\mathbb{Q}[p_1, p_2, p_3, \dots]$

product: $\mu(f \otimes g) = fg \quad \mu(s_\lambda \otimes s_\mu) = \sum_\nu c_{\lambda\mu}^\nu s_\nu$

co-product: $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$
 $\Delta(s_\nu) = \sum_{\lambda, \mu} c_{\lambda\mu}^\nu s_\lambda \otimes s_\mu$

antipode: $S(p_k) = (-1)^k p_k \quad S(s_\lambda) = (-1)^{|\lambda|} s_{\lambda'}$

grading: $R^q(p_k) = q^k p_k \quad R^q(s_\lambda) = q^{|\lambda|} s_\lambda$

Linear operators

$$B_m(s_\lambda) = s_{(m, \lambda)}$$

Theorem: Jing

$$\tilde{B}_{\mu_1}^q (\tilde{B}_{\mu_2}^q (\dots \tilde{B}_{\mu_k}^q (1) \dots)) = \sum_{\lambda} K_{\lambda\mu}(q) s_\lambda$$

$q = 0$

$$B_{\mu_1} (B_{\mu_2} (\dots B_{\mu_k} (1) \dots)) = s_\mu$$

$q = 1$

$$B_{\mu_1}(1) B_{\mu_2}(1) \dots B_{\mu_k}(1) = h_{\mu_1} h_{\mu_2} \dots h_{\mu_k} = \sum_{\lambda} K_{\lambda\mu} s_\lambda$$

Application: Generalized Kostka polynomials (q -Littlewood-Richardson coefficients)

Theorem: Shimozono-Z

Instead we take as our linear operators

$$B_\mu = B_{\mu_1} B_{\mu_2} \cdots B_{\mu_k}$$

$$B_\mu(s_\lambda) = s_{(\mu \cdot \lambda)}$$

then for a sequence of partitions

$$\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(k)}$$

the q -analog is a family of symmetric functions whose coefficients at $q = 1$ are the Littlewood-Richardson coefficients

$$\tilde{B}_{\mu^{(1)}}^q (\tilde{B}_{\mu^{(2)}}^q (\cdots \tilde{B}_{\mu^{(k)}}^q (1) \cdots)) = \sum_{\lambda} K_{\lambda; (\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(k)})} (q) s_{\lambda}$$

Shimozono-Weyman

$q = 0$

$q = 1$

$$B_{\mu^{(1)}} (B_{\mu^{(2)}} (\cdots B_{\mu^{(k)}} (1) \cdots))$$

$$= s_{(\mu^{(1)} \cdot \mu^{(2)} \cdots \mu^{(k)})}$$

$$B_{\mu^{(1)}} (1) B_{\mu^{(2)}} (1) \cdots B_{\mu^{(k)}} (1)$$

$$= s_{\mu^{(1)}} s_{\mu^{(1)}} \cdots s_{\mu^{(1)}}$$

Example 5: q,t-Kostka polynomials

Hopf algebra: Symmetric functions $\mathbb{Q}[p_1, p_2, p_3, \dots]$

product: $\mu(f \otimes g) = fg \quad \mu(s_\lambda \otimes s_\mu) = \sum_\nu c_{\lambda\mu}^\nu s_\nu$

co-product: $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$
 $\Delta(s_\nu) = \sum_{\lambda, \mu} c_{\lambda\mu}^\nu s_\lambda \otimes s_\mu$

antipode: $S(p_k) = (-1)^k p_k \quad S(s_\lambda) = (-1)^{|\lambda|} s_{\lambda'}$

grading: $R^q(p_k) = q^k p_k \quad R^q(s_\lambda) = q^{|\lambda|} s_\lambda$

Linear operators

Add a column to Hall-Littlewood SFs

$$B_m^t(H_\mu^t) = H_{\mu+1^m}^t$$

Theorem: Noumi-Kirillov

$$\widetilde{B}_{\mu_1}^t \left(\widetilde{B}_{\mu_2}^t \left(\dots \widetilde{B}_{\mu_k}^t (1) \dots \right) \right) = \sum_{\lambda} K_{\lambda\mu'}(q, t) s_{\lambda}$$

$q = 0$

$q = 1$

$$B_{\mu_1}^t \left(B_{\mu_2}^t \left(\dots B_{\mu_k}^t (1) \dots \right) \right) = \sum_{\lambda} K_{\lambda\mu'}(t) s_{\lambda}$$

$$B_{\mu_1}^t (1) B_{\mu_2}^t (1) \dots B_{\mu_k}^t (1) = h_{\mu} \left[\frac{X}{1-t} \right] (t; t)_{\mu}$$

Applications to q, t -Kostka polynomials

1. Polynomiality of coefficients

Macdonald's definition only immediately implied that coefficients were rational functions in q and t

t -Kostka are polynomials $\implies q, t$ -Kostka are polynomials

2. Recurrences

Recursively calculate coefficients from smaller partitions

Action of B_m^t on the Schur functions known
"ribbon rule" for t -Kostka



Derive generalized ribbon rule for q, t -Kostka

Theorem: a "Macdonald-Morris" recurrence

positivity?

Non-commutative symmetric functions

$$\mathbb{Q}\langle h_1, h_2, h_3, \dots \rangle$$

linear basis $h_\alpha := h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_k}$

where α is a composition with non-zero parts

$D(\alpha)$ = descent set of α

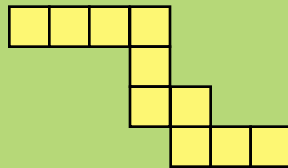
$$= \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell(\alpha)-1}\}$$

$$\alpha \geq \beta \text{ if } D(\alpha) \subseteq D(\beta)$$

Example

$$\alpha = (4, 1, 2, 3)$$

$$D(\alpha) = \{4, 5, 7\}$$



$$\beta = (4, 3, 3)$$

$$D(\beta) = \{4, 7\}$$

$$\alpha \leq \beta$$

Analog of Schur functions

$$s_\alpha = \sum_{\beta \geq \alpha} (-1)^{\ell(\alpha) + \ell(\beta)} h_\beta$$

Example 6: NC-Hall-Littlewood functions

Hopf algebra: NC-symmetric functions $\mathbb{Q}\langle h_1, h_2, h_3, \dots \rangle$

product: $\mu(f \otimes g) = fg$

co-product: $\Delta(h_k) = \sum_{i=0}^k h_i \otimes h_{k-i}$

antipode: $S(h_k) = \sum_{\alpha \models k} (-1)^{\ell(\alpha)} h_\alpha$

grading: $R^q(h_k) = q^k h_k$

Linear operators

$$B_m(\mathfrak{1}_\alpha) = \mathfrak{1}_{(\alpha, m)}$$

Theorem: Bergeron-Z

$$\tilde{B}_{\alpha_k}^q (\dots \tilde{B}_{\alpha_2}^q (\tilde{B}_{\alpha_1}^q (1)) \dots) = \sum_{\beta \geq \alpha} q^{c(\alpha, \beta)} \mathfrak{1}_\beta$$

$$c(\alpha, \beta) = \sum_{i \in D(\alpha) \cap D(\beta)} i$$

$q = 0$

$q = 1$

$$B_{\alpha_k} (\dots B_{\alpha_2} (B_{\alpha_1} (1)) \dots) = \mathfrak{1}_\alpha$$

$$\begin{aligned} B_{\alpha_1} (1) B_{\alpha_2} (1) \dots B_{\alpha_k} (1) \\ = h_\alpha = \sum_{\beta \geq \alpha} \mathfrak{1}_\beta \end{aligned}$$