THE REPRESENTATION THEORY OF THE SYMMETRIC GROUP REVISITED

joint work with Rosa Orellana

$$\begin{array}{|c|c|c|}\cline{1-1} 6 \\ \hline 2 & 4 & 7 \\ \hline 1 & 3 & 5 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\cline{1-1} 67 \\ \cline{1-2} 12 & 4 \\ \hline & 3 & 5 \\ \hline\end{array}$$

Problem:

Describe the maps from the symmetric group to the general linear group (the possible linear actions of the symmetric group on a vector space).

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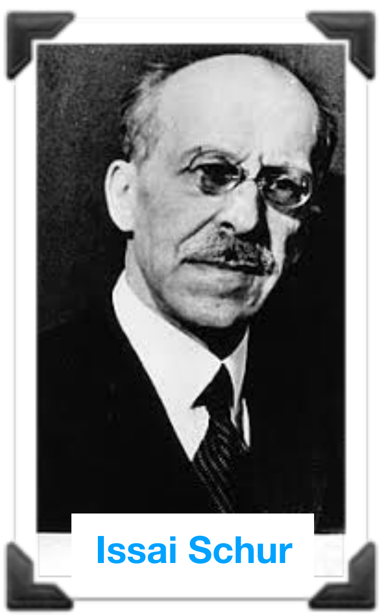
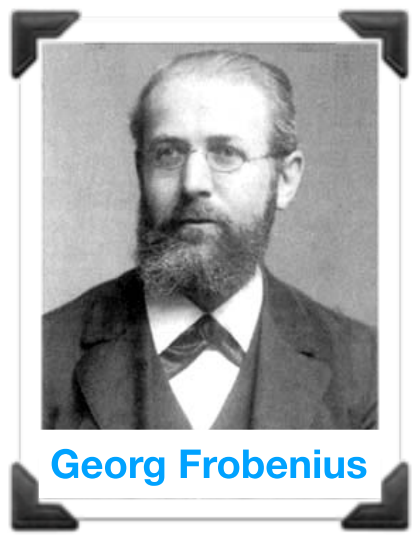
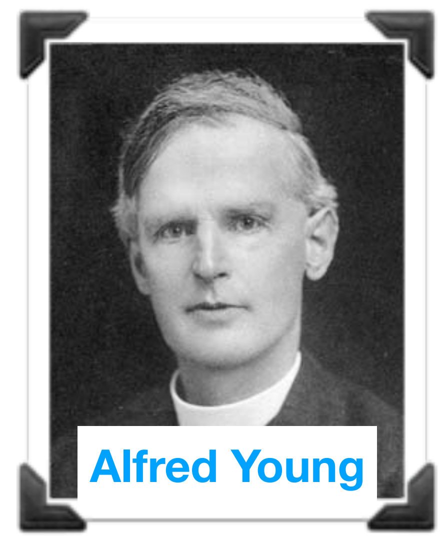


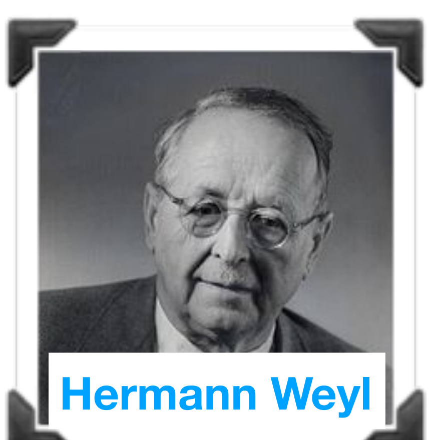
Issai Schur



Georg Frobenius

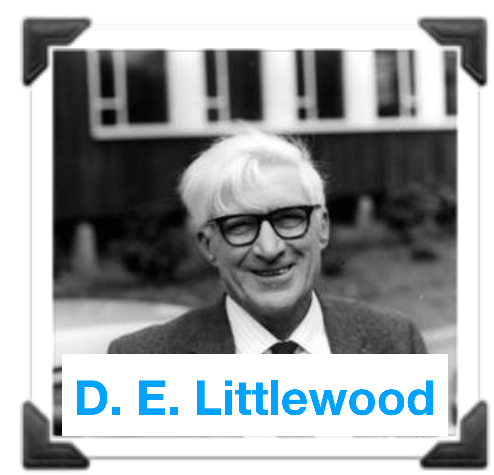


Alfred Young



Hermann Weyl

Gilbert de Beauregard Robinson



D. E. Littlewood

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Representation theory : a combinatorial viewpoint

Author: Amritanshu Prasad.

Publication info: Delhi : Cambridge University Press, 2015.

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- All representations are (up to change of basis) a direct sum of irreducible components
- There are only a finite number of possible irreducible components and there are the same number as the number of conjugacy classes
- The representations are characterized by the "characters"

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The dimension of an irreducible module indexed by a partition M^λ *is the number of standard tableaux of shape* λ

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$$span\{\sigma : \sigma \in S_n\} \simeq span\{e_{T_1,T_2} : T_1, T_2 \text{ standard tableaux same shape}\}$$

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Robinson-Schensted-Knuth

$$n! = \sum_{\lambda \vdash n} (\#\text{ pairs of std tableaux of shape } \lambda)$$

$$5361274 \quad \longleftrightarrow \quad \left(\begin{array}{|c|c|c|} \hline 5 \\ \cline{1-1} 3 & 6 & 7 \\ \hline 1 & 2 & 4 \\ \hline \end{array}\, , \begin{array}{|c|c|c|} \hline 4 \\ \cline{1-1} 2 & 5 & 7 \\ \hline 1 & 3 & 6 \\ \hline \end{array} \right)$$

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The representation theory of the symmetric group.....visited

$$Sym := \mathbb{Q}[p_1[X_n], p_2[X_n], p_3[X_n], \ldots]$$

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$$

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$\{p_\lambda[X_n]\}$ $\{h_\lambda[X_n]\}$ $\{e_\lambda[X_n]\}$ $\{s_\lambda[X_n]\}$

are all bases of the ring of symmetric functions

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$\{p_\lambda[X_n]\}$ $\{h_\lambda[X_n]\}$ $\{e_\lambda[X_n]\}$ $\{s_\lambda[X_n]\}$

are all bases of the ring of symmetric functions

The Littlewood-Richardson rule

$$s_\lambda[X_n] s_\mu[X_n] = \sum_{\nu : |\nu| = |\lambda| + |\mu|} c^\nu_{\lambda\mu} s_\nu[X_n]$$

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Notes and references

The Littlewood–Richardson rule (9.2) was first stated, but not proved, in [L13] (p. 119). The proof subsequently published by Robinson [R5], and reproduced in Littlewood's book ([L9], pp. 94–6) is incomplete, and it is this proof that we have endeavoured to complete.

Complete proofs of the rule first appeared in the 1970s ([S7], [T4]).† Since then, many other formulations, proofs and generalizations have appeared, some of which are covered by the following references: Bergeron and Garsia [B4]; James [J7]; James and Peel [J10]; James and Kerber [J9]; Kerov [K8]; Littelmann [L7], [L8]; White [W3]; and Zelevinsky [Z2], [Z3].

† Gordon James [J8] reports that he was once told that 'the Littlewood–Richardson rule helped to get men on the moon, but it was not proved until after they had got there. The first part of this story might be an exaggeration.'

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…bases of the ring of …tric functions

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The representation theory of the general linear group.....visited

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The representation theory of the general linear group.....visited

The dimension of an irreducible module indexed by a partition W^λ is the number of column strict tableaux of shape λ

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The representation theory of the general linear group.....visited

$$V_n^{\otimes k} \simeq \bigoplus_{\lambda \vdash k} W^\lambda \otimes M^\lambda$$

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The representation theory of the general linear group.....visited

$$V_n^{\otimes k} \simeq \bigoplus_{\lambda \vdash k} W^\lambda \otimes M^\lambda$$

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$$span\{v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k} : 1 \leq i_1, i_2, \ldots, i_k \leq n\} \simeq$$

$$span\{w_{T,S} : T \text{ column strict}, S \text{ standard}, sh(T) = sh(S) = \lambda\}$$

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$$span\{w_{T,S} : T \text{ column strict}, S \text{ standard}, sh(T) = sh(S) = \lambda\}$$

character of irreducible representation

$$char_A W^\lambda = s_\lambda[X_n]$$

where $x_1, x_2, \ldots, x_n$
are the eigenvalues of the matrix A

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$$V_n^{\otimes k} \simeq \bigoplus_{\lambda \vdash k} W^\lambda \otimes M^\lambda$$

$$p_\mu[X_n] = \sum_{\lambda \vdash k} \chi^\lambda(\mu) s_\lambda[X_n]$$

$$V_n^{\otimes k} \simeq \bigoplus_{\lambda \vdash k} W^\lambda \otimes M^\lambda \qquad\qquad p_\mu[X_n] = \sum_{\lambda \vdash k} \chi^\lambda(\mu) s_\lambda[X_n]$$

symmetric group characters are the change of basis coefficients power $\rightarrow$ Schur

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symmetric group characters are the change of basis coefficients power $\to$ Schur

	(1)(2)	(12)
$\ydiagram{2}$	1	1
$\ydiagram{1,1}$	1	-1

	(1)(2)(3)	(12)(3)	(123)
$\ydiagram{3}$	1	1	1
$\ydiagram{2,1}$	2	0	-1
$\ydiagram{1,1,1}$	1	-1	1

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
$\ydiagram{4}$	1	1	1	1	1
$\ydiagram{3,1}$	3	1	-1	0	-1
$\ydiagram{2,2}$	2	0	2	-1	0
$\ydiagram{2,1,1}$	3	-1	-1	0	1
$\ydiagram{1,1,1,1}$	1	-1	1	1	-1

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symmetric group characters are the change of basis coefficients power → Schur

general linear group characters are the evaluations of Schur functions at eigenvalues of matrices

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The representation theory of the symmetric group revisited

general linear group characters are the evaluations of Schur functions at eigenvalues of matrices

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$(1,1)$	1	-1

	(1)(2)(3)	(12)(3)	(123)
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	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
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$(2,1,1)$	3	-1	-1	0	1
$(1,1,1,1)$	1	-1	1	1	-1

The representation theory of the symmetric group revisited

$$S_n \subseteq Gl_n \qquad A_{(134)(25)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

general linear group characters are the evaluations of Schur functions at eigenvalues of matrices

	(1)(2)	*(12)*
(2)	1	1
$(1,1)$	1	-1

	(1)(2)(3)	*(12)(3)*	*(123)*
(3)	1	1	1
$(2,1)$	2	0	-1
$(1,1,1)$	1	-1	1

	(1)(2)(3)(4)	*(12)(3)(4)*	*(12)(34)*	*(123)(4)*	*(1234)*
(4)	1	1	1	1	1
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The representation theory of the symmetric group revisited

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The representation theory of the symmetric group revisited

ARE symmetric group characters are evaluations of some functions at eigenvalues of matrices?

general linear group characters are the evaluations of Schur functions at eigenvalues of matrices

	(1)(2)	(12)
(2)	1	1
$(1,1)$	1	-1

	(1)(2)(3)	(12)(3)	(123)
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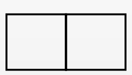
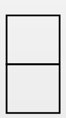
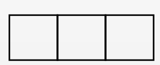
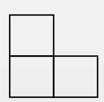
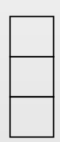
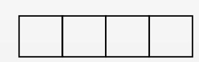
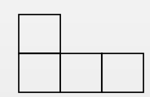
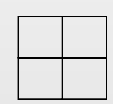
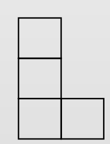
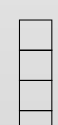
$(1,1,1,1)$	1	-1	1	1	-1

The representation theory of the symmetric group revisited

general linear group characters are the evaluations of Schur functions at eigenvalues of matrices

	(1)(2)	(12)
	1	1
	1	-1

	(1)(2)(3)	(12)(3)	(123)
	1	1	1
	2	0	-1
	1	-1	1

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
	1	1	1	1	1
	3	1	-1	0	-1
	2	0	2	-1	0
	3	-1	-1	0	1
	1	-1	1	1	-1

The representation theory of the symmetric group revisited

$$\chi^{(n)}[X_n] = 1$$

general linear group characters are the evaluations of Schur functions at eigenvalues of matrices

	(1)(2)	(12)
	1	1
	1	-1

	(1)(2)(3)	(12)(3)	(123)
	1	1	1
	2	0	-1
	1	-1	1

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
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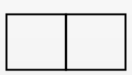
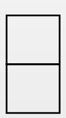
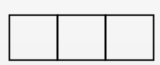
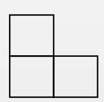
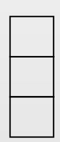
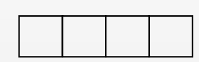
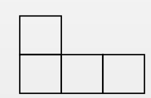
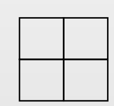
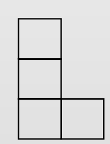
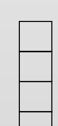
	1	1
	1	-1

	(1)(2)(3)	(12)(3)	(123)
	1	1	1
	2	0	-1
	1	-1	1

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
	1	1	1	1	1
	3	1	-1	0	-1
	2	0	2	-1	0
	3	-1	-1	0	1
	1	-1	1	1	-1

The representation theory of the symmetric group revisited

$$\chi^{(n)}[X_n] = 1$$

$$\chi^{(n-1,1)}[X_n] = s_1[X_n] - 1$$

general linear group characters are the evaluations of Schur functions at eigenvalues of matrices

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	1	1
	1	-1

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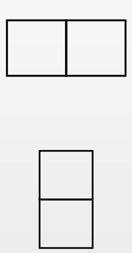
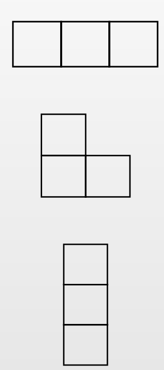
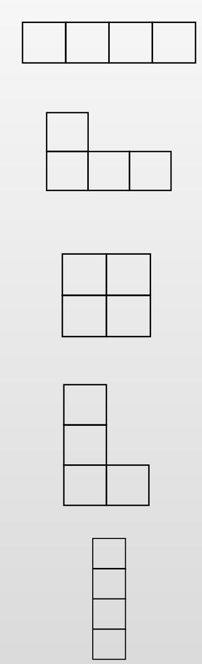


	(1)(2)	(12)
	1	1
	1	-1



	(1)(2)(3)	(12)(3)	(123)
	1	1	1
	2	0	-1
	1	-1	1



	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
	1	1	1	1	1
	3	1	-1	0	-1
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	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
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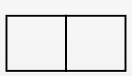
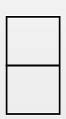
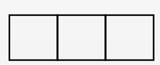
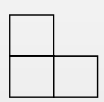
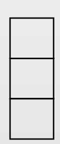
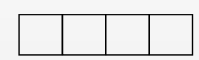
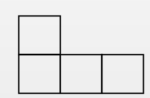
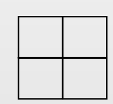
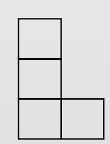
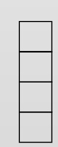
(2,1,1)	3	-1	-1	0	1
(1,1,1,1)	1	-1	1	1	-1

The representation theory of the symmetric group revisited

symmetric group characters are evaluations of some functions at eigenvalues of matrices.

	(1)(2)	(12)
	1	1
	1	-1

	(1)(2)(3)	(12)(3)	(123)
	1	1	1
	2	0	-1
	1	-1	1

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
	1	1	1	1	1
	3	1	-1	0	-1
	2	0	2	-1	0
	3	-1	-1	0	1
	1	-1	1	1	-1

The representation theory of the symmetric group revisited

$$\tilde{s}_{()}[X_n] = 1$$

$$\tilde{s}_1[X_n] = s_1[X_n] - 1$$

$$\tilde{s}_2[X_n] = s_2[X_n] - 2s_1[X_n]$$

$$\tilde{s}_{1,1}[X_n] = s_{1,1}[X_n] - s_1[X_n] + 1$$

symmetric group characters are evaluations of some functions at eigenvalues of matrices.

	(1)(2)	(12)
(2)	1	1
$(1,1)$	1	-1

	(1)(2)(3)	(12)(3)	(123)
(3)	1	1	1
$(2,1)$	2	0	-1
$(1,1,1)$	1	-1	1

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
(4)	1	1	1	1	1
$(3,1)$	3	1	-1	0	-1
$(2,2)$	2	0	2	-1	0
$(2,1,1)$	3	-1	-1	0	1
$(1,1,1,1)$	1	-1	1	1	-1

The representation theory of the symmetric group revisited

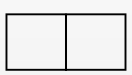
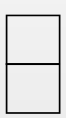
$$\tilde{s}_{()}[X_n] = 1 \qquad s_{()}[X_n] = \tilde{s}_{()}[X_n]$$

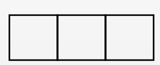
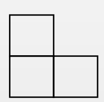
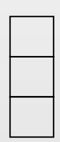
$$\tilde{s}_1[X_n] = s_1[X_n] - 1 \qquad s_1[X_n] = \tilde{s}_1[X_n] + \tilde{s}_{()}[X_n]$$

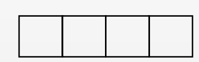
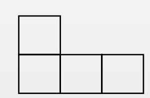
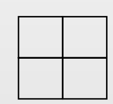
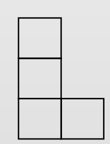
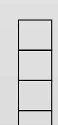
$$\tilde{s}_2[X_n] = s_2[X_n] - 2s_1[X_n] \qquad s_2[X_n] = \tilde{s}_2[X_n] + 2\tilde{s}_1[X_n] + 2\tilde{s}_{()}[X_n]$$

$$\tilde{s}_{1,1}[X_n] = s_{1,1}[X_n] - s_1[X_n] + 1 \qquad s_{1,1}[X_n] = \tilde{s}_{1,1}[X_n] + \tilde{s}_1[X_n]$$

symmetric group characters are evaluations of some functions at eigenvalues of matrices.

	(1)(2)	(12)
	1	1
	1	-1

	(1)(2)(3)	(12)(3)	(123)
	1	1	1
	2	0	-1
	1	-1	1

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
	1	1	1	1	1
	3	1	-1	0	-1
	2	0	2	-1	0
	3	-1	-1	0	1
	1	-1	1	1	-1

The representation theory of the symmetric group revisited

$$s_{()}[X_n] = \tilde{s}_{()}[X_n]$$

$$s_1[X_n] = \tilde{s}_1[X_n] + \tilde{s}_{()}[X_n]$$

$$s_2[X_n] = \tilde{s}_2[X_n] + 2\tilde{s}_1[X_n] + 2\tilde{s}_{()}[X_n]$$

$$s_{1,1}[X_n] = \tilde{s}_{1,1}[X_n] + \tilde{s}_1[X_n]$$

symmetric group characters are evaluations of some functions at eigenvalues of matrices.

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(1,1)	1	-1

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(3)	1	1	1
(2,1)	2	0	-1
(1,1,1)	1	-1	1

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
(4)	1	1	1	1	1
(3,1)	3	1	-1	0	-1
(2,2)	2	0	2	-1	0
(2,1,1)	3	-1	-1	0	1
(1,1,1,1)	1	-1	1	1	-1

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$$W^{\lambda}\downarrow^{Gl_n}_{S_n} \simeq \bigoplus_{\mu} (M^{(n-|\mu|,\mu)})^{\oplus r_{\lambda\mu}}$$

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$$s_{1,1}[X_n] = \tilde{s}_{1,1}[X_n] + \tilde{s}_1[X_n]$$

symmetric group characters are evaluations of some functions at eigenvalues of matrices.

	(1)(2)	(12)
$\ydiagram{2}$	1	1
$\ydiagram{1,1}$	1	-1

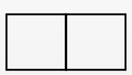
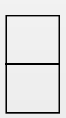
	(1)(2)(3)	(12)(3)	(123)
$\ydiagram{3}$	1	1	1
$\ydiagram{2,1}$	2	0	-1
$\ydiagram{1,1,1}$	1	-1	1

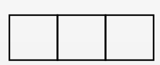
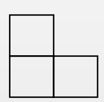
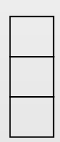
	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
$\ydiagram{4}$	1	1	1	1	1
$\ydiagram{3,1}$	3	1	-1	0	-1
$\ydiagram{2,2}$	2	0	2	-1	0
$\ydiagram{2,1,1}$	3	-1	-1	0	1
$\ydiagram{1,1,1,1}$	1	-1	1	1	-1

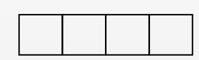
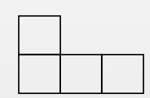
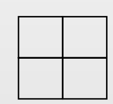
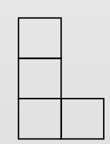
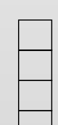
The representation theory of the symmetric group revisited

$$W^{\lambda}\downarrow^{Gl_n}_{S_n} \simeq \bigoplus_{\mu} (M^{(n-|\mu|,\mu)})^{\oplus r_{\lambda\mu}} \qquad\qquad s_\lambda[X_n] = \sum_{\mu} r_{\lambda\mu}\tilde{s}_\mu[X_n]$$

symmetric group characters are evaluations of some functions at eigenvalues of matrices.

	(1)(2)	(12)
	1	1
	1	-1

	(1)(2)(3)	(12)(3)	(123)
	1	1	1
	2	0	-1
	1	-1	1

	(1)(2)(3)(4)	(12)(3)(4)	(12)(34)	(123)(4)	(1234)
	1	1	1	1	1
	3	1	-1	0	-1
	2	0	2	-1	0
	3	-1	-1	0	1
	1	-1	1	1	-1

The representation theory of the symmetric group revisited

$$W^\lambda \downarrow_{S_n}^{Gl_n} \simeq \bigoplus_\mu (M^{(n-|\mu|,\mu)})^{\oplus r_{\lambda\mu}} \qquad s_\lambda[X_n] = \sum_\mu r_{\lambda\mu} \tilde{s}_\mu[X_n]$$

symmetric group characters are evaluations of some functions at eigenvalues of matrices.

The representation theory of the symmetric group revisited

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symmetric group characters are evaluations of some functions at eigenvalues of matrices.

Finding a "nice" combinatorial formula for the coefficients $r_{\lambda\mu}$ *will take some development of the linear algebra and combinatorics of this basis. This is known as the "restriction problem." The answer is closely related to (inner and outer) plethysm.*

Connection with the partition algebra

$$p_\mu[X_n] = \sum_{\lambda \vdash k} \chi^\lambda(\mu) s_\lambda[X_n]$$

Connection with the partition algebra

$$p_\mu[X_n] = \sum_{\lambda \vdash k} \chi^\lambda(\mu) s_\lambda[X_n]$$
$$= \sum_{\lambda : |\lambda| \leq k} \chi^\lambda_{P_k(n)}(\mu) \tilde{s}_\lambda[X_n]$$

$P_k(n)$ *partition algebra - Martin and Jones from 90's,*

Halverson, Ram, Benkart from 2000's

Connection with the partition algebra

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Halverson, Ram, Benkart from 2000's

	•	(1)	$(1,1)$	(2)	$(1,1,1)$	$(2,1)$	(3)
•	1	1	2	2	5	3	2
(1)	0	1	3	1	10	4	1
(2)	0	0	1	1	6	2	0
$(1,1)$	0	0	1	-1	6	0	0
(3)	0	0	0	0	1	1	1
$(2,1)$	0	0	0	0	2	0	-1
$(1,1,1)$	0	0	0	0	1	-1	1

Connection with the partition algebra

$$p_\mu[X_n] = \sum_{\lambda \vdash k} \chi^\lambda(\mu) s_\lambda[X_n]$$
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The dimension of an irreducible module indexed by a partition P^λ *is the number of set valued tableaux of shape* $(n - |\lambda|, \lambda)$

$P_k(n)$ *partition algebra - Martin and Jones from 90's, Halverson, Ram, Benkart from 2000's*

	$\bullet$	(1)	$(1,1)$	(2)	$(1,1,1)$	$(2,1)$	(3)
$\bullet$	1	1	2	2	5	3	2
(1)	0	1	3	1	10	4	1
(2)	0	0	1	1	6	2	0
$(1,1)$	0	0	1	-1	6	0	0
(3)	0	0	0	0	1	1	1
$(2,1)$	0	0	0	0	2	0	-1
$(1,1,1)$	0	0	0	0	1	-1	1

Connection with the partition algebra

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$P_k(n)$ *partition algebra - Martin and Jones from 90's, Halverson, Ram, Benkart from 2000's*

	•	$\ydiagram{1}$	$\ydiagram{1,1}$	$\ydiagram{2}$	$\ydiagram{1,1,1}$	$\ydiagram{2,1}$	$\ydiagram{3}$
•	1	1	2	2	5	3	2
$\ydiagram{1}$	0	1	3	1	10	4	1
$\ydiagram{2}$	0	0	1	1	6	2	0
$\ydiagram{1,1}$	0	0	1	-1	6	0	0
$\ydiagram{3}$	0	0	0	0	1	1	1
$\ydiagram{2,1}$	0	0	0	0	2	0	-1
$\ydiagram{1,1,1}$	0	0	0	0	1	-1	1

$$P_k(n) \simeq \bigoplus_{\lambda : |\lambda| \le k} P^\lambda \otimes P^\lambda$$

Connection with the partition algebra

$$p_\mu[X_n] = \sum_{\lambda \vdash k} \chi^\lambda(\mu) s_\lambda[X_n]$$
$$= \sum_{\lambda : |\lambda| \le k} \chi^\lambda_{P_k(n)}(\mu) \tilde{s}_\lambda[X_n]$$

The dimension of an irreducible module indexed by a partition P^λ *is the number of set valued tableaux of shape* $(n-|\lambda|, \lambda)$

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(1)	0	1	3	1	10	4	1
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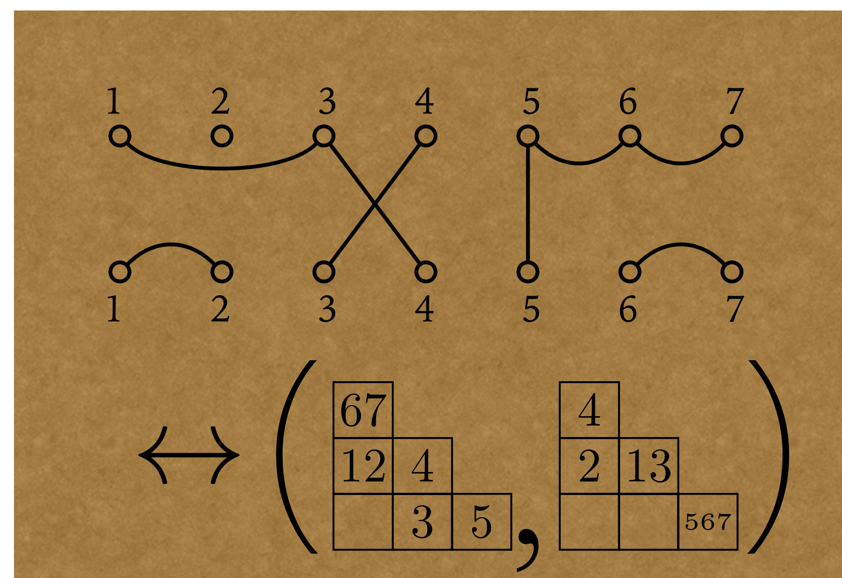
(3)	0	0	0	0	1	1	1
$(2,1)$	0	0	0	0	2	0	-1
$(1,1,1)$	0	0	0	0	1	-1	1

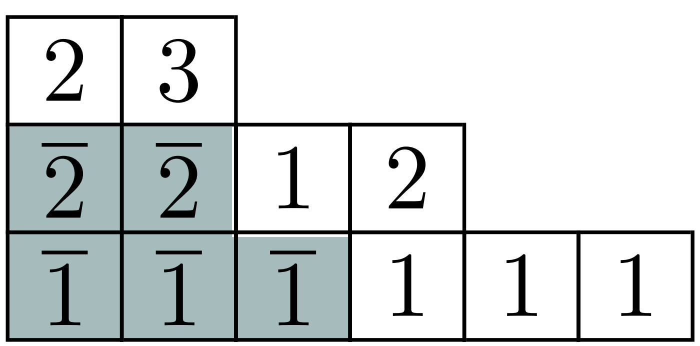
$$P_k(n) \simeq \bigoplus_{\lambda : |\lambda| \le k} P^\lambda \otimes P^\lambda$$



The Littlewood-Richardson rule

$$s_\lambda[X_n]s_\mu[X_n] = \sum_{\nu:|\nu|=|\lambda|+|\mu|} c^{\nu}_{\lambda\mu} s_\nu[X_n]$$

$$W^\lambda \otimes W^\mu \simeq \bigoplus_\gamma (W^\gamma)^{\oplus c^{\gamma}_{\lambda\mu}}$$



The Littlewood-Richardson rule

$$s_\lambda[X_n]s_\mu[X_n] = \sum_{\nu:|\nu|=|\lambda|+|\mu|} c^{\nu}_{\lambda\mu} s_\nu[X_n]$$

Reduced Kronecker coefficients

$$\tilde{s}_\lambda[X_n]\tilde{s}_\mu[X_n] = \sum_{\gamma} \overline{g}_{\lambda\mu\gamma}\tilde{s}_\gamma[X_n]$$

$$W^\lambda \otimes W^\mu \simeq \bigoplus_\gamma (W^\gamma)^{\oplus c^{\gamma}_{\lambda\mu}}$$

$$M^{(n-|\lambda|,\lambda)} \otimes M^{(n-|\mu|,\mu)} \simeq \bigoplus_\gamma (M^{(n-|\gamma|,\gamma)})^{\oplus \overline{g}_{\lambda\mu\gamma}}$$

$$\begin{array}{|c|c|c|c|c|c|}
\hline
2 & 3 \\
\hline
\overline{2} & \overline{2} & 1 & 2 \\
\hline
\overline{1} & \overline{1} & \overline{1} & 1 & 1 & 1 \\
\hline
\end{array}$$

Theorem - combinatorial interpretation

$$\tilde{s}_{\mu_1}[X_n]\tilde{s}_{\mu_2}[X_n]\cdots\tilde{s}_{\mu_\ell}[X_n]\tilde{s}_\lambda[X_n] \qquad \leq \qquad s_{\mu_1}[X_n]s_{\mu_2}[X_n]\cdots s_{\mu_\ell}[X_n]\tilde{s}_\lambda[X_n]$$

$\bar{1}$	$\bar{1}$	$\bar{2}1$	$\bar{2}12$

$\bar{1}$	$\bar{1}$	$\bar{2}1$	$\bar{2}2$	
				1

$\bar{1}$	$\bar{1}$	$\bar{2}11$	$\bar{2}2$

✦*set entries*

- *column strict and satisfies a lattice condition*
- *have shape* $(r, \gamma)/(\gamma_1)$
- *content* λ *in the barred entries*
- *content* μ *in the unbarred entries*
- *at most one barred entry per cell*
- *first row cannot have only barred entries*

✦*first row cannot have sets of size 1*

$$\tilde{s}_{\mu_1}[X_n]\tilde{s}_{\mu_2}[X_n]\cdots\tilde{s}_{\mu_\ell}[X_n]\tilde{s}_\lambda[X_n] \leq s_{\mu_1}[X_n]s_{\mu_2}[X_n]\cdots s_{\mu_\ell}[X_n]\tilde{s}_\lambda[X_n]$$

$$\geq \qquad\qquad \geq$$

$$\tilde{s}_\mu[X_n]\tilde{s}_\lambda[X_n] \leq s_\mu[X_n]\tilde{s}_\lambda[X_n]$$

Reduced Kronecker coefficients and the "restriction problem" seem to be closely related. There should be a notion of 'lattice' on these families of tableaux which simultaneously solves both of these problems

Merci!

Mon cher LaCIM, c'est à ta ton tour

de te laisser parler d'amour….

Bon 50ième anniversaire!