Non-commutative symmetric functions III:
A representation theoretical approach

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Joint work with Nantel Bergeron, Mercedes Rosas, Christophe Hohlweg, and others
The structure of Sym
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- Sym is a self dual graded Hopf algebra with bases indexed by partitions
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- Sym is commutative and cocommutative
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- Sym is commutative and cocommutative.

- It is freely generated by its primitive elements and there is one at each degree. $\mathbb{Q}[p_1, p_2, p_3, \ldots]$.

- There are various bases of Sym that are linked with the representation theory of $\text{Sn/Gln}$: monomial, homogeneous, elementary, power, Schur, Hall-Littlewood, Macdonald, etc.
properties of NCSym

• Hopf algebra of set partitions

• much bigger than Sym

• non-commutative but co-commutative
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NCSym
algebra of set partitions

\{\{1, 4, 9\}, \{2, 3, 7, 8\}, \{5, 6\}\}

Sym
algebra of partitions

\[
\begin{align*}
\text{VS}
\end{align*}
\]
The join operation on set partitions

\[ A \vdash [n], \ B \vdash [m] \]

\[ A|B = \{ A_1, A_2, \ldots, A_{\ell(A)}, B_1 + n, B_2 + n, \ldots, B_{\ell(B)} + n \} \]
\[ \{\{1,4\}, \{2\}, \{3,5\}\}|\{\{1,3,4\}, \{2\}\} = \{\{1,4\}, \{2\}, \{3,5\}, \{6,8,9\}, \{7\}\} \]
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An analogue of the power basis by Rosas-Sagan

\[ p_A = \sum_{B \geq A} m_B \]

\[ A \leq B \quad \text{if for each } i \text{ there is a } j \text{ with } A_i \subseteq B_j \]
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Analogue of the power basis by Rosas-Sagan

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Proposition: \[ p_A \cdot p_B = p_{A|B} \]
NCSym is free

We say that a set partition $A$ is *atomic* if

$$A \neq B \mid C$$

for non-empty set partitions $B, C$ or

$$A_1 \uplus A_2 \uplus \cdots \uplus A_i \neq [k]$$

for $k < n$
NCSym is free

We say that a set partition $A$ is atomic if

$$A \not\equiv B \mid C$$

for non-empty set partitions $B, C$ or

$$A_1 \uplus A_2 \uplus \cdots \uplus A_i \not\equiv [k]$$

for $k < n$

Example: There are 5 set partitions of $[3]$

$$p\{\{1\} , \{2\} , \{3\}\} = p\{\{1\}\} \cdot p\{\{1\}\} \cdot p\{\{1\}\}$$

$$p\{\{1,2\} , \{3\}\} = p\{\{1,2\}\} \cdot p\{\{1\}\}$$

$$p\{\{1\} , \{2,3\}\} = p\{\{1\}\} \cdot p\{\{1,2\}\}$$

$$p\{\{1,2,3\}\} , p\{\{1,3\}\} , \{2\}$$

are atomic
NCSym is free

NCSym is freely generated by $p_A$, $A$ atomic

\[ NCSym = \mathbb{Q}\langle p\{\{1\}\}, \ p\{\{1,2\}\}, \ p\{\{1,3\}, \{2\}\}, p\{\{1,2,3\}\}, \ p\{\{1,4\}, \{2\}, \{3\}\}, p\{\{1,3\}, \{2,4\}\}, p\{\{1,4\}, \{2,3\}\}, \ p\{\{1,2,4\}, \{3\}\}, p\{\{1,3,4\}, \{2\}\}, p\{\{1,2,3,4\}\}, \ldots \rangle \]

number of generators at each degree

1, 1, 2, 6, 22, 92, 426, 2146, ...
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number of generators at each degree

1, 1, 2, 6, 22, 92, 426, 2146, ...

\[ \frac{1}{1 - (t + t^2 + 2t^3 + 6t^4 + 22t^5 + 92t^6 + \cdots)} = 1 + t + 2t^2 + 5t^3 + 15t^4 + 52t^5 + 203t^6 + \cdots \]
The split of a set partition

For $A$ set partition

Let $A^i = (A^{(1)}, A^{(2)}, \ldots, A^{(\ell)})$

where $A^{(i)}$ is atomic

and $A = A^{(1)} | A^{(2)} | \cdots | A^{(\ell)}$

$$p_A = p_{A^{(1)}} \cdot p_{A^{(2)}} \cdots p_{A^{(\ell)}}$$
The split of a set partition

For $A$ set partition

Let $A^! = (A^{(1)}, A^{(2)}, \ldots, A^{(\ell)})$

where $A^{(i)}$ is atomic

and $A = A^{(1)}|A^{(2)}|\cdots|A^{(\ell)}$

$$p_A = p_{A^{(1)}} \cdot p_{A^{(2)}} \cdots p_{A^{(\ell)}}$$

Fix an order on all atomic set partitions

We say $A$ is Lyndon if

$$A^! <_{\text{lex}} (A^{(i)}, A^{(i+1)}, \ldots, A^{(\ell)}) \quad \forall i > 1$$
Example of a split of a Lyndon s.p.

\[ A = \{\{1\}, \{2, 4\}, \{3\}, \{5\}, \{6, 7, 8\}\} \]

\[ \{\{1\}\} < \{\{1, 2\}\} < \{\{1, 2\}, \{3\}\} < \{\{1, 2, 3\}\} < \cdots \]
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Example of a split of a Lyndon s.p.

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\[ A^1 = (\{\{1\}\}, \{\{1, 3\}, \{2\}\}, \{\{1\}\}, \{\{1, 2, 3\}\}) \]

\[ \{\{1\}\} < \{\{1, 2\}\} < \{\{1, 2\}, \{3\}\} < \{\{1, 2, 3\}\} < \cdots \]

\[ A \text{ is Lyndon} \]
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\[ {\{1\}} < {\{1, 2\}} < {\{1, 2, 3\}} < {\{1, 2, 3\}} < \cdots \]

\[ B = \{\{1\}, \{2, 4\}, \{3\}, \{5\}, \{6, 8\}, \{7\}\} \]

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\[ B^! = (\{\{1\}\}, \{\{1, 3\}, \{2\}\}, \{\{1\}\}, \{\{1, 3\}, \{2\}\}) \]

\( A \) is Lyndon \hspace{1cm} B \) is not Lyndon
NCSym is co-free

Basis for primitive elements is indexed by Lyndon set partitions

The number of primitive elements at each degree
1, 1, 3, 9, 34, 135, 610, 2965, ...
NCSym is co-free

Basis for primitive elements is indexed by Lyndon set partitions

The number of primitive elements at each degree

1, 1, 3, 9, 34, 135, 610, 2965, ...

\[
\frac{1}{(1-t)(1-t^2)(1-t^3)^3(1-t^4)^9(1-t^5)^{34}(1-t^6)^{135}...} = 1 + t + 2t^2 + 5t^3 + 15t^4 + 52t^5 + 203t^6 + \cdots
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- NCSym and the dual algebra are freely generated.
The structure of $\text{NCSym}$

- $\text{NCSym}$ is a graded Hopf algebra (not self dual) with bases indexed by set partitions.
- $\text{NCSym}$ is non-commutative and co-commutative.
- $\text{NCSym}$ and the dual algebra are freely generated.
- (Rosas-Sagan) There are analogues of monomial, homogeneous, elementary, power as vector space bases, but not(?) a satisfactory Schur basis.
Open question:

Is there a representation theoretical model analogous to what happens with Sym?
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\[ \bigoplus_{n \geq 0} k\mathfrak{S}_n \] is a graded algebra with irreps in 1-1 correspondence with partitions.

\[ G_0(k\mathfrak{S}, \circ) \] ring of isomorphism classes of representations of \[ \bigoplus_{n \geq 0} k\mathfrak{S}_n \]
Grothendeick ring of representations of $\bigoplus \mathbf{k}\mathcal{S}_n$

$G_0(\mathbf{k}\mathcal{S}, \circ)$ ring of isomorphism classes of representations of

product $M \otimes N \mapsto \text{Ind}_{\mathbf{k}\mathcal{S}_{m+n}}^{\mathbf{k}\mathcal{S}_m \times \mathbf{k}\mathcal{S}_n} M \otimes N$

coproduct $M \mapsto \bigoplus_{i=0}^{n} \text{Res}_{\mathbf{k}\mathcal{S}_i \times \mathbf{k}\mathcal{S}_{n-i}}^{\mathbf{k}\mathcal{S}_n} M$

internal product diagonal action on $M \otimes N$
Grothendieck ring of representations of $\bigoplus k\Sigma_n$

$G_0(k\Sigma, \circ)$ ring of isomorphism classes of representations of

product $M \otimes N \mapsto Ind_{k\Sigma_m \times k\Sigma_n}^{k\Sigma_{m+n}} M \otimes N$

coproduct $M \mapsto \bigoplus_{i=0}^{n} Res_{k\Sigma_i \times k\Sigma_{n-i}}^{k\Sigma_n} M$

internal product diagonal action on $M \otimes N$

$G_0(k\Sigma, \circ) \cong Sym$
Internal product on Sym?

\[ f \mapsto f[XY] \]

is a ‘natural’ internal coproduct on Sym and

\[ Sym \cong Sym^* \]

by duality, this internal coproduct coincides with the internal coproduct on \( G_0(k\mathcal{S}, \circ) \)

\[ G_0(k\mathcal{S}, \circ) \cong Sym^* \]
Make this isomorphism explicit

irreducible $\mathfrak{S}_n$ module $M^\lambda$

induction of $\mathfrak{S}_n$ module and an $\mathfrak{S}_m$ module to $\mathfrak{S}_{n+m}$

restriction of an $\mathfrak{S}_n$ module

internal tensor product

$s_\lambda$

$f \cdot g$

$\Delta(f)$

$f \circ g$
Internal (co)product on NCSym

By ‘replacing one set of variables by two’ we define an internal coproduct on NCSym.

\[ \Delta^\circ (m_A) = \sum_{B \wedge C = A} m_B \otimes m_C \]

where \( B \wedge C \) is the set partition finer than both \( B \) and \( C \)
Internal (co)product on NCSym

By ‘replacing one set of variables by two’ we define an internal coproduct on NCSym.

\[ \Delta^\odot (m_A) = \sum_{B \wedge C = A} m_B \otimes m_C \]

where \( B \wedge C \) is the set partition finer than both B and C.

Proposition:

\[ \Delta^\odot (p_A) = p_A \otimes p_A \]
Is there a ‘tower’ of algebras whose representations are indexed by set partitions?

\[(k\Pi_n, \wedge)\] the algebra of set partitions of \([n]\) with the meet product

\[\{\{1, 3, 4\}, \{2, 5\}, \{6, 7, 8\}\} \wedge \{\{1, 3, 5, 7\}, \{2, 4, 6, 8\}\}\]

\[= \{\{1, 3\}, \{2\}, \{4\}, \{5\}, \{6, 8\}, \{7\}\}\]
Is there a ‘tower’ of algebras whose representations are indexed by set partitions?

\((k\Pi_n, \land)\) the algebra of set partitions of \([n]\) with the meet product

\begin{align*}
\{\{1, 3, 4\}, \{2, 5\}, \{6, 7, 8\}\} \land \{\{1, 3, 5, 7\}, \{2, 4, 6, 8\}\}
\end{align*}

\begin{align*}
= \{\{1, 3\}, \{2\}, \{4\}, \{5\}, \{6, 8\}, \{7\}\}
\end{align*}

this algebra is semi-simple and, since it is commutative, all irreducible modules are of dimension 1 (hence there is one for every set partition).
Refinement order is a lattice

$$A \leq B$$
iff for each $i$, there is a $j$

$$A_i \subseteq B_j$$
What are the simple modules?

\[ C \land e_A = \begin{cases} e_A & \text{if } C \geq A \\ 0 & \text{else} \end{cases} \]
What are the simple modules?

\[ C \wedge e_A = \begin{cases} 
  e_A & \text{if } C \geq A \\
  0 & \text{else} 
\end{cases} \]

\[ e_A \otimes e_B \cong e_{A \vee B} \]  

diagonal action
What are the simple modules?

\[ C \wedge e_A = \begin{cases} e_A & \text{if } C \geq A \\ 0 & \text{else} \end{cases} \]

\[ e_A \otimes e_B \simeq e_{A \vee B} \]

\[ \text{diagonal action} \]

\[ \text{Ind}_{k\Pi_n \times k\Pi_m}^{k\Pi_{n+m}} e_A \otimes e_B \simeq e_{A \mid B} \]
Is there a basis of NCSym* which matches this?

\[ x_A x_B = x_{A|B} \]

\[ \Delta^\circ (x_A) = \sum_{B \lor C = A} x_B \otimes x_C \]

Put this in the computer and solve
Is there a basis of NCSym* which matches this?

\[ x_A x_B = x_A | B \]

\[ \Delta^\odot(x_A) = \sum_{B \lor C = A} x_B \otimes x_C \]

Put this in the computer and solve.
there is essentially one solution to this…

\[ x_A = \sum_{B \leq A} \mu(B, A) p_B \]
Is \( x_A \) an NCSym analogue of Schur functions?

Whatever it is, it is definitely something interesting.
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- the coproduct is difficult to explain and has +/- signs, but still seems curious

- Restriction is dual to induction (and is commutative). No explanation for meaning of coproduct.
Directions and open problems

• Understand better the structure of particular bases. Product, coproduct, antipode. What role do these play with respect to representation theory of \((k\Pi, \wedge)\)

• Needs some applications. Lift structures from Sym to find new attacks on plethysm, inner tensor product, positivity questions.

• Understand better non-commutative invariants of finite reflection groups and relationship to commutative invariants.