Non-commutative symmetric functions II: Combinatorics and coinvariants

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joint work with Nantel Bergeron, Anouk Bergeron-Brlek, Emmanuel Briand, Christophe Hohlweg, Christophe Reutenauer, Mercedes Rosas

NCSym = Hopf algebra of set partitions

$A \vdash [n]$ set partition of $[n] = \{1, 2, \ldots, n\}$

$A = \{A_1, A_2, \ldots, A_k\}$

A_i nonempty subset of $[n]$

$A_1 \uplus A_2 \uplus \cdots \uplus A_k = [n]$

e.g. $\{\{1, 3, 4\}, \{2, 8\}, \{5, 6\}, \{7\}\} \vdash [8]$

$$NCSym = \bigoplus_{n \geq 0} \mathcal{L}\{m_A : A \vdash [n]\}$$

Comparison Sym to NCSym

Sym^n $NCSym^n$

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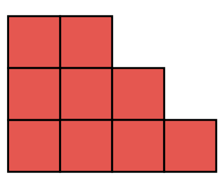
algebra of partitions 	

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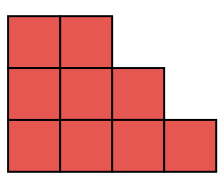

NCSym^n

$\mathfrak{S}_n$-invariants of $T(V^*) \simeq \mathbb{Q}\langle X_n \rangle$

algebra of set partitions

$\{\{1,4,9\},\{2,3,7,8\},\{5,6\}\}$

Comparison Sym to NCSym

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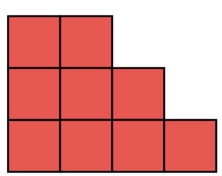
algebra of partitions 	algebra of set partitions $\{\{1,4,9\},\{2,3,7,8\},\{5,6\}\}$

Linear span of sum of $\mathfrak{S}_n$ orbits of monomials

$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$	$x_{i_1} x_{i_2} \cdots x_{i_k}$

Comparison Sym to NCSym

Sym^n has graded dimensions given by

$$\sum_{k=0}^{n} \frac{q^k}{(1-q)(1-q^2)\cdots(1-q^k)} = \prod_{i=1}^{n} \frac{1}{1-q^i}$$

1, 1, 2, 3, 5, 7, 11, ...

NCSym^n has graded dimensions given by

$$\sum_{k=0}^{n} \frac{q^k}{(1-q)(1-2q)\cdots(1-kq)}$$

1, 1, 2, 5, 15, 52, 203, ...

Harmonics of $\mathfrak{S}_n$

$$\mathcal{H}_n = \{P(X_n) \in \mathbb{Q}[X_n] : f(\partial_n)P(X_n) = 0, \forall f(X_n) \in Sym^n\}$$

the space of polynomials which are killed by all symmetric differential operators

$f(\partial_n)$ means replace variables in polynomial by differential operators

$$f(\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$$

e.g. $x_i - x_j \in \mathcal{H}_n$

Properties of the harmonics of $\mathfrak{S}_n$

$$dim\ \mathcal{H}_n = n!$$

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$$\mathcal{H}_n \otimes Sym^n \simeq \mathbb{Q}[X_n]$$

$$\mathcal{F}rob(char\ \mathcal{H}_n) = h_n\left[\frac{X}{1-q}\right](1-q)(1-q^2)\cdots(1-q^n)$$

Coinvariants of $\mathfrak{S}_n$

$$\mathbb{Q}[X_n] / \langle f(X_n) \in Sym^n : f(0) = 0 \rangle \simeq \mathcal{H}_n$$

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with respect to this scalar product

$$\mathcal{H}_n = \left\langle (Sym^n)^+ \right\rangle^{\perp} \simeq \mathbb{Q}[X_n]/\left\langle (Sym^n)^+ \right\rangle$$

Non-commutative setting

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What is meant by non-commutative derivative?

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What is meant by non-commutative derivative?

Left ideal or two-sided ideal?

A non-commutative derivative

$$d_{x_r} x_{i_1} x_{i_2} \cdots x_{i_k} = \begin{cases} x_{i_2} x_{i_3} \cdots x_{i_k} & \text{if } r = i_1 \\ 0 & \text{if } r \neq i_1 \end{cases}$$

$$\langle x_r x_{i_1} x_{i_2} \cdots x_{i_k}, x_{j_1} x_{j_2} \cdots x_{j_k} \rangle = \langle x_{i_1} x_{i_2} \cdots x_{i_k}, d_{x_r} x_{j_1} x_{j_2} \cdots x_{j_k} \rangle$$

where

$$\langle x_{i_1} x_{i_2} \cdots x_{i_k}, x_{j_1} x_{j_2} \cdots x_{j_k} \rangle = \begin{cases} 1 & \text{if } \vec{i} = \vec{j} \\ 0 & \text{otherwise} \end{cases}$$

Note in the commutative setting :

$$\left\langle x_i x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n} \right\rangle = \left\langle x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \partial_{x_i} x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n} \right\rangle$$

Another non-commutative derivative

$$\partial_{x_r} x_{i_1} x_{i_2} \cdots x_{i_k} = \sum_{i_d = r} x_{i_1} \cdots x_{i_{d-1}} x_{i_{d+1}} \cdots x_{i_k}$$

$$\chi : \mathbb{Q}\langle X_n \rangle \longrightarrow \mathbb{Q}[X_n]$$

if $\chi(x_{i_1} x_{i_2} \cdots x_{i_k}) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$

then $\chi(\partial_{x_r} x_{i_1} x_{i_2} \cdots x_{i_k}) = \partial_{x_r} x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$

Two types of non-commutative harmonics

$$MHar_n = \{P(X_n) \in \mathbb{Q}\langle X_n\rangle : f(\partial_{X_n})P(X_n) = 0, \forall f(X_n) \in NCSym^n\}$$

$$NCHar_n = \{P(X_n) \in \mathbb{Q}\langle X_n\rangle : f(d_{X_n})P(X_n) = 0, \forall f(X_n) \in NCSym^n\}$$

the space of non-commutative polynomials which are killed by all symmetric differential operators

Non-commutative coinvariants

Because of the duality between product and differentiation

$$\langle x_r x_{i_1} x_{i_2} \cdots x_{i_k}, x_{j_1} x_{j_2} \cdots x_{j_k} \rangle = \langle x_{i_1} x_{i_2} \cdots x_{i_k}, d_{x_r} x_{j_1} x_{j_2} \cdots x_{j_k} \rangle$$

we have that

$$NCHar_n = \{ P(X_n) \in \mathbb{Q}\langle X_n \rangle : f(d_{X_n}) P(X_n) = 0, \forall f(X_n) \in NCSym^n \}$$

$$NCHar_n \simeq \mathbb{Q}\langle X_n \rangle / \langle (NCSym^n)^+ \rangle$$

where $\langle (NCSym^n)^+ \rangle$ is the left ideal generated by elements of NCSym with no constant term

More non-commutative coinvariants

But what about ?

$$MHar_n = \{P(X_n) \in \mathbb{Q}\langle X_n \rangle : f(\partial_{X_n})P(X_n) = 0, \forall f(X_n) \in NCSym^n\}$$

shuffle $\sqcup\!\sqcup$ is a commutative product on $\mathbb{Q}\langle X_n \rangle$

$$\left\langle \partial_{x_r} x_{i_1} x_{i_2} \cdots x_{i_k}, x_{j_1} x_{j_2} \cdots x_{j_{k-1}} \right\rangle = \left\langle x_{i_1} x_{i_2} \cdots x_{i_k}, x_r \sqcup\!\sqcup x_{j_1} x_{j_2} \cdots x_{j_{k-1}} \right\rangle$$

$$MHar_n \simeq \mathbb{Q}\langle X_n \rangle / \left\langle (NCSym^n)^+ \right\rangle_{\sqcup\!\sqcup}$$

Properties of non-commutative harmonics

as $\mathfrak{S}_n$ modules

NC-Chevalley like theorem

$$MHar_n \otimes Sym^n \simeq \mathbb{Q}\langle X_n \rangle$$

$$NCHar_n \otimes NCSym^n \simeq \mathbb{Q}\langle X_n \rangle$$

Proof idea:

Exhibit explicit isomorphisms of LHS into $\mathbb{Q}\langle X_n \rangle$

Sym^n is realized by using the shuffle product

Consequences of these isomorphisms

$$MHar_n \otimes Sym^n \simeq \mathbb{Q}\langle X_n \rangle$$

$$\mathcal{F}rob(char\ MHar_n) = (q;q)_n \sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} h_{(n-d,1^d)}[X]$$

$$\mathcal{F}rob(char\ NCHar_n) = \left(\sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} \right)^{-1} \sum_{d=0}^{n} \frac{q^d}{\{q,q\}_d} h_{(n-d,1^d)}[X]$$

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$$\mathcal{F}rob(MHar_n) \odot \frac{h_n[X]}{(q;q)_n} = \mathcal{F}rob(\mathbb{Q}\langle X_n \rangle)$$

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$$\frac{1}{(q;q)_n}\mathcal{F}rob(MHar_n) = \mathcal{F}rob(\mathbb{Q}\left\langle X_n \right\rangle)$$

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Properties of non-commutative harmonics

$$dim_q \ MHar_n = \left(\sum_{k=0}^{n} \frac{q^k(1-nq)}{(q;q)_k} \right)^{-1}$$

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Non-commutative polynomials

$\mathbb{Q}\langle X_n \rangle =$ universal enveloping algebra of free Lie algebra $\mathcal{L}_n$

free Lie algebra $\mathcal{L}_n = X_n \oplus [\mathcal{L}_n, \mathcal{L}_n]$

$\mathcal{A}'_n =$ universal enveloping algebra of $[\mathcal{L}_n, \mathcal{L}_n]$

by PBW theorem $\mathbb{Q}\langle X_n \rangle \simeq \mathcal{A}'_n \otimes \mathbb{Q}[X_n]$

can also show that this holds as $Gl_n(\mathbb{C})$ modules

Some interesting isomorphisms

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$$\boxed{\mathcal{A}'_n \otimes \mathcal{H}_n \simeq MHar_n}$$

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- The analogous quotient of quasi-symmetric functions $\mathbb{Q}[X_n] / \langle (QSym^n)^+ \rangle$ has Catalan dimension. What happens in the non-commutative case? $\mathbb{Q}\langle X_n \rangle / \langle (NCQSym^n)^+ \rangle$

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- General theory with other finite subgroups of $Gl_n(\mathbb{C})$