k-Schur functions indexed by a maximal rectangle

Mike Zabrocki - York University

joint work with Chris Berg, Nantel Bergeron, Hugh Thomas
Lapointe-Morse (2005)

$$\Lambda^{(k)} = \mathbb{Q}[h_1, h_2, \ldots, h_k]$$

definition of a basis \( \{s^{(k)}_\lambda\}_\lambda \) indexed by partitions

\( \lambda \) partitions \( \lambda_1 \leq k \)

\[ \lambda = p(\gamma) \]

Example:

\((k+1)\)-cores = partitions with no \((k+1)\)-hooks

\[ \mathbf{c}(\lambda) = \gamma \]

\[ k = 4 \]
Lapointe-Morse definition of $k$-Schur functions

\[ \{ s^{(k)}_{\lambda} \}_{\lambda} \text{ basis of algebra } \Lambda^{(k)} = \mathbb{Q}[h_1, h_2, \ldots, h_k] \]

satisfying

\[ h_r s^{(k)}_{\lambda} = \sum_{\mu} s^{(k)}_{\mu} \]

\( c(\mu)/c(\lambda) \) is a horizontal strip, \( \lambda \leq \mu \)

\[ |\mu| = |\lambda| + r \]
This is a recursive definition because of trianguularity considerations

Example: \( k=3 \) to calculate \( s^{(3)}_{(2,2,1)} \)

the 3-Pieri rule says:

\[
h_2 s^{(3)}_{(2,1)} = s^{(3)}_{(2,2,1)} + s^{(3)}_{(3,1,1)}
\]

We may assume (inductively) that expansions of \( s^{(3)}_{(3,1,1)} \) and \( s^{(3)}_{(2,1)} \) are known in terms of the generators

In particular, if hook \( \lambda \) is small (less or equal \( k \)) then

\[
s^{(k)}_{\lambda} = s_{\lambda}
\]
Affine symmetric group $W$ of type $\widetilde{A}_k$

$W$ generated by elements $\{s_0, s_1, s_2, \ldots, s_k\}$

$s_i^2 = 1$

$s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ \hspace{1cm} $i, i+1 \ (mod \ k+1)$

$s_is_j = s_js_i$ \hspace{1cm} $i - j \neq k, 0, 1 \ (mod \ k+1)$

$W_0$ is the subgroup generated by $\{s_1, s_2, \ldots, s_k\}$

$W/W_0 = \text{cosets of } W_0 \text{ are in bijection with}$

$k$-bounded partitions/$(k+1)$-cores
$k=2$
Weak order on 3-cores
3-cores/2-bounded partitions

Cosets of $S_3$ as subgroup of affine $S_3$

minimal length coset representatives

Paths in the weak order from the empty core

Reduced word for minimal length coset representatives

$S_1S_0S_2S_1S_2S_0$
Weak order on 4-cores
Limit as $k \to \infty$
Young’s lattice =
order by inclusion

Limit as $k \to \infty$
Young’s lattice =
order by inclusion
Consider elements of the affine nil-Coxter algebra

\[ u_i^2 = 0 \]
\[ u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} \quad i, i + 1 \ (mod \ k + 1) \]
\[ u_i u_j = u_j u_i \quad i - j \neq k, 0, 1 \ (mod \ k + 1) \]

\[ h_r = \sum_{|A| = r} u_A \quad 1 \leq r \leq k \]
\[ A \subseteq \{0, 1, 2, \ldots k\} \]

\[ u_A \text{ cyclically decreasing word with content } A \]

if \( i, i + 1 \in A \), \( u_i \) comes before \( u_{i+1} \)
Let $\gamma$ be a $(k+1)$-core

$\mathcal{U}_i$ acts on $\gamma$ by adding $i$-addable corner if possible

the result is 0 otherwise
$u_A$ cyclically decreasing word with content

if $i, i + 1 \in A$ $u_i$ comes before $u_{i+1}$

acting by all cyclically decreasing words adds all possible horizontal strips

$$h_r(\gamma) = \sum_\nu$$

$k=3$

adds a horizontal strip
\[ \Lambda^{(k)} = \mathbb{Q}[h_1, h_2, \ldots, h_k] \cong \mathbb{Q}[h_1, h_2, \ldots, h_k] \]

\[ u : \mathbb{Q}[h_1, h_2, \ldots, h_k] \rightarrow \mathbb{Q}[h_1, h_2, \ldots, h_k] \]

\[ s_{\lambda}^{(k)} = u(s_{\lambda}^{(k)}) \]

Say then that we determine:

\[ s_{\lambda}^{(k)} = \sum_{w} c_w w \]

\( w \) is in the affine Nil-Coxeter algebra

\( c_w \) coefficients
k-Littlewood-Richardson coefficients:

\[ s^{(k)}_\lambda s^{(k)}_\mu = \sum_\nu c^{(k)}_{\lambda\mu} s^{(k)}_\nu \]

Viewing this in terms of actions on cores:

\[ s^{(k)}_\mu \emptyset = c(\mu) \]

\[ s^{(k)}_\lambda c(\mu) = \sum_\nu c^{(k)}_{\lambda\mu} c(\nu) \quad \text{with} \quad s^{(k)}_\lambda = \sum_w c_w w \]

\[ c^{(k)}_{\lambda\mu} \]

is equal to \( c_w \) if there exists a \( w \) s.t.

\[ wc(\mu) = c(\nu) \]
We haven’t come up with a k-LR rule, but can reduce it to a more manageable problem

Let $R$ be a rectangle with hook = $k$

$$s_R s^{(k)}_\lambda = s^{(k)}_{R \cup \lambda}$$

$$s^{(k)}_\lambda = s_{R_1} s_{R_2} \cdots s_{R_d} s^{(k)}_{\tilde{\lambda}}$$

where each of the $R_i$ are rectangles with hook = $k$ and the partition $\tilde{\lambda}$ contains less than $k+2-r$ parts of size $r$
combinatorial formula #1

$$R = \left( (k + 1 - r)^r \right)$$

$$S_R = \sum_{\lambda \subseteq R} u_{\lambda}$$

grey = contents + $k+1-r$
white = contents

Example: $k=4$  $R=(2,2,2)$

$$u_4u_3u_0u_4u_1u_0 + u_2u_4u_3u_0u_4u_1 + u_3u_2u_4u_3u_0u_4 + u_1u_2u_4u_3u_0u_1 + u_1u_3u_2u_4u_3u_0 + u_2u_1u_3u_2u_4u_3 + u_0u_1u_2u_4u_0u_1 + u_0u_1u_3u_2u_4u_0 + u_0u_2u_1u_3u_2u_4 + u_1u_0u_3u_1u_3u_2$$
### Combinatorial Formula #2

$$R = \left( \left( k + 1 - r \right)^r \right)$$

$$s_R = \sum u_A u_{A+1} u_{A+2} \cdots u_{A+r-1}$$

where $|A| = k + 1 - r$

**Example:**  $k=4$, $R=(2,2,2)$

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>r-3</td>
<td>r-2</td>
<td>r-1</td>
<td>r</td>
<td>r+1</td>
<td>r+2</td>
</tr>
</tbody>
</table>

```plaintext

<table>
<thead>
<tr>
<th>k=4</th>
<th>R=(2,2,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1 2 3 4</td>
<td>0 1 2 3 4</td>
</tr>
<tr>
<td>1 2 3 4 0</td>
<td>1 2 3 4 0</td>
</tr>
<tr>
<td>2 3 4 0 1</td>
<td>2 3 4 0 1</td>
</tr>
<tr>
<td>0 1 2 3 4</td>
<td>0 1 2 3 4</td>
</tr>
<tr>
<td>1 2 3 4 0</td>
<td>1 2 3 4 0</td>
</tr>
<tr>
<td>2 3 4 0 1</td>
<td>2 3 4 0 1</td>
</tr>
</tbody>
</table>
```
More geometric formula

\[ \Gamma = \{ (a_1, a_2, \ldots, a_{k+1}) : a_i \in \{0, 1\}, \sum a_i = k + 1 - r \} \]

\[ s_R = \sum_{\gamma \in \Gamma} \text{pseudo-translation by } \gamma \]

Example: \( k=2 \)

\[ R = \]

\[ \begin{array}{c}
\end{array} \]
So then what remains to give an explicit $k$-Littlewood Richardson rule is to give more explicit formulas for $s^{(k)}_{\tilde{\lambda}}$ where $\tilde{\lambda}$ contains no rectangles with a $k$-hook.

For a fixed $k$ there are $k!$ such partitions.