On symmetric group and partition algebra characters

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Schur-Weyl Duality

$V$ is an $n$-dimensional $Gl_n$ module

$A \in Gl_n \quad (v_i)A = \sum_{j=1}^{n} a_{ij}v_j$

$T^k(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{s_k \text{ action}} \simeq \sum_{\lambda \vdash k} S^\lambda \otimes M^\lambda$

centralizer algebra $kS_k$

Frobenius image of character

$s_\lambda = \frac{1}{k!} \sum_{\sigma \in S_k} \text{char}_S(\sigma)p_{\text{type}}(\sigma)$

diagonal $Gl_n$ action

irreducible characters

$S_\lambda(z_1, z_2, \ldots, z_n) = \text{char}_{M^\lambda}(A)$

Schur functions
Schur-Weyl Duality

$V$ is an $n$-dimensional module

$A \in \frac{O_{2n}}{Sp_{2n}} \frac{O_{2n+1}}{O_{2n+1}}$

$(v_i)A = \sum_{j=1}^{n} a_{ij}v_j$

$T^k(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{\text{Brauer algebra action}} \simeq \sum_{\lambda \vdash k} S^\lambda \otimes M^\lambda$

centralizer is Brauer algebra

diagonal $\frac{O_{2n}}{Sp_{2n}} \frac{O_{2n+1}}{O_{2n+1}}$ action

irreducible characters

universal character functions

$(\Omega[-s_2] \frac{1}{s_\lambda})(z_1, z_2, \ldots, z_n, z_1^{-1}, z_2^{-1}, \ldots, z_n^{-1})$

$(\Omega[-s_{11}] \frac{1}{s_\lambda})(z_1, z_2, \ldots, z_n, z_1^{-1}, z_2^{-1}, \ldots, z_n^{-1})$

$(\Omega[-s_2] \frac{1}{s_\lambda})(z_1, z_2, \ldots, z_n, z_1^{-1}, z_2^{-1}, \ldots, z_n^{-1}, 1)$

$= \text{char}_{M^\lambda}(A)$
Schur-Weyl Duality

$V$ is an $n$-dimensional $S_n$ module

$A \in S_n \quad (v_i)A = \sum_{j=1}^{n} a_{ij} v_j$

$T^k(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{\text{partition algebra action}} \simeq \sum_{\lambda \vdash k} S^\lambda \otimes M^\lambda$

centralizer is partition algebra

diagonal $S_n$ action

irreducible characters
The irreducible characters of the symmetric group form a basis of the symmetric functions.

\[ \Xi_r := 1, \zeta_r, \zeta_r^2, \ldots, \zeta_r^{r-1} \quad \zeta_r = e^{2\pi i / r} \]

\[ \Xi_\mu := \Xi_{\mu_1}, \Xi_{\mu_2}, \ldots, \Xi_{\mu_{\ell(\mu)}} \]

eigenvalues of a permutation matrix with cycle structure \( \mu \)

\[ \tilde{s}_\lambda(\Xi_\mu) = \chi^{(|\mu|-|\lambda|,\lambda)}(\mu) \]
The irreducible characters of the symmetric group are built from induced trivial characters multi-set partitions of a multi-set

\[ h_\nu = \sum_{\pi \vdash \{1^{\nu_1}, \ldots, \ell^{\nu_\ell}\}} \tilde{h}_{\tilde{m}(\pi)} \]

\( \tilde{m}(\pi) = \) vector of number of times sets repeat in multi-set partition

\[ \tilde{h}_\lambda(\Xi_\mu) = \langle h_{(|\mu|-|\lambda|,\lambda)}, p_\mu \rangle \]
Theorem

The coefficient of $\tilde{h}_\lambda$ in $h_\mu$

is the number of multiset partitions of a multiset

$$h_{31} = \tilde{h}_1 + 3\tilde{h}_{11} + \tilde{h}_{111} + \tilde{h}_{21} + \tilde{h}_{31}$$

- $\{\{1, 1\}, \{1, 2\}\}$
- $\{\{1, 1, 1\}, \{2\}\}$
- $\{\{1\}, \{1, 1, 2\}\}$
- $\{\{1\}, \{1\}, \{1, 2\}\}$
Theorem
The coefficient of $\tilde{s}_\lambda$ in $h_\mu$
is the number of column strict tableaux of shape $(r, \lambda)$ and content $\mu$ whose entries are multisets

$$h_{31} = 7\tilde{s}_{()} + 14\tilde{s}_1 + 8\tilde{s}_{11} + \tilde{s}_{111} + 10\tilde{s}_2 + 4\tilde{s}_{21} + 4\tilde{s}_3 + \tilde{s}_{31} + \tilde{s}_4$$
Structure coefficients are Kronecker

For $\lambda, \mu \vdash n$ where $n$ is sufficiently large

$$\tilde{s}_\lambda \tilde{s}_\mu = \sum_{\nu \vdash n} k_{\lambda \mu \nu} \tilde{s}_\nu$$

where the $k_{\lambda \mu \nu}$ are the coefficients in the Kronecker product

$$s_\lambda \ast s_\mu = \sum_{\nu \vdash n} k_{\lambda \mu \nu} s_\nu$$
Useful tool in expressions for characters of representations that depend on $n$ of $S_n$ and stabilize as the size of $n$ increases.

Kronecker

plethysm

inner plethysm

certain quotients
Kronecker product of complete/homogeneous

\[ h_{\lambda} \ast h_{\mu} = \sum_{M: \text{row}(M) = \lambda, \text{col}(M) = \mu} h_{\text{read}(M)} \]

Example:

\[ \lambda = (3, 2) \quad \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} \]

\[ h_{32} \ast h_{41} = h_{311} + h_{221} \]
Usual product of induced trivial character

\[ \tilde{h}_{\tilde{m}(\pi)} \tilde{h}_{\tilde{m}(\tau)} = \sum_{\gamma \in \pi \# \tau} \tilde{h}_{\tilde{m}(\gamma)} \]

\[ \pi \# \tau = \text{set of } \{ S_{i_1}, \ldots, S_{i_k}, T_{j_1}, \ldots, T_{j_r}, S_{i_1}' \cup T_{j_1}', \ldots, S_{i_r}' \cup T_{j_r}' \} \]

\[ \pi = \{ \{1\}, \{1\} \} \]
\[ \tau = \{ \{2\} \} \]
\[ \tilde{h}_2 \tilde{h}_1 = \tilde{h}_{11} + \tilde{h}_{21} \]
\[ \pi \# \tau \]
\[ \{ \{1\}, \{1, 2\} \} \]
\[ \{ \{1\}, \{1\}, \{2\} \} \]
Implementation in Sage 6.10

```python
sage: Sym = SymmetricFunctions(QQ)
sage: st = Sym.irreducible_symmetric_group_character()
sage: st
Symmetric Functions over Rational Field in the irreducible symmetric group character basis
sage: s = Sym.Schur()

sage: s(st[3,2]) # expand irreducible character in the Schur basis

sage: st(s[3,2]) # expand Schur function in the irreducible basis

sage: st[2]*st[2,1]

sage: s[7,2].kronecker_product(s[6,2,1])
```
Partition algebra characters

transition coefficients from $\mathbf{p}_\mu$ to $\tilde{s}_\lambda$ are partition algebra characters

$$p_\mu = \sum_{|\lambda| \leq |\mu|} \chi_{P_{|\mu|}}(|\mu|-|\lambda|, \lambda) (d_\mu) \tilde{s}_\lambda$$

$sage$: \text{st(p[2,1])} \\

(Halverson 2000)

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Character polynomials give power sum expansion

\[ p_k [\Xi_1 m_1 2m_2 \ldots r m_r ] = \sum_{d \mid k} dm_d \]

\[ \tilde{s}_\lambda \left. \left| p_k \rightarrow \sum_{d \mid k} dm_d \right. \right. = q_\lambda (m_1, m_2, m_3, \ldots) \]

\[ k m_k = \sum_{d \mid k} \mu(k/d) p_d \]

\[ q_\lambda \left( p_1, \frac{p_2 - p_1}{2}, \frac{p_3 - p_1}{3}, \ldots \right) = \tilde{s}_\lambda \]
Character polynomials give power sum expansion

\[ \tilde{s}_\lambda = \sum_{\gamma \vdash \mid \lambda \mid} \chi^\lambda(\gamma) \frac{p_\gamma}{z_\gamma} \]

where

\[ p_{ir} = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} \binom{r}{k} \left( \frac{1}{i} \sum_{d|i} \mu(i/d) p_d \right)_k \]

\[ p_\gamma := \prod_{i \geq 1} p_{i^{m_i(\gamma)}} \]