Irreducible characters of the symmetric group as symmetric functions

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$S_n \subseteq Gl_n$

permutation matrices contained in invertible matrices

Open problem:

irreducible $Gl_n$ representation $V^\lambda$

how does $V^\lambda$ decompose into irreducible symmetric group representations?
Answer using characters of $\text{Gl}_n$

character of $V^\lambda$ is $s_\lambda(x_1, x_2, \ldots, x_n)$

$$\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix} \sim A \begin{bmatrix} x_1 & 0 & \cdots & 0 \\
0 & x_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & x_n
\end{bmatrix} A^{-1}$$

$$\Xi_r := 1, \zeta_r, \zeta_r^2, \ldots, \zeta_r^{r-1} \quad \zeta_r = e^{2\pi i / r}$$

$$\Xi_\mu := \Xi_{\mu_1}, \Xi_{\mu_2}, \ldots, \Xi_{\mu_{\ell(\mu)}}$$

eigenvalues of a permutation matrix with cycle structure $\mu$
Frobenius image:

\[ \phi_n(f) = \sum_{\mu \vdash n} f[\Xi_{\mu}] \frac{p_{\mu}}{z_{\mu}} \]

The multiplicity of an \( S_n \) irreducible \( M^\gamma \)

in the irreducible \( Gl_n \) module \( V^\lambda \)

is equal to the coefficient of \( s_\gamma \) in \( \phi_n(s_\lambda) \)
What are the irreducible characters of the symmetric group?

\[ \tilde{s}_\lambda = \phi_{n}^{-1}(s(n-|\lambda|,\lambda)) \]

The irreducible characters of the symmetric group form a basis of the symmetric functions.

\[ \tilde{s}_\lambda(\Xi_\mu) = \chi^{(|\mu|-|\lambda|,\lambda)}(\mu) \]
Theorem

The coefficient of $\tilde{s}_\lambda$ in $h_\mu$

is the number of column strict tableaux of shape $(r, \lambda)$ and content $\mu$ whose entries are multisets

$$h_{31} = 7\tilde{s}(\) + 14\tilde{s}_1 + 8\tilde{s}_{11} + \tilde{s}_{111} + 10\tilde{s}_2$$
$$+ 4\tilde{s}_{21} + 4\tilde{s}_3 + \tilde{s}_{31} + \tilde{s}_4$$
Intermediate basis-induced trivial characters

\[ \tilde{\eta}_\lambda = \phi_n^{-1}(h_{(n-|\lambda|,\lambda)}) \]

\[ \tilde{\eta}_\lambda [\Xi_\mu] = \langle h_{(|\mu| - |\lambda|,\lambda)}, \rho_\mu \rangle \]

Combinatorial expansion

\[ h_\lambda = \sum_{\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \ldots, \ell^{\lambda_\ell}\}} \tilde{\eta}_{\tilde{m}(\pi)} \cdot \]

\[ h_{31} = \tilde{h}_1 + 3\tilde{h}_{11} + \tilde{h}_{111} + \tilde{h}_{21} + \tilde{h}_{31} \]

\{1112\}, {111|2}, {112|1}, {11|12}, {11|1|2}, {12|1|1}, {1|1|1|2}
There exists a Hopf algebra of multi-set partitions...

- Contains Hopf algebra of set partitions (NCSym) symmetric functions in non-commuting variables
- Monomial and power sum bases defined exactly as in NCSym. Commutative image of monomial is induced trivial character, power is complete basis
- Product and coproduct give interesting combinatorial interpretation to the Kronecker product of complete symmetric functions
Structure coefficients are Kronecker

For $\lambda, \mu \vdash n$ where $n$ is sufficiently large

$$\tilde{s}_\lambda \tilde{s}_\mu = \sum_{\nu \vdash n} k_{\lambda \mu \nu} \tilde{s}_\nu$$

where the $k_{\lambda \mu \nu}$ are the coefficients in the Kronecker product

$$s_\lambda \ast s_\mu = \sum_{\nu \vdash n} k_{\lambda \mu \nu} s_\nu$$
Positive structure coefficients (reduced Kronecker)

Positive coproduct structure coefficients

Combinatorial tools for working with symmetric group and partition algebra characters

Orthonormal basis with respect to character scalar product

\[
\langle f, g \rangle_@ = \sum_{\mu | n} \frac{f[\Xi_{\mu}]g[\Xi_{\mu}]}{z_{\mu}} = \langle \phi_n(f), \phi_n(g) \rangle 
\]

\( \{ \tilde{s}_\lambda \}_\lambda \) is an orthonormal basis with respect to \( \langle \cdot, \cdot \rangle_@ \)

\( \tilde{s}_\lambda = s_\lambda + \) terms of lower degree
homogeneous \rightarrow \text{column strict tableaux} \rightarrow \text{multiset tableaux} \rightarrow \text{irreducible character} \rightarrow \text{Schur} \rightarrow \text{column strict tableaux} \rightarrow \text{multiset tableaux} \rightarrow \text{homogeneous}
Implementation in Sage

```
sage: Sym = SymmetricFunctions(QQ)
sage: st = Sym.irreducible_symmetric_group_character()
sage: st
Symmetric Functions over Rational Field in the irreducible symmetric group character basis
sage: s = Sym.Schur()

sage: s(st[3,2])

sage: st(s[3,2])

sage: st[2]*st[2,1]

sage: s[7,2].kronecker_product(s[6,2,1])
```