CHAPTER 5: EXPONENTIAL GENERATING FUNCTIONS

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1. What is an exponential generating function?

Sequences that satisfy a certain type of recurrence or can be decomposed into "pairs" satisfying something like $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ work well with generating funtions. However recurrences are sometimes (often) more complex. There are two other types of generating functions which work well with sequences satisfying other types of formulae called Dirichlet generating functions and exponential generating functions.

We won't get into Dirichlet generating functions. They work well with sequences satifying equations like $c_n = \sum_{d \text{ divides } n} a_d b_{n/d}$.

Exponential generating functions work well with sequences that satisfying relations like $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$. Given a sequence $a_0, a_1, a_2, a_3, \ldots$, the exponential generating function is

$$A(x) = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + a_4 \frac{x^4}{4!} + \cdots .$$

Now that we need to distinguish between the generating function of a sequence and the exponential generating function for a sequence, we refer to generating function as its 'ordinary generating function.' Exponential generating function will be abbreviated 'e.g.f.' and ordinary generating function will be abbreviated 'o.g.f.'

Below is a list of common sequences with their exponential generating functions. Those with a ??? in the entry don't have a simple algebraic formula for the (exponential) generating function.:

sequence	e.g.f.	o.g.f.
$1, 1, 1, 1, 1, \ldots$	e^x	$\frac{1}{1-x}$
$0, 1, 2, 3, 4, \ldots$	xe^x	$\frac{x}{(1-x)^2}$
$0^2, 1^2, 2^2, 3^2, 4^2, \ldots$	$(x + x^2)e^x$	$\frac{x+x^2}{(1-x)^3}$
$0^3, 1^3, 2^3, 3^3, 4^3, \ldots$	$(x + 3x^2 + x^3)e^x$	$\frac{x+4x^2+x^3}{(1-x)^4}$
$0!, 1!, 2!, 3!, 4!, \ldots$	$\frac{1}{1-x}$	???
$\binom{0}{k}, \binom{1}{k}, \binom{2}{k}, \binom{3}{k}, \ldots$	$\frac{x^k}{k!} e^x$	$\frac{x^k}{(1-x)^k}$
$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots$	???	$(1 + x)^n$
$2^0, 2^1, 2^2, 2^3, 2^4, \ldots$	e^{2x}	$\frac{1}{1-2x}$

Example 1. Recall in the section on ordinary generating functions we considered the exercise of creating new generating functions from old ones by algebraic operations. We list below sequences which are related to $a_0, a_1, a_2, a_3, \ldots$. The variable n refers to the term of the sequence (i.e. the n^{th} term of the sequence starting at $n = 0$). Exercise: for each of the sequences below determine an algebraic expression in terms of $A(x)$ which is the generating function for that sequence. Some of the operations on the sequences don't have a 'nice' algebraic operation on the generating function. The answers are at the end of the chapter.

(1) shift right - $0, a_0, a_1, a_2, a_3, a_4, a_5, \ldots$
(2) shift left - $a_1, a_2, a_3, a_4, a_5, a_6, \ldots$
(3) add/subtract one - $a_0 \pm 1, a_1 \pm 1, a_2 \pm 1, a_3 \pm 1, a_4 \pm 1, a_5 \pm 1, a_6 \pm 1, \ldots$
(4) multiply by 2 - $2a_0, 2a_1, 2a_2, 2a_3, 2a_4, 2a_5, 2a_6, \ldots$
(5) multiply by 2^n - $a_0, 2a_1, 4a_2, 8a_3, 16a_4, 32a_5, 64a_6, \ldots$
(6) multiply by n - $0, a_1, 2a_2, 3a_3, 4a_4, 5a_5, 6a_6, \ldots$
(7) multiply by $\binom{n}{k}$ - $\binom{0}{k}a_0, \binom{1}{k}a_1, \binom{2}{k}a_2, \binom{3}{k}a_3, \binom{4}{k}a_4, \binom{5}{k}a_5, \binom{6}{k}a_6, \ldots$
(8) remove odd terms - $a_0, 0, a_2, 0, a_4, 0, a_6, \ldots$
(9) remove even terms - $0, a_1, 0, a_3, 0, a_5, 0, \ldots$
(10) weighted partial sums - $a_0, \binom{1}{0}a_0 + \binom{1}{1}a_1, \binom{2}{0}a_0 + \binom{2}{1}a_1 + \binom{2}{2}a_2, \binom{3}{0}a_0 + \binom{3}{1}a_1 + \binom{3}{2}a_2 + \binom{3}{3}a_3, \binom{4}{0}a_0 + \binom{4}{1}a_1 + \binom{4}{2}a_2 + \binom{4}{3}a_3 + \binom{4}{4}a_4, \ldots$

2. A sequence with an interesting exponential generating function

In working with ordinary generating functions, the Fibonaci numbers were a good example of a sequence that had a nice ordinary generating function. Just as ordinary generating functions work well with partitions of integers, exponential generating functions seem to work well sequences which count numbers of permutations and set partitions.

Sometimes the ordinary generating function of a sequence of integers just doesn't have a nice expression for the generating function. One such sequence is the Bell numbers: $B_0 = 1$, $B_1 = 1$ and $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$ for $n > 1$ which is equal to the number of set partitions of $n + 1$. We can calculate the next few values as $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, $B_5 = 52$.

The problem is that the expression $\sum_{k=0}^{n} \binom{n}{k} B_k$ is not of the form $\sum_{k=0}^{n} a_k b_{n-k}$. Why? If I set $B(x) = \sum_{n\geq 0} B_n x^n$, then when I multiply $B(x)A(x)|_{x^n}$ is $\sum_{k=0}^{n} a_{n-k} B_k$ and I can't find a generating function where $a_{n-k} = \binom{n}{k}$. It just doesn't seem to work.

There is a way around this. We can define a new type of generating function $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ and if we take a second to $B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$ and multiply these together then we see that

$$A(x)B(x) = \sum_{n\geq 0} \sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} x^n = \sum_{n\geq 0} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a_k b_{n-k} \frac{x^n}{n!} = \sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \frac{x^n}{n!}$$

.

This gives us a new principle to work with.

Principle 2. The coefficient of $x^n/n!$ in the product of $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ and $B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$ is equal to

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}. \tag{1}$$

I mention this because in the recurrence for B_{n+1} if we set $a_k = B_k$ and $b_{n-k} = 1$ then it is of this form. Therefore it seems as though we might be able to write down a generating function of this form. We call $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ the exponential generating function for a sequence. Consider the exponential generating function for the sequence $1, 1, 1, 1, 1, 1, \ldots,$

$$\sum_{n\geq 0} 1\frac{x^n}{n!} = \sum_{n\geq 0} \frac{x^n}{n!} = e^x \ .$$

The exponential generating function for the sequence $0, 1, 2, 3, 4, 5, 6, \ldots,$ is equal to

$$\sum_{n\geq 0} n\frac{x^n}{n!} = \sum_{n\geq 1} \frac{x^n}{(n-1)!} = xe^x \ .$$

Now consider the sequence $\binom{0}{k}, \binom{1}{k}, \binom{2}{k}, \binom{3}{k}, \binom{4}{k}, \binom{5}{k}, \ldots,$ where k is fixed. We calculate that the exponential generating function is equal to

$$\sum_{n\geq 0} \binom{n}{k} \frac{x^n}{n!} = \sum_{n\geq k} \frac{n!}{k!(n-k)!} \frac{x^n}{n!} = \sum_{n\geq k} \frac{1}{k!} \frac{x^n}{(n-k)!} = \frac{x^k}{k!} \sum_{n\geq k} \frac{x^{n-k}}{(n-k)!} = \frac{x^k}{k!} e^x \ .$$

Now lets apply what we know to finding a formula for the exponential generating function for $B(x) = \sum_{n\geq 0} B_n \frac{x^n}{n!}$ where $B_0 = B_1 = 1$ and $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. Lets work it out as we normally do except with exponential generating functions.

$$\begin{aligned}
B(x) &= \sum_{n\geq 0} B_n \frac{x^n}{n!} \\
&= 1 + \sum_{n\geq 1} B_n \frac{x^n}{n!} \\
&= 1 + \sum_{n\geq 1} \sum_{k=0}^{n-1} \binom{n-1}{k} B_k \frac{x^n}{n!} \\
&= 1 + B_0 \frac{x}{1!} + \left(\binom{1}{0} B_0 + \binom{1}{1} B_1 \right) \frac{x^2}{2!} + \left(\binom{2}{0} B_0 + \binom{2}{1} B_1 + \binom{2}{2} B_2 \right) \frac{x^3}{3!} + \cdots
\end{aligned}$$

Now those coefficients that are appearing in this sum should look very familiar. They are exactly those that appear in equation (1) except that $a_k = B_k$ and $b_{n-k} = 1$. Therefore if we calculate $B(x)e^x$ we see

$$B(x)e^x = B_0 + \left(\binom{1}{0} B_0 + \binom{1}{1} B_1\right) \frac{x^1}{1!} + \left(\binom{2}{0} B_0 + \binom{2}{1} B_1 + \binom{2}{2} B_2\right) \frac{x^2}{2!} + \cdots$$

We can make the expression for $B(x)e^x$ look exactly like the expression that comes after the 1+ in the expression for $B(x)$ by integrating one time. What this means is that

$$B(x) = 1 + \int B(x)e^x dx$$

or also

$$B'(x) = B(x)e^x \ .$$

For a moment consider any exponential generating function that satisfies $A'(x) = A(x)e^x$, then if we take the coefficient of $\frac{x^n}{n!}$ on both sides of the equation then $a_{n+1} = \sum_{k=0}^{n} \binom{n}{k} a_k$. Therefore an exponential generating function $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$

$$A(x) \text{ satistifes } A'(x) = A(x)e^x \text{ if and only if } a_{n+1} = \textstyle\sum_{k=0}^{n} \binom{n}{k} a_k$$

and in this case, $A(0) = a_0 = 1$ implies that $A(x) = \sum_{n\geq 0} B_n \frac{x^n}{n!}$ where B_n are the Bell numbers.

It is not trivial to solve for $B(x)$ given the differential equation $B'(x) = B(x)e^x$ but this is one of the first techniques that appears in a course on differential equations. Note that if we take $B(x) = e^{e^x-1}$, then this equation does satisfy the differential equation and $B(0) = e^{e^0-1} = e^0 = 1$. We conclude that

$$e^{e^x-1} = \sum_{n\geq 0} B_n \frac{x^n}{n!} \ .$$

In fact if I use "sage" to compute the Taylor expansion of e^{e^x-1}, then I see that

```
sage: taylor(exp(exp(x)-1), x, 0, 6)
203/720*x^6 + 13/30*x^5 + 5/8*x^4 + 5/6*x^3 + x^2 + x + 1
```

If I rewrite this with the $n!$ in the dominators (no simplification of the fractions) then I see that

$$e^{e^x-1} = 1 + \frac{x}{1!} + 2\frac{x^2}{2!} + 5\frac{x^3}{3!} + 15\frac{x^4}{4!} + 52\frac{x^5}{5!} + 203\frac{x^6}{6!} + \cdots$$

and this agrees with what we calculated earlier with B_0 through B_5.

I can also use sage to help me with the algebra of verifying that $B'(x) = \frac{d}{dx}(e^{e^x-1}) = e^{e^x-1}e^x = B(x)e^x$.

```
sage: diff(exp(exp(x)-1),x)
e^(x + e^x - 1)
sage: exp(x)*exp(exp(x)-1)
e^(x + e^x - 1)
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`do the q-refinement for the Stirling numbers of the second kind` ToDo

3. Combinatorial calculations using exponential generating functions

Let me state precisely the multiplication principle of exponential generating functions.

Theorem 3. *(the multiplication principle of exponential generating functions) If* $A(x) = \sum_{n\geq 0} a_n \frac{x_n}{n!}$ *and* $B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$ *then the coefficient of* $\frac{x^n}{n!}$ *in* $A(x)B(x)$ *is equal to*

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \ .$$

In combinatorial terms if a_n *is equal to the number of widgets of size* n *and* b_n *is equal to the number of doodles of size* n*, then* $A(x)B(x)$ *is equal to the exponential generating function for the number of triples* (S, x, y) *where* x *is a widget of size* k*,* y *is a doodle of size* $n-k$ *and* S *is a subset of* $\{1, 2, \ldots, n\}$ *of size* k*.*
(the addition principle of exponential generating functions) The coefficient of $\frac{x^n}{n!}$ *in* $A(x)+B(x)$ *is* $a_n + b_n$ *and the interpretation for this coefficient is the number of objects which are either a widget or a doodle of size* n *(just as we saw in the case of ordinary generating functions).*

This is stated more generally this may be stated as the following:

Theorem 4. *(The Multiplication Principle of Exponential Generating Functions) Let* $A_i(x) = \sum_{n\geq 0} a_n^{(i)} \frac{x^n}{n!}$*, then*

$$A_1(x)A_2(x)\cdots A_d(x) = \sum_{n\geq 0} \left(\sum_{i_1+i_2+\cdots+i_d=n} \binom{n}{i_1, i_2, \ldots, i_d} a_{i_1}^{(1)} a_{i_2}^{(2)} \cdots a_{i_d}^{(d)} \right) \frac{x^n}{n!} \ .$$

Alternatively the coefficient of $\frac{x^n}{n!}$ *in* $A_1(x)A_2(x)\cdots A_d(x)$ *is equal to*

$$\sum_{i_1+i_2+\cdots+i_d=n} \binom{n}{i_1, i_2, \ldots, i_d} a_{i_1}^{(1)} a_{i_2}^{(2)} \cdots a_{i_d}^{(d)} \ .$$

I will continue to expand on the use of exponential generating functions. What we will need to do is develop tools for creating libraries of generating functions as we did for ordinary generating functions. For instance, if I give you the exponential generating function $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$, then I expect you to be able to give me expressions for $\sum_{n\geq 0} a_{n+2} \frac{x^n}{n!}$, $\sum_{n\geq 0} n a_n \frac{x^n}{n!}$, $\sum_{n\geq 0} a_n \frac{x^{n+2}}{(n+2)!}$.
There is another class of problems that is useful for the Mulitplication Principle of Exponential generating functions that I discussed last time. Consider problems like:

How many words (rearrangements of the letters) in the alphabet $\{a, b, c, d\}$ are there of length n?

Since our words are of length n, there are 4^n possible words with letters in $\{a, b, c, d\}$, each letter of the word has 4 choices. The exponential generating function for the number of these words is $\sum_{n\geq 0} 4^n \frac{x^n}{n!} = e^{4x}$. But what is kind of surprising is that I can also place restrictions on the letters and write down the exponential generating function for the sequence. Say that I consider the set of words

How many words are there in the alphabet $\{a, b, c, d\}$ such that there an even number of a's and b's (total) and at most 8 c's?

If we were to enumerate this using the multiplication principle and the addition priciple, then we would choose i spots from n for the a's and b's, choose a word in the a's and b's of length i, choose j of the remaining $n - i$ for the c's such that there are at most 8 c's, then the remaining $n - i - j$ spaces are where we place the d's. By the addition and multiplication principle of generating functions, we have

(2)

$$\sum_{i+j\leq n} \binom{n}{i} \binom{\text{\# words length } i \text{ in } a \text{ and } b}{\text{with an even \# } a\text{'s \&} b\text{'s}} \binom{n-i}{j} \binom{\text{\# words of length } j}{\text{in } c \text{ with } \leq 8\ c\text{'s}} \binom{\text{\# words of length}}{n-i-j \text{ in } d}$$

If we combine the binomials $\binom{n}{i}$ and $\binom{n-i}{j}$ and note that it is equal to $\binom{n}{i, j, n-i-j} = \binom{n}{i}\binom{n-i}{j}$.

Last time I presented the multiplication principle of exponential generating functions. I will restate it here with multiple generating functions (while the last time it was a product of two).

Principle 5. (The Multiplication Principle of Exponential Generating Functions) Let $A_i(x) = \sum_{n\geq 0} a_n^{(i)} \frac{x^n}{n!}$, then

$$A_1(x)A_2(x)\cdots A_d(x) = \sum_{n\geq 0} \left(\sum_{i_1+i_2+\cdots+i_d=n} \binom{n}{i_1, i_2, \ldots, i_d} a_{i_1}^{(1)} a_{i_2}^{(2)} \cdots a_{i_d}^{(d)} \right) \frac{x^n}{n!}\ .$$

Alternatively the coefficient of $\frac{x^n}{n!}$ in $A_1(x)A_2(x)\cdots A_d(x)$ is equal to

$$\sum_{i_1+i_2+\cdots+i_d=n} \binom{n}{i_1, i_2, \ldots, i_d} a_{i_1}^{(1)} a_{i_2}^{(2)} \cdots a_{i_d}^{(d)}\ .$$

You should recognize that (2) is a special case of a coefficient of one of these coefficients. The expression in (2) is equal to the coefficient of $x^n/n!$ in the product

(3)

$$\binom{\text{g.f. for words length } in \text{ in } a \text{ and } b}{\text{with an even \# } a\text{'s \&} b\text{'s}} \binom{\text{g.f. for words of length } n}{\text{in } c \text{ with } \leq 8\ c\text{'s}} \binom{\text{g.f. for words of length}}{n \text{ in } d}$$

Now I note that since there is precisely 1 word of length n using only the letter d then

$$\left(\begin{matrix}\text{g.f. for words of length} \\ n \text{ in } d\end{matrix}\right) = \sum_{n\geq 0} \frac{x^n}{n!} = e^x$$

Since there is one word of length n in the letters c unless $n > 8$, then

$$\left(\begin{matrix}\text{g.f. for words of length } n \\ \text{in } c \text{ with} \leq 8\ c\text{'s}\end{matrix}\right) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^8}{8!}$$

Now if we insist that there are an even number of a's and b's then the there are 4 words of length 2 (aa, ab, ba, bb), there are 16 words of length 4 ($aaaa$, $aaab$, $aaba$, ..., $bbbb$). In general, the number of words of length n is 2^n if n is even and 0 if n is odd, hence the exponential generating function is equal to

$$\left(\begin{matrix}\text{g.f. for words length } in \text{ in } a \text{ and } b \\ \text{with an even \# } a\text{'s \&}b\text{'s}\end{matrix}\right) = 1+4\frac{x^2}{2!}+16\frac{x^4}{4!}+64\frac{x^6}{6!}+\cdots = \frac{1}{2}\left(e^{2x} + e^{-2x}\right) = cosh(2x)$$

Therefore putting this together with (3) we have that the coefficient of $x^n/n!$ in

$$cosh(2x)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^8}{8!}\right)e^x$$

is equal to the number of words in the alphabet $\{a, b, c, d\}$ such that there an even number of a's and b's (total) and at most 8 c's.

For example for the words of length 1 there is only c and d, for the words of length 2 we can have aa, bb, ab, ba, cc, cd, dc, dd so there are 8 words of length 2. For words of length 3 we can have caa, aca, aac, cbb, bcb, bbc, cab, acb, abc, cba, bca, bac, another 12 with a,b and ds and then 8 more are words in c and d (32 in total). In total there are We should then see that the series expands as $1 + 2\frac{x}{1!} + 8\frac{x^2}{2!} + 32\frac{x^3}{3!} + \cdots$. I will check this on the computer to show you how it is done.

```
sage: taylor(exp(x)*cosh(2*x)*sum(x^n/factorial(n) for n in range(9)),x,0,4)
16/3*x^4 + 16/3*x^3 + 4*x^2 + 2*x + 1
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In general we have that ordinary generating functions used for counting problems that can be reduced to integer sum problems and exponential generating functions are useful for enumerating problems that can be reduced to enumerating words. It is also sometimes said that ordinary generating functions are good for enumerating "unlabeled" objects and exponential generating functions are good for enumerating "labeled" objets. This is a vague rule and hard to tell why this might be correct until we come with more examples of uses for ordinary and exponential generating functions. For example, we looked at the exponential generating function for the number of set partitions of n and this was e^{e^x-1} (this is a "labeled" object), we also started to look at partitions and ordinary generating functions.

4. More examples

I wanted to show an example of what I expected for a problem on generating functions for the homework. I showed an example of an exponential generating function instead of an ordinary generating function because the principle was the same and you had something to compare how exponential and ordinary generating functions differ.

I took a problem that was extremely similar to one of the homework problems, but differed because it was enumerating a set of words (structures with order) rather than distributions of loonies and twonies to people. Please refer to Homework # 2 problem 5 for the comparison.

Count the number of words of length 20 with the letters A, B, C, D satisfying the following properties.

(1) no restriction
(2) there are an even number of A's and B's in total
(3) there are at most a total of 6 A's and Bs.
(4) the number of A's and B's is even and there are at most 6.
(5) the number of A's and B's is even or there are at most 6 in total.

Lemma 6. *The exponential generating function for the number of words of length n with C's and D's (or A's and B's) is equal to* $e^{2x} = \sum_{n\geq 0} 2^n \frac{x^n}{n!}$.

Proof. There is one word of length n with C's only (or D's only). The exponential generating function for the number of words is equal to $e^x = \sum_{n\geq 0} \frac{x^n}{n!}$. Every word of length n with C's and D's is isomorphic to the number of triples (S, x, y) consisting of a subset S of $\{1, 2, \ldots, n\}$ representing the positions of the C's, a word x of length $|S|$ of C's and a word y of D's of length $n - k$. Therefore the exponential generating function for the number of these words will be the product of exponential generating functions for the words of C's and the exponential generating functions for the words of D's. Their product is $e^x e^x = e^{2x}$. □

or better

Proof. There are 2^n words of length n with C's and D's because for each letter of the word there are two choices. Therefore the exponential generating function is $\sum_{n\geq 0} 2^n \frac{x^n}{n!} = e^{2x}$. □

Lemma 7. *The exponential generating function for the number of words with A's and Bs where there an even number of A's and B's is* $(e^{2x} + e^{-2x})/2$.

Proof. The exponential generating function for the words with A's and B's with an even number of letters is equal to the terms with even exponents in the exponential generating function e^{2x} for all words with A's and B's. This is $(e^{2x} + e^{-2x})/2$. □

Lemma 8. *The exponential generating function for the number of words with A's and Bs with at most 6 letters in total is* $1 + 2x + 2x^2 + 4/3x^3 + 2/3x^4 + 4/15x^5 + 4/45x^6$.

Proof. A Sage calculation yields the first 6 terms of the exponential generating function for the words with A's and B's, namely $e^{2x} = 1 + 2x + 2x^2 + 4/3x^3 + 2/3x^4 + 4/15x^5 + 4/45x^6 + \cdots$. □

Lemma 9. *The exponential generating function for the number of words with A's and Bs with at most 6 letters in total and an even number is $1 + 2x^2 + 2/3x^4 + 4/45x^6$.*

Proof. The even terms from the expression in Lemma 8 is the exponential generating function for the number of terms of even length and length less than or equal to 6. □

Lemma 10. *The exponential generating function for the number of words with A's and Bs with at most 6 letters in total or an even number is $(e^{2x} + e^{-2x})/2 + 2x + 4/3x^3 + 4/15x^5$.*

Proof. The odd terms from the expression in Lemma 8 is equal to the exponential generating function for the words which are of length less than or equal to 6 and of odd length and this is equal to $2x + 4/3x^3 + 4/15x^5$. The disjoint union of this set of words and those that are of even length is equal to the words with A's and Bs with at most 6 letters in total or an even number. By the addition principle of exponential generating functions, this is equal to the sum of the expression for the exponential generating function for the number of words with A's and Bs where there an even number of A's and B's and the exponential generating function for the words which are of length less than or equal to 6 and of odd length and this is equal to $(e^{2x} + e^{-2x})/2 + 2x + 4/3x^3 + 4/15x^5$. □

Note that now a lot of my answer is cut and paste:

Answer to part (1): Therefore the exponential generating function for the number of words with the letters A, B, C, and D is equal to the product of the exponential generating functions for the words with A's and B's times the exponential generating functions for the words with C's and D's. Their product is $e^{2x}e^{2x} = e^{4x}$.

The number of words with the letters A, B, C, and D of length 20 is equal to 20! times the coefficient of x^{20} in e^{4x} which is equal to 4^{20}.

Answer to part (2): The exponential generating function for the number of words with the letters A, B, C, D where there are an even number of A's and B's is equal to the product of the exponential generating function for the number of words with A's and Bs where there an even number of A's and B's (from Lemma 7) and the exponential generating function for the number of words with C's and D's. This is equal to $e^{2x}(e^{2x} - e^{-2x})/2 = (e^{4x} - 1)/2$.

The number of words with the letters A, B, C, and D where there are an even number of A's and B's of length 20 is equal to 20! times the coefficient of x^{20} in $(e^{4x} - 1)/2$ which is equal to $4^{20}/2$.

Answer to part (3): The exponential generating function for the number of words with the letters A, B, C, D where there are at most 6 A's and B's in total is equal to the product of the exponential generating function for the number of words with A's and Bs where there at most 6 A's and B's (from Lemma 8) and the exponential generating function for the number of words with C's and D's. This is equal to $e^{2x}(1 + 2x + 2x^2 + 4/3x^3 + 2/3x^4 + 4/15x^5 + 4/45x^6)$.

The number of words with the letters A, B, C, and D where there are at most 6 A's and B's of length 20 is equal to 20! times the coefficient of x^{20} in $e^{2x}(1 + 2x + 2x^2 + 4/3x^3 + 2/3x^4 + 4/15x^5 + 4/45x^6)$ which (according to Sage) is equal to 63396904960.

Answer to part (4): The exponential generating function for the number of words with the letters A, B, C, D where there are at most 6 A's and B's in total and an even number is equal to the product of the exponential generating function for the number of words with A's and Bs where there at most 6 A's and B's and an even number (from Lemma 9) and the exponential generating function for the number of words with C's and D's. This is equal to $e^{2x}(1 + 2x^2 + 2/3x^4 + 4/45x^6)$.

The number of words with the letters A, B, C, and D where there are at most 6 A's and B's and an even number of length 20 is equal to 20! times the coefficient of x^{20} in $e^{2x}(1 + 2x^2 + 2/3x^4 + 4/45x^6)$ which (according to Sage) is equal to 45923434496.

Answer to part (5): The exponential generating function for the number of words with the letters A, B, C, D where there are less than or equal to 6 A's and B's in total or an even number of A's and B's is equal to the product of the exponential generating function for the number of words with A's and Bs where there at most 6 A's and B's or an even number (from Lemma 10) and the exponential generating function for the number of words with C's and D's. This is equal to $e^{2x}((e^{2x} + e^{-2x})/2 + 2x + 4/3x^3 + 4/15x^5)$.

The number of words with the letters A, B, C, and D where there are at less than or equal to 6 A's and B's in total or an even number of A's and B's of length 20 is equal to 20! times the coefficient of x^{20} in $e^{2x}((e^{2x} + e^{-2x})/2 + 2x + 4/3x^3 + 4/15x^5)$ which (according to Sage) is equal to 567229284352.

5. Exercises

(1) Find a formula for the exponential generating function of the number of odd set partitions of n (that is, the set partitions where each set in the partition has an odd number of elements). If we let B_n^{odd} = to the number of odd set partitions of n, then $B_0^{odd} = 1$ and for $n \geq 0$,

$$B_{n+1}^{odd} = \sum_{k=0}^{\lceil n/2 \rceil} \binom{n}{2k} B_{n-2k}^{odd}.$$

Show that this exponential generating function is $e^{sinh(x)}$.

(2) Let $B_k(x) = \sum_{n \geq 0} \binom{n}{k} \frac{x^n}{n!}$. Show that

$$B_k(x) B_\ell(x) = \frac{1}{2^{k+\ell}} \binom{k+\ell}{\ell} B_{k+\ell}(2x).$$

Take the coefficient of $x^n/n!$ on both sides of the equation and use it to give a binomial coefficient identity. Verify that your identity holds for $n = 6$, $k = 3$, $\ell = 1$.

(3) Find the exponential generating function for the number of words of length n using letters $a, b, c, d, e, f, g, h, i$ such that

(a) all 9 letters occur without restriction
(b) at least one of the first 6 letters appears
(c) the first 6 letters each appear at least once and the last three each appear an even number of times
(d) the first 6 letters each appear at least once and the last six each appear an even number of times
(e) at least one of the first 6 letters appears and the total number of the last 6 letters is even

Use your generating function to find the number of number of words of length 20 with the restrictions above.
As a hint, for length 10 the number of words for part (a) is 3486784401, (b) 3486725352, (c) 45465840, (d) 680400, (e) 1743362676.

(4) Given the generating function $A(x) = \sum_{n\geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$, find a formula for the generating function

$$\tilde{A}(x) = a_1 + a_0 x + a_3 x^2 + a_2 x^3 + a_5 x^4 + a_4 x^5 + \cdots$$

If $B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!} = b_0 + b_1 \frac{x}{1!} + b_2 \frac{x^2}{2!} + \cdots$ is an exponential generating function, find a formula for the generating function

$$\tilde{B}(x) = b_1 + b_0 \frac{x}{1!} + b_3 \frac{x^2}{2!} + b_2 \frac{x^3}{3!} + b_5 \frac{x^4}{4!} + b_4 \frac{x^5}{5!} + \cdots$$

(5) Recall that the unsigned Stirling number of the first kind is denoted $s'(n,k)$ and is equal to the number of permutations of n into k cycles. Define $T_k(x) = \sum_{n\geq 0} s'(n,k) \frac{x^n}{n!}$.
(a) Show that $T_1(x) = -log(1-x)$.
(b) Use the multiplication principle of exponential generating functions to explain why

$$k! T_k(x) = T_1(x)^k \ . \qquad (\&)$$

(c) Take the coefficient of $\frac{x^n}{n!}$ in $T_k(x)$ and show that

$$s'(n,k) = \frac{1}{k!} \sum_{a_1+a_2+\cdots+a_k=n} \frac{n!}{a_1 a_2 \cdots a_k} \qquad (\%)$$

where the sum is over solutions to $a_1 + a_2 + \cdots + a_k = n$ with $a_i > 0$.
(d) By taking the derivative of both sides of equation (&) we see that $k! T_k'(x) = k T_1(x)^{k-1} T_1'(x)$ and since $T_1(x)^{k-1} = (k-1)! T_{k-1}(x)$ we have that $T_k'(x) = T_{k-1}(x) T_1'(x)$. Take the coefficient of $\frac{x^n}{n!}$ in both sides of the equation $T_k'(x) = T_{k-1}(x) T_1'(x)$ to arrive at a recursive formula for the unsigned Stirling numbers of the first kind.

6. Answers to example 1

(1) shift right - $0, a_0, a_1, a_2, a_3, a_4, a_5, \ldots$ - antiderivative with constant term equal to 0.

$$\int A(x)dx = \sum_{n\geq 0} a_n \int \frac{x^n}{n!}dx = \sum_{n\geq 0} a_n \frac{x^{n+1}}{(n+1)!}$$

(2) shift left - $a_1, a_2, a_3, a_4, a_5, a_6, \ldots$ - derivative

$$\frac{d}{dx}A(x) = \frac{d}{dx}(a_0 + \sum_{n\geq 1} a_n \frac{x^n}{n!}) = \sum_{n\geq 1} a_n \frac{x^{n-1}}{(n-1)!}$$

(3) add/subtract one - $a_0 \pm 1, a_1 \pm 1, a_2 \pm 1, a_3 \pm 1, a_4 \pm 1, a_5 \pm 1, a_6 \pm 1, \ldots$ - add or subtract e^x

$$A(x) \pm e^x = \sum_{n\geq 0} a_n \frac{x^n}{n!} \pm \sum_{n\geq 0} \frac{x^n}{n!} = \sum_{n\geq 0}(a_n \pm 1)\frac{x^n}{n!}$$

(4) multiply by 2 - $2a_0, 2a_1, 2a_2, 2a_3, 2a_4, 2a_5, 2a_6, \ldots$

$$2A(x) = \sum_{n\geq 0} 2a_n \frac{x^n}{n!}$$

(5) multiply by 2^n - $a_0, 2a_1, 4a_2, 8a_3, 16a_4, 32a_5, 64a_6, \ldots$

$$A(2x) = \sum_{n\geq 0} 2^n a_n \frac{x^n}{n!}$$

(6) multiply by n - $0, a_1, 2a_2, 3a_3, 4a_4, 5a_5, 6a_6, \ldots$

$$x\frac{d}{dx}A(x) = \sum_{n\geq 1} x a_n \frac{x^{n-1}}{(n-1)!} = \sum_{n\geq 1} n a_n \frac{x^n}{n!}$$

(7) multiply by $\binom{n}{k}$ - $\binom{0}{k}a_0, \binom{1}{k}a_1, \binom{2}{k}a_2, \binom{3}{k}a_3, \binom{4}{k}a_4, \binom{5}{k}a_5, \binom{6}{k}a_6, \ldots$

$$\frac{x^k}{k!}\frac{d^k}{dx^k}A(x) = \sum_{n\geq 0} a_n \frac{x^k}{k!}\frac{d^k}{dx^k}\frac{x^n}{n!} = \sum_{n\geq k} a_n \frac{x^k}{k!}\frac{x^{n-k}}{(n-k)!} = \sum_{n\geq k} a_n \frac{n!}{k!(n-k)!}\frac{x^n}{n!}$$

(8) remove odd terms - $a_0, 0, a_2, 0, a_4, 0, a_6, \ldots$

$$\frac{A(x)+A(-x)}{2} = \frac{1}{2}\left(\sum_{n\geq 0} a_n \frac{x^n}{n!} + \sum_{n\geq 0} a_n(-1)^n \frac{x^n}{n!}\right) = \sum_{n\geq 0} a_{2n}\frac{x^{2n}}{(2n)!}$$

(9) remove even terms - $0, a_1, 0, a_3, 0, a_5, 0, \ldots$

$$\frac{A(x)-A(-x)}{2} = \frac{1}{2}\left(\sum_{n\geq 0} a_n \frac{x^n}{n!} - \sum_{n\geq 0} a_n(-1)^n \frac{x^n}{n!}\right) = \sum_{n\geq 0} a_{2n+1}\frac{x^{2n+1}}{(2n+1)!}$$

(10) weighted partial sums - a_0, $\binom{1}{0}a_0+\binom{1}{1}a_1$, $\binom{2}{0}a_0+\binom{2}{1}a_1+\binom{2}{2}a_2$, $\binom{3}{0}a_0+\binom{3}{1}a_1+\binom{3}{2}a_2+\binom{3}{3}a_3$, $\binom{4}{0}a_0+\binom{4}{1}a_1+\binom{4}{2}a_2+\binom{4}{3}a_3+\binom{4}{4}a_4, \ldots$

$$e^x A(x) = \left(\sum_{n\geq 0} \frac{x^n}{n!}\right)\left(\sum_{n\geq 0} a_n \frac{x^n}{n!}\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n}\binom{n}{k}a_k\right)\frac{x^n}{n!}$$