Find me a basis that . . .

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June 17, 2013

Definition: Symmetric Functions

$$\mathsf{Sym} = \mathbb{Q}[h_1, h_2, h_3, \ldots]$$

Degree $h_r = r$, Sym is a polynomial ring graded by the degree of the monomials.

Starting basis

For λ a partition of n

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}}$$

Sym is an algebra with basis elements indexed by a combinatorial object.

product

The multiplication is given by the product of monomials in the polynomial ring.

$$h_\lambda \cdot h_\mu = h_{sort(\lambda \cup \mu)}$$

coproduct

There is also a coproduct operation $\Delta : \mathsf{Sym} \to \mathsf{Sym} \otimes \mathsf{Sym}$

$$\Delta(h_r) = h_r \otimes 1 + h_{r-1} \otimes h_1 + h_{r-2} \otimes h_2 + \cdots + 1 \otimes h_r$$

$$\Delta(h_\lambda) = \Delta(h_{\lambda_1})\Delta(h_{\lambda_2}) \cdots \Delta(h_{\lambda_{\ell(\lambda)}})$$

Properties of Sym

1. bases indexed by partitions
2. graded
3. dimension of space of degree 0 is 1 (connected)
4. has an internal product $h_\lambda * h_\mu = \sum_{\nu \vdash |\lambda|} C_{\lambda,\mu,\nu} h_\nu$
5. related to representation theory
6. self-dual
7. commutative and cocommutative
8. can be realized as a polynomial algebra

Other examples of CHA's

compositions, permutations, binary trees, set partitions, set compositions, words, matroids, etc.

WHY?

A combinatorial Hopf algebra allows you to play with the combinatorics of an infinite graded set plus examine the relationship of combinatorial operations on these objects with linear algebra + product + coproduct + internal product . . .

representation theory, geometry, homological algebra, etc.

If in addition you can show there is an isomorphism with other types of calculations you would like to do, you can translate computations that can be quite abstract to (linear) algebra calculations within the CHA.

How do we define/identify a combinatorial Hopf algebra.

Minimum requirements

1. bases indexed by a combinatorial object
2. has a product and coproduct
3. graded
4. dimension of space of degree 0 is 1 (connected)

Adddditional conditions

1. has distinguished basis which has positive product and coproduct structure coefficients
2. has an internal product with graded pieces is a subalgebra
3. is related to representation theory
4. has multiplicative linear functional (character) mapping Hopf algebra elements to field
5. can be realized as a subalgebra of polynomial algebras with an infinite number of variables

Fix a CHA: find me a basis...

"ordinary" : Bases in an algebra are common...

"awesome" : Bases with a positive product that is easy to describe are rare

"totally awesome" : Bases with a positive product and coproduct are rarer still

"super totally awesome" : Bases with a positive product and coproduct and internal (co)product

General problem:

Given a combinatorial Hopf algebra with a distinguished "super totally awesome" basis, find all the ("super" +) "totally awesome" bases.

For a combinatorial Hopf algebra let $A = \{A_\alpha\}$ and $B = \{B_\alpha\}$ be bases of this CHA.

partial order on bases

Say that $A \geq B$ if $A_\alpha = \sum_\beta c_{\alpha\beta} B_\beta$

for some set of coefficients $c_{\alpha\beta} \in \mathbb{N}$.

Say that a basis is integral monomial if when it is realized as a subalgebra of polynomial algebra it has integral coefficients (or $\geq M$ if that makes sense).

Conjecture: The smallest integral monomial "super totally awesome" basis of *Sym* basis is the Schur basis

Conjecture: The smallest integral monomial "totally awesome" basis of $Sym^{(k)} = \mathbb{Q}[h_1, h_2, \ldots, h_k]$ is the k-Schur basis ($q = 1$).

Problem: Given a CHA and an analogue of the complete basis (or just any "super totally awesome" basis), give an algorithm for calculating all (and in particular, the smallest "super"+) "totally awesome" bases.

NSym is *the* combinatorial Hopf algebra of compositions

$$\text{Let } NSym = \mathbb{Q}\langle H_1, H_2, H_3, \ldots \rangle$$

$$\text{with } deg(H_r) = r \text{ and } H_iH_j \neq H_jH_i \text{ for } i \neq j.$$

$$NSym_n = \{H_\alpha : \alpha \text{ composition of } n\}$$

$$NSym = \bigoplus_{n \geq 0} NSym_n$$

$$H_\alpha \cdot H_\beta = H_{(\alpha,\beta)}$$

$$\Delta(H_r) = H_r \otimes 1 + H_{r-1} \otimes H_1 + H_{r-2} \otimes H_2 + \cdots + 1 \otimes H_r$$

$$H_\alpha * H_\beta = \sum_\gamma \tilde{C}_{\alpha,\beta,\gamma} H_\gamma$$

Problem: Find the smallest "super totally awesome" basis of *NSym* or prove that it is the ribbon basis.

$$R_\alpha = \sum_{\beta \geq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} H_\beta$$

$$H_\alpha = \sum_{\beta \geq \alpha} R_\beta$$

Problem: Identify all the "totally awesome" bases smaller than the complete basis.

Given a combinatorial Hopf algebra $\mathcal{H} = \sum_{n\geq 0} \mathcal{H}_n$ (with an internal product) say that $B = \{B_\tau\}$ is a basis of $\bigoplus_{r=0}^{n} \mathcal{H}_r \subseteq \mathcal{H}$.

B is n-"partially awesome" if $B_\tau \cdot B_\theta$ has an $\mathbb{N}$-expansion in the B-basis.

B is n-"partially totally awesome" if in addition, $\Delta(B_\tau)$ has an $\mathbb{N}$-$B \otimes B$-expansion

B is n-"partially super totally awesome" if in addition, $B_\tau * B_\theta$ has an $\mathbb{N}$-B-expansion

One way of solving the problem (in particular for *NSym*):

1. for each $n-1$-“partially“ (+“super”+) “totally awesome” basis
2. first set $H_\alpha = \sum_\beta c_{\alpha\beta} B_\beta$ for unknown coefficients $c_{\alpha\beta}$ for all $\alpha, \beta \models n$
3. compute (in terms of $c_{\alpha\beta}$)
   - product coefficients $B_\alpha B_\beta = \sum_\gamma d^\gamma_{\alpha\beta} B_\gamma$,
   - coproduct coefficients $\Delta(B_\gamma) = \sum_{\alpha,\beta} e^\alpha_{\alpha\beta} B_\gamma$
   - internal product coefficients $B_\alpha * B_\beta = \sum_\gamma f_{\alpha\beta\gamma} B_\gamma$
4. Solve equations $c_{\alpha\beta}, d^\gamma_{\alpha\beta}, e^\alpha_{\alpha\beta}, f_{\alpha\beta\gamma} \in \mathbb{N}$
5. Find all solutions of n-“partially“ (+“super”+) “totally awesome” bases or decide that there are none for this $n-1$-partial basis.

Ideas for future projects for Sage

Species: Make the species code robust enough that given a species, you can produce a combinatorial Hopf algebra (construction via recipe of Aguiar). (Martin Ruby (here Thursday Friday))

Hopf algebras: Pick up some of the combinatorial Hopf algebras (e.g. set partitions) that are on the sage-combinat queue and bring them to life (FQSym - Florent, Supercharacters - Franco (not here), FQSym, WQSym, PQSym, ... - Jean-Baptiste Priez, BWSym and related Hopf algebras - Olivier Mallet, Ali Chouria)

NSym/QSym: Add code to make these Hopf algebras more robust, efficient and complete. (me, Chris, Darij +)

Let α be a composition and r a positive integer, then for a set of elements $\{\mathbb{S}_\alpha\}$ of *NSym* we will call a right Pieri-like rule is a product rule of the form

$$\mathbb{S}_\alpha H_r = \sum_\beta \mathbb{S}_\beta$$

over some sum of compositions β of $|\alpha| + r$ where $\beta_i \geq \alpha_i$.

Repeated applications of this Pieri rule where we record the cells that were added as we multiply H_{α_i} with an i gives a notion of a composition tableau so that

$$H_\alpha = \sum_\gamma \mathcal{K}_{\gamma\alpha} \mathbb{S}_\gamma$$

where the the $\mathcal{K}_{\gamma\alpha}$ are the number of composition tableau of shape γ and content α.

Example 1: Dual quasi-Schur basis

4	5	5			
3	3	3	3	5	5
2	2	4	4		
1	1	1	2	2	3

Example 2: Immaculate basis

4					
3	3	3	3	4	4
2	2	3			
1	1	1	2	2	3

Example 3: column strict

4					
3	3	3	3	4	4
2	2	2			
1	1	1	2	2	3

Question: What is a minimal set of conditions do we need to put a Pieri-like rule such that...

1. $\mathbb{S}_\alpha$ is a basis of *NSym*
2. the commutative image of $\mathbb{S}_\lambda$ for λ partition is s_λ