Combinatorics of characters of symmetric group as symmetric functions

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joint work with Rosa Orellana
The ring of symmetric functions’ dual role in representation theory

$\text{Sym}_{X_n}$ is the ring of characters of $\text{Gl}_n(\mathbb{C})$

$s_\lambda(x_1, x_2, \ldots, x_n)$

$\text{Sym}$ is isomorphic to the ring of characters of $\bigoplus_{k \geq 0} S_k$

$\mathcal{F}_{S_k}(\chi^\lambda) = s_\lambda$

$\lambda$ partition of $k$ and $\chi^\lambda$ is an irreducible $S_k$ character
The ring of symmetric functions’ dual role in representation theory

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$\bigcup$

$\text{Sym}_{X_n}$ is the ring of characters of $S_n$

$\tilde{s}_{\lambda}(x_1, x_2, \ldots, x_n)$

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The ring of symmetric functions’ dual role in representation theory

\( Sym_{X_n} \) is the ring of characters of \( GL_n(\mathbb{C}) \)
\[ s_\lambda(x_1, x_2, \ldots, x_n) \]

\( Sym_{X_n} \) is the ring of characters of \( S_n \)
\[ \tilde{s}_\lambda(x_1, x_2, \ldots, x_n) \]

\[ \tilde{s}_\lambda[\text{eigenvals of permutation matrix } \mu] = \chi^{(n-|\lambda|,\lambda)}(\mu) \]

\[ \tilde{s}_\lambda \tilde{s}_\nu = \sum_\gamma \overline{k}_{\lambda \nu \gamma} \tilde{s}_\gamma \]
Theorem

The coefficient of $\tilde{s}_\lambda$ in $h_\mu$

is the number of column strict tableaux of shape $(r, \lambda)$ and content $\mu$ whose entries are multi-sets

$$h_{31} = 7\tilde{s}_{()} + 14\tilde{s}_1 + 8\tilde{s}_{11} + \tilde{s}_{111} + 10\tilde{s}_2$$

$$+ 4\tilde{s}_{21} + 4\tilde{s}_3 + \tilde{s}_{31} + \tilde{s}_4$$
Discovered and rediscovered….

**Littlewood 1958**

The characters of the symmetric group can be obtained from those of the full linear group in a similar manner to that used for the orthogonal group, namely by considering a tensor corresponding to any partition \( \lambda \) of any integer \( n \), and removing all possible contractions with the fundamental forms \( (2, p. 392) \). The remainder when all contractions are removed is an irreducible character, provided that \( n - \rho \gg \lambda_1 \), and it is not difficult to see that it is in fact the character of the symmetric group corresponding to the partition \( (n - \rho, \lambda_1, \ldots, \lambda_i) \). It is convenient to represent by \( [\lambda] \) not this character, but the corresponding S-function

\[
[\lambda] = [n - \rho, \lambda_1, \ldots, \lambda_i].
\]

\[
\]

**Butler, King 1973**

The symmetric groups are thus treated quite differently from the linear and other continuous groups: the orthogonal, rotation, and symplectic groups. The characters of these groups are known in terms of \( S \) functions and the usual method of calculating such things as Kronecker products of the representations of these groups is to use \( S \)-functional expressions for their characters and the powerful algebra of \( S \) functions associated with the \( n \)-independent outer product rule. The labels that arise from this approach are the same as those that arise from tensorial arguments.\(^7\)\(^9\) The aim of this paper is to show that the symmetric groups, \( \Sigma_n \), may be treated in an \( n \)-independent manner similar to that used for the restricted groups \( O_n \) and \( S_{0n} \), rather than in the usual \( n \)-dependent manner requiring a development of the somewhat complicated algebra of inner products of \( S \) functions.\(^10\)

Some specific examples of (3.4) are of interest, namely:

\[
L_{n-1} \rightarrow \Sigma_n \{1\} \rightarrow \{1\}
\]

\[
\{2\} \rightarrow \{2\} + \{1\} + \{0\}
\]

\[
\{1^2\} \rightarrow \{1^2\}
\]

\[
\{1^k\} \rightarrow \{1^k\}
\]

\[
\{1^{n-1}\} \rightarrow \{1^{n-1}\}
\]
Why? Applications

Church-Farb representation stability

representation theory of symmetric group and the partition algebra

Kronecker and reduced/stable Kronecker product

restriction/branching from irreducible Gln to Sn

plethysm

Combinatorics of multi-set partitions, multi-set tableaux and Hopf algebras
partition algebra irrep dimensions
oscillating tableaux

symmetric group irrep dimensions
standard tableaux

general linear group irrep dimensions
column strict tableaux

Brauer algebra irrep dimensions
oscillating tableaux
\[ \dim \text{ irreducible } Gl_n = F_{\lambda,n} \]

\[ V = \mathcal{L}\{v_1, v_2, \ldots, v_n\} \]

\[ V \otimes k \cong \bigoplus_{\lambda \vdash k} (\text{irreducible } Gl_n)^{\oplus f_{\lambda}} \]
\[ \dim \text{ irreducible } \mathcal{G}_n = F_{\lambda, n} \]

\[ V = \mathcal{L}\{v_1, v_2, \ldots, v_n\} \]

- Dimension
- Character

\[ F_{1,6} = 6 \]
\[ s_1 = h_1 \]

\[ F_{2,6} + F_{11,6} = 36 \]
\[ s_2 + s_{11} = h_{11} \]

\[ F_{3,6} + 2F_{21,6} + F_{111,6} = 216 \]
\[ s_3 + 2s_{21} + s_{111} = h_{111} \]

\[ F_{4,6} + 3F_{31,6} + 2F_{22,6} + 3F_{211,6} + F_{1111,6} = 6^4 \]
\[ s_4 + 3s_{31} + 2s_{22} + 3s_{211} + s_{111} = h_{1111} \]

\[ F_{5,6} + 4F_{41,6} + 5F_{32,6} + 6F_{311,6} + 5F_{221,6} + 4F_{2111,6} + F_{15,6} = 6^5 \]
\[ s_5 + 4s_{41} + 5s_{32} + 6s_{311} + 5s_{221} + 4s_{2111} + s_{15} = h_{15} \]
\[ \emptyset \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \begin{array}{c} 2 \\ 1 \\ 3 \\ 4 \end{array} \]

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\[ \emptyset \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \square \rightarrow \begin{array}{c} 4 \\ 1 \\ 2 \\ 3 \end{array} \]
\[ V = \mathcal{L}\{v_1, v_2, \ldots, v_n\} \]

\[ V^\otimes k \cong \bigoplus_{\lambda \vdash k} \left( \text{irreducible } Gl_n \text{ subspace } \lambda \right)^{\oplus f_\lambda} \]

\[ n^k = \sum_{\lambda \vdash k} \left( \text{\# of column strict tableaux of shape } \lambda \right) \left( \text{\# of standard tableaux of shape } \lambda \right) \]

\[ h_{1k}(x_1, x_2, \ldots, x_n) = \sum_{\lambda \vdash k} f_\lambda s_\lambda(x_1, x_2, \ldots, x_n) \]
\[ V = \mathcal{L}\{v_1, v_2, \ldots, v_n\} \]

\[ V \otimes^k \simeq \bigoplus_{\lambda: |\lambda| \leq k} \left( \text{irreducible subspace } S_{\lambda} \right)^{\oplus n_\lambda} \]

\[ \eta^k = \sum_{\lambda \vdash n} \binom{\text{# of standard tableaux shape}}{\lambda} \left( \text{paths in a Bratteli diagram} \right) \]

\[ h_{1k}(x_1, x_2, \ldots, x_n) = \sum_{\lambda: |\lambda| \leq k} n_\lambda \tilde{s}_\lambda(x_1, x_2, \ldots, x_n) \]
\[ f_6 = 1 \]
\[ \tilde{s}(\cdot) = h(\cdot) \]
\[ f_6 + f_{51} = 6 \]
\[ \tilde{s}(\cdot) + \tilde{s}_1 = h_1 \]
\[ 2f_6 + 3f_{51} + f_{42} + f_{411} = 36 \]
\[ 2\tilde{s}(\cdot) + 3\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_{11} = h_{11} \]
\[ 5f_6 + 10f_{51} + 6f_{42} + 6f_{411} + f_{33} + 2f_{321} + f_{3111} = 216 \]
\[ 5\tilde{s}(\cdot) + 10\tilde{s}_1 + 6\tilde{s}_2 + 6\tilde{s}_{11} + \tilde{s}_3 + \tilde{s}_{21} + \tilde{s}_{111} = h_{111} \]
\[
\eta^k = \sum_{\lambda \vdash n} \left( \frac{\text{# of standard tableaux of shape } \lambda}{\lambda \vdash n} \right) \left( \text{# of standard set tableaux in } \lambda_1, \lambda_2, \ldots, \lambda_k \text{ of shape } \lambda \right)
\]
Summary.....

- The dimensions of the irreducible partition algebra representations are equal to the number of standard set valued tableaux.

- There is a bijection with the (previously known) combinatorial interpretation (oscillating tableaux) and there is an RSK bijection which explains

\[ n^k = \sum_{\lambda + n} \left( \text{# of standard tableaux of shape } \lambda \right) \left( \text{# of standard set tableaux in } \lambda_1, \lambda_2, \ldots, \lambda_k \right) \]

\[ \text{# of set partitions of } \xi_1, \xi_2, \ldots, 2n \]  

\[ = \sum_{|\lambda| \leq n} \left( \text{# of standard set valued tableaux in } \xi_1, \xi_2, \ldots, n \text{ of shape } (k, \lambda) \right)^2 \]