Non-Commutative Symmetric Functions I: A zoo of Hopf algebras

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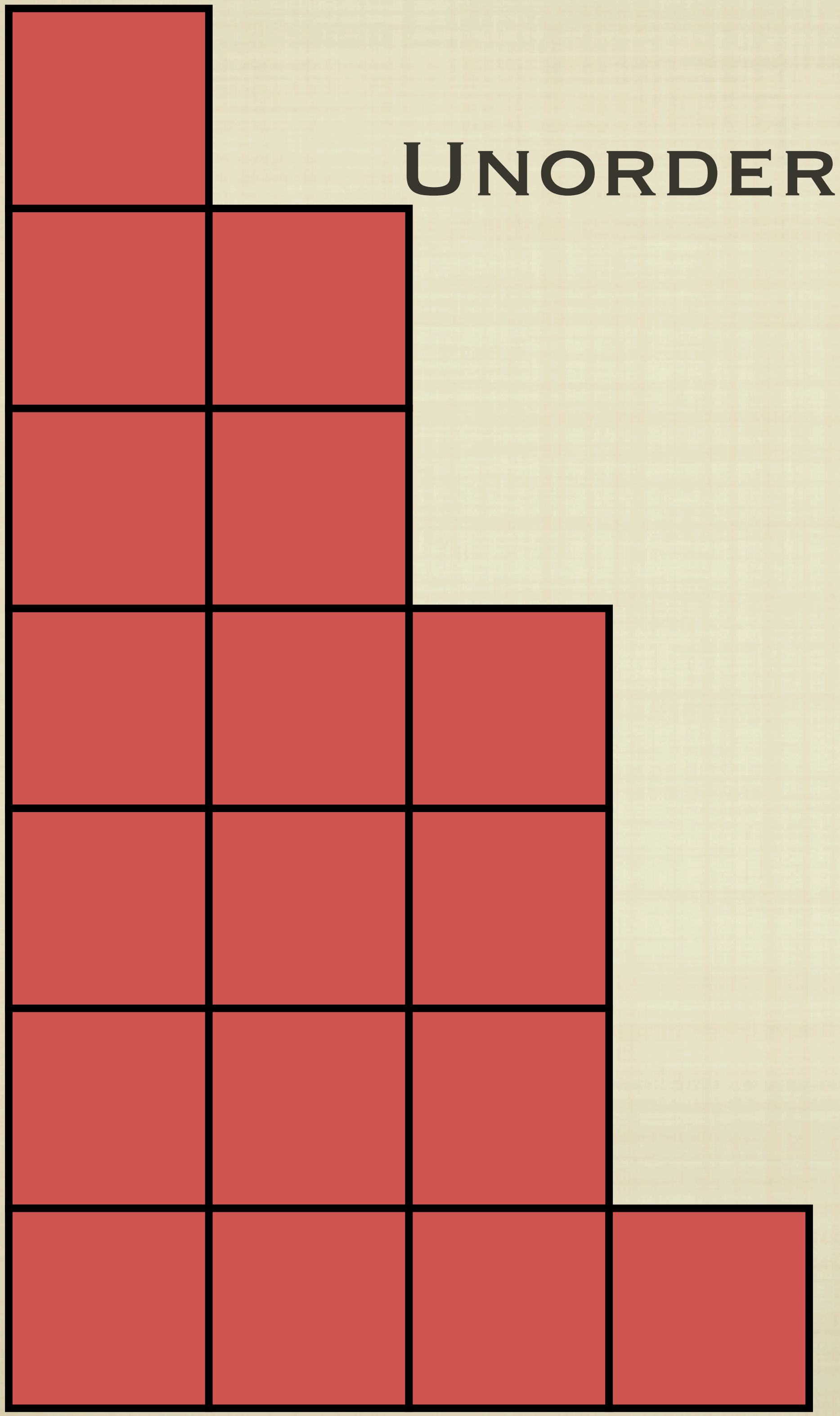
York University

Joint work with Nantel Bergeron, Anouk Bergeron-Brlek, Emmanurel Briand, Christophe Hohlweg, Christophe Reutenauer, Mercedes Rosas and others...

Partitions

Unordered lists of non-negative integers

Partitions



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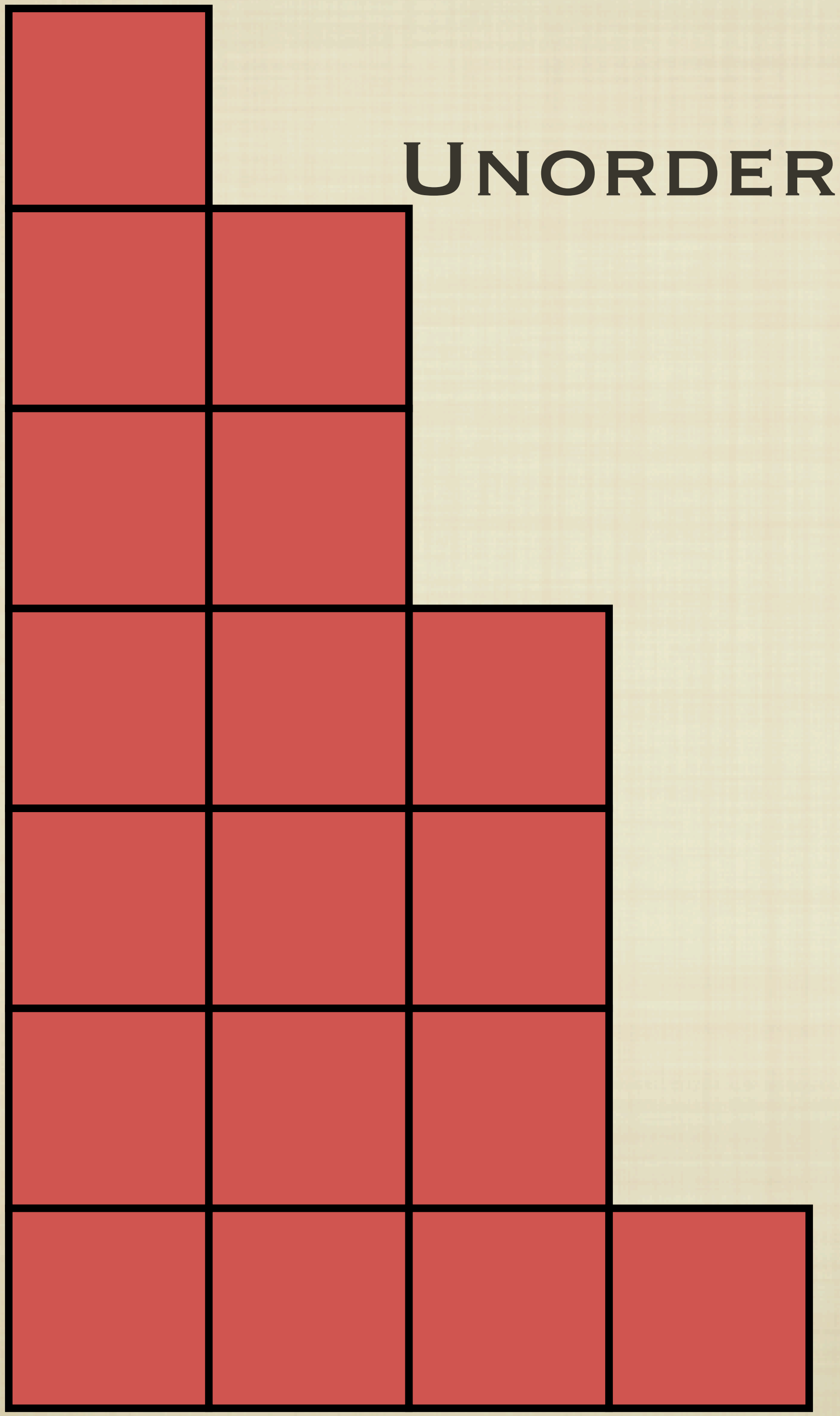

Unordered lists of non-negative integers

$$(4, 3, 3, 3, 2, 2, 1)$$

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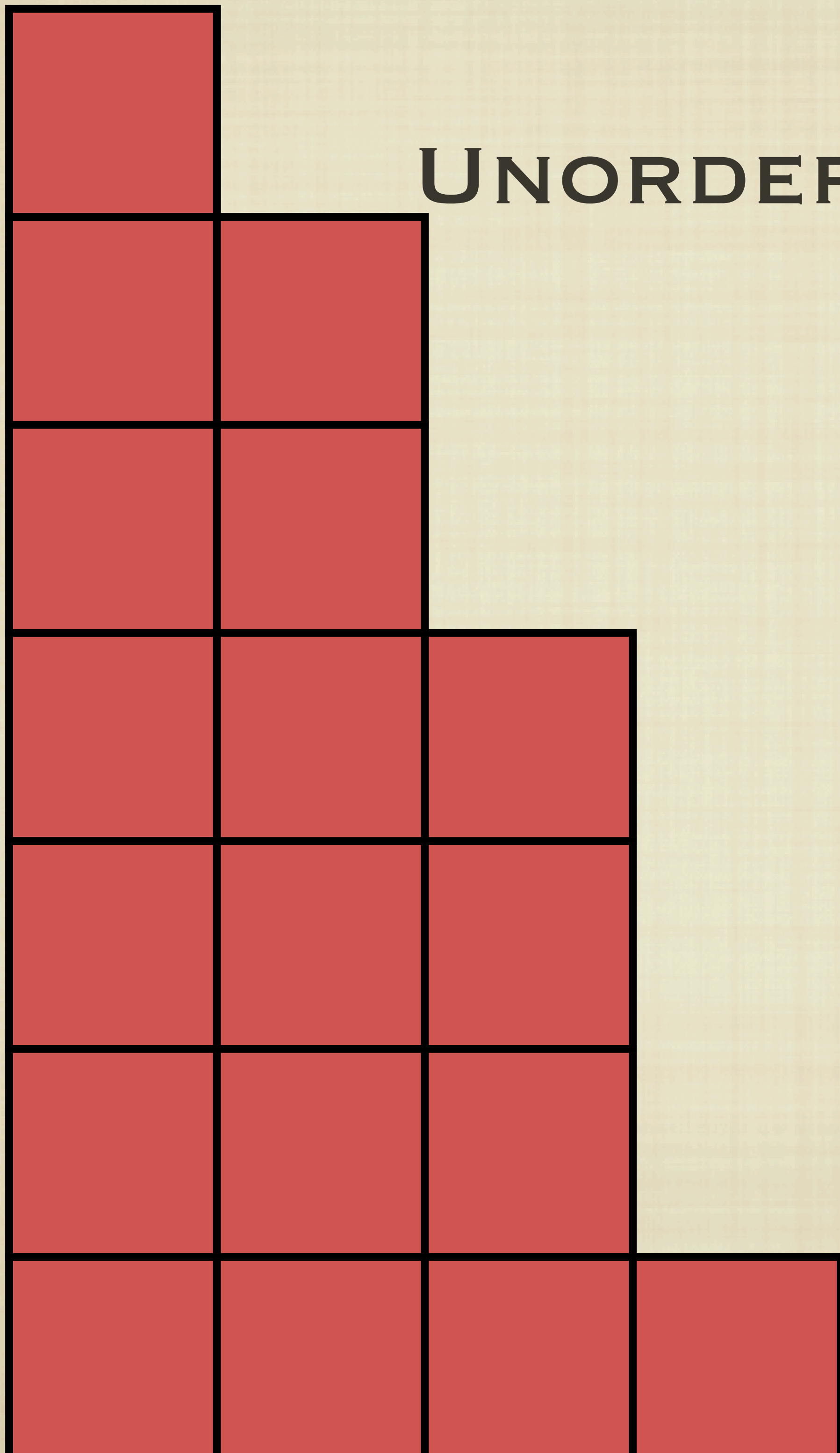

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$$4 + 3 + 3 + 3 + 2 + 2 + 1$$

Create an algebra

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- linearly spanned by partitions

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- Commutative product μ

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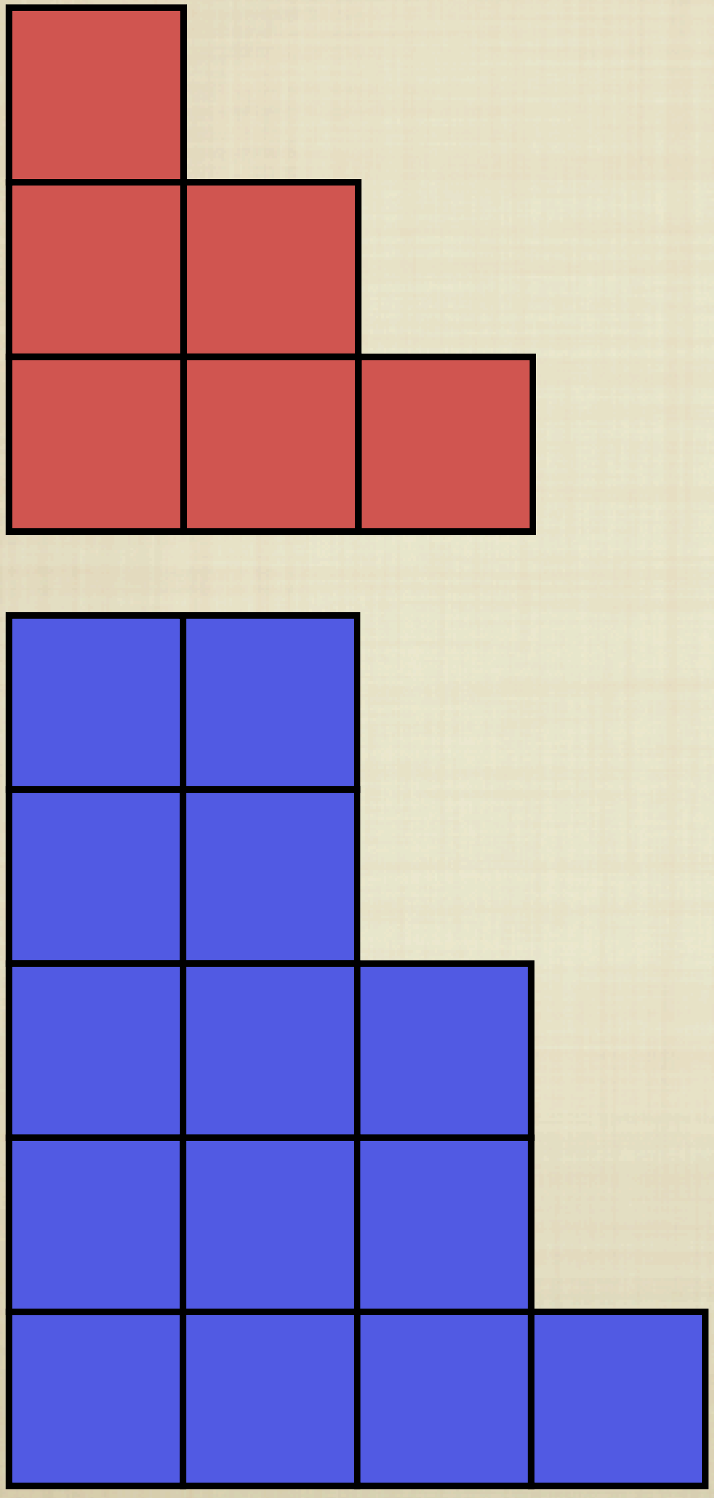
$$\mu : \Lambda_i \otimes \Lambda_j \longrightarrow \Lambda_{i+j}$$

Define a commutative product



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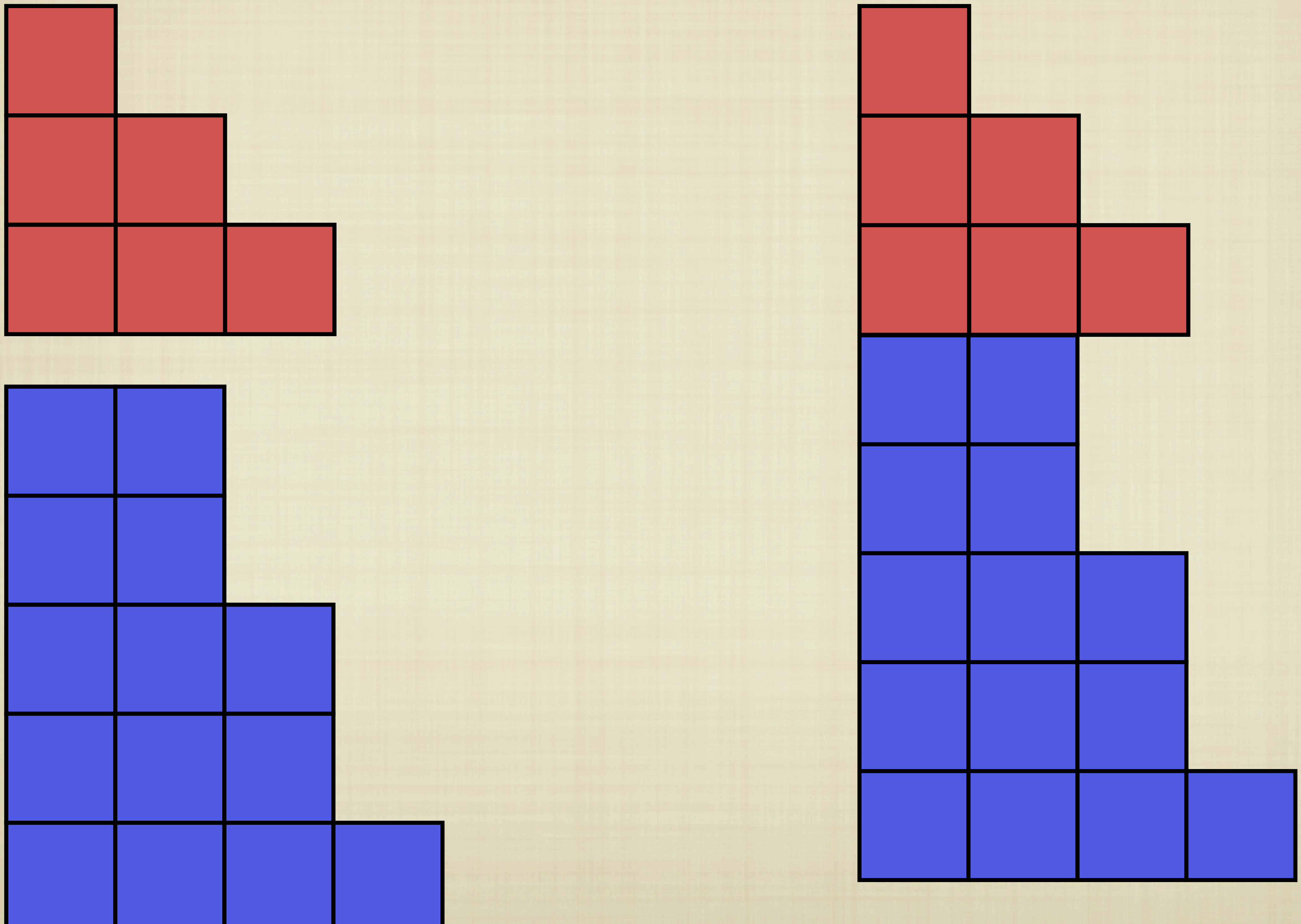

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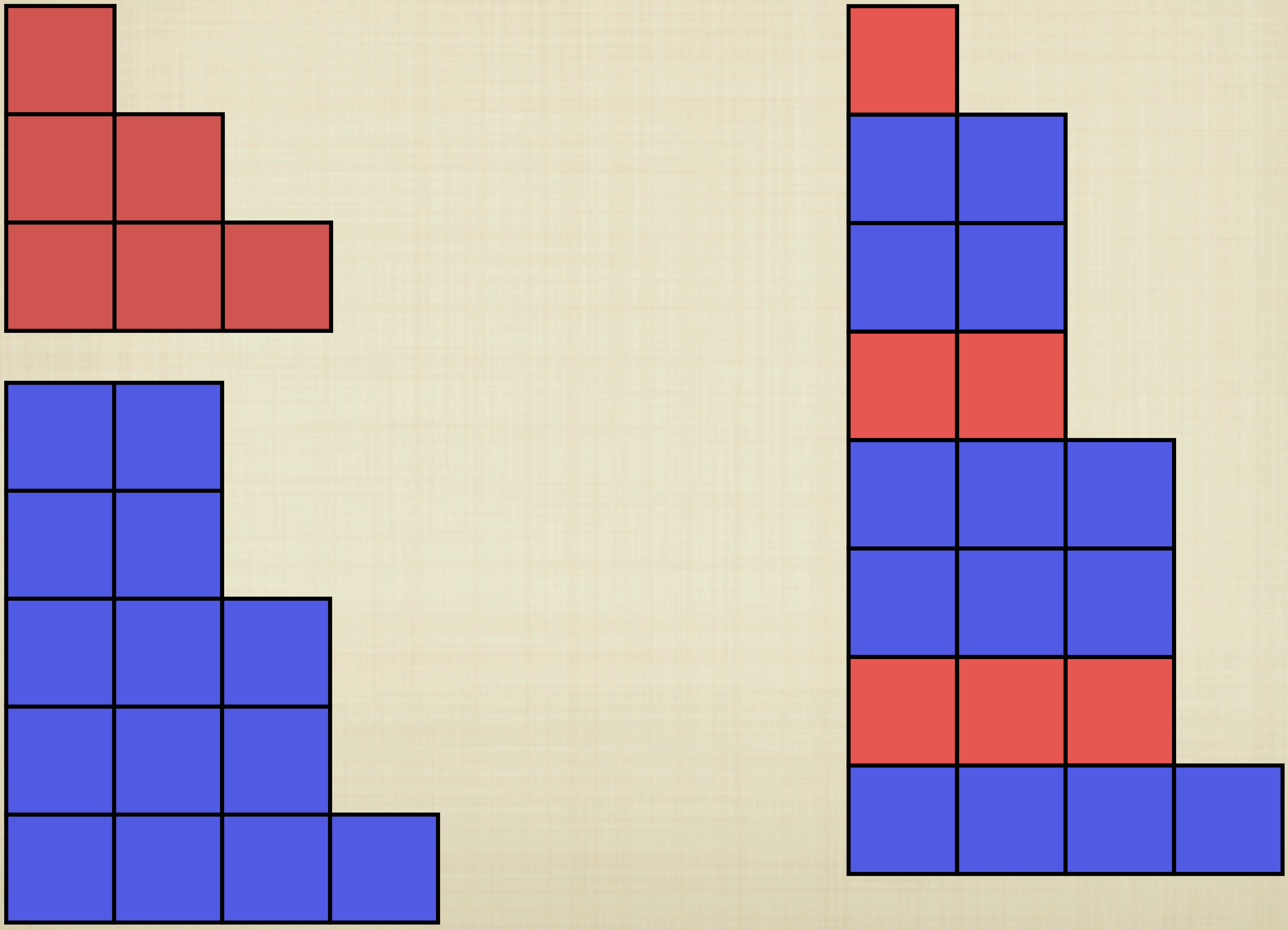

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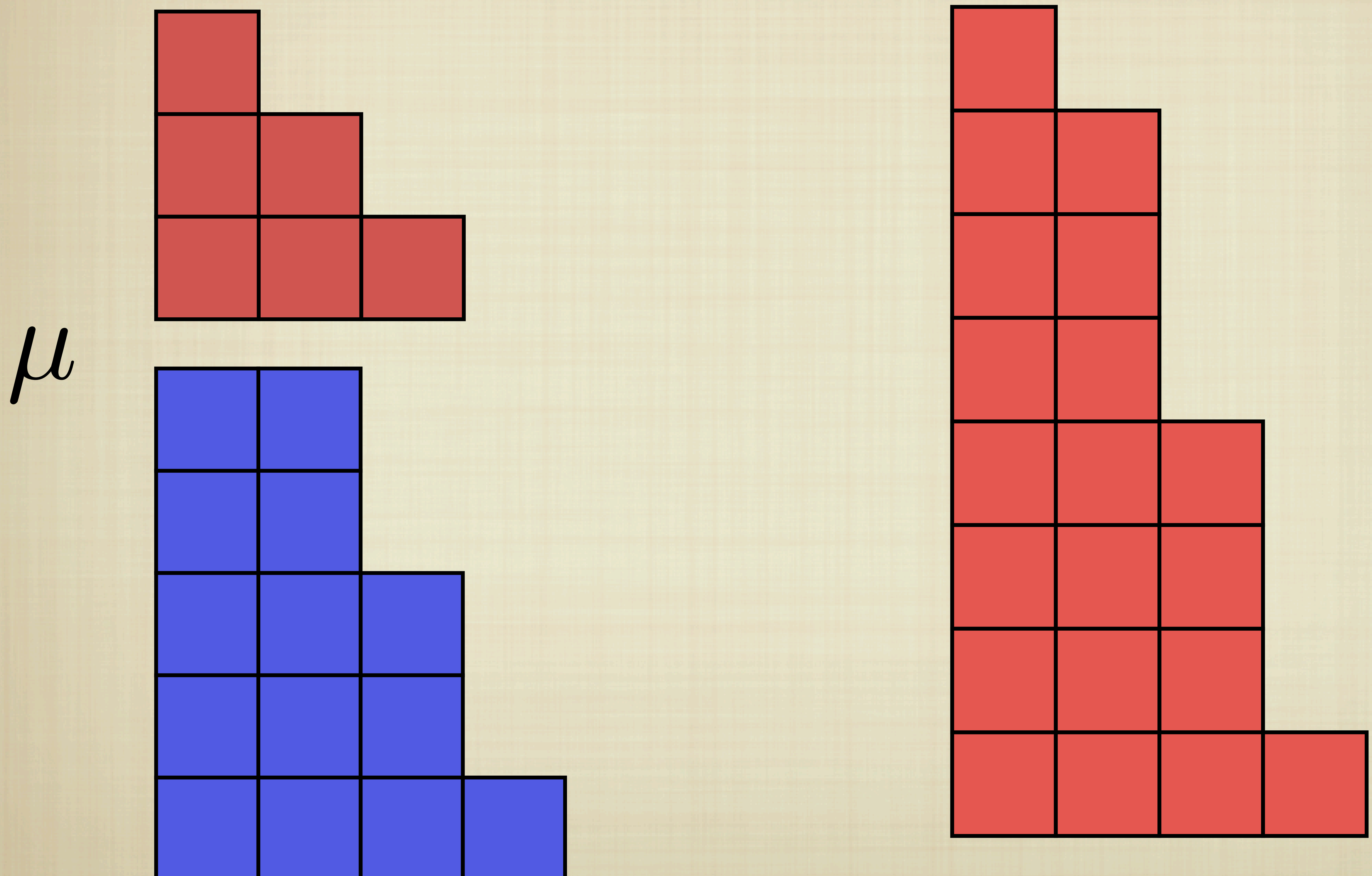
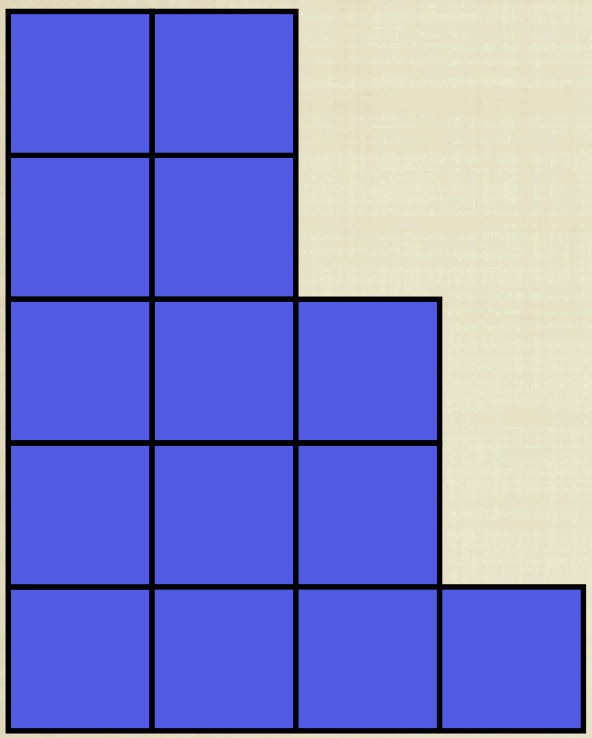
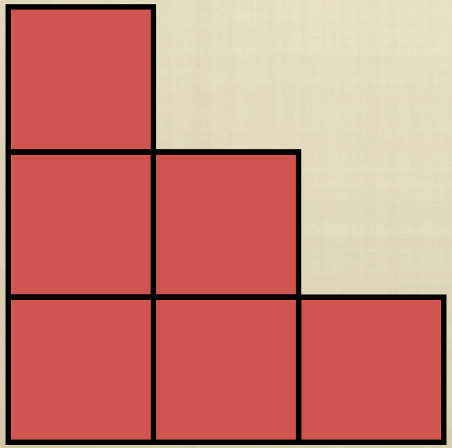


PROPERTIES OF THIS ALGEBRA

- Commutative and graded





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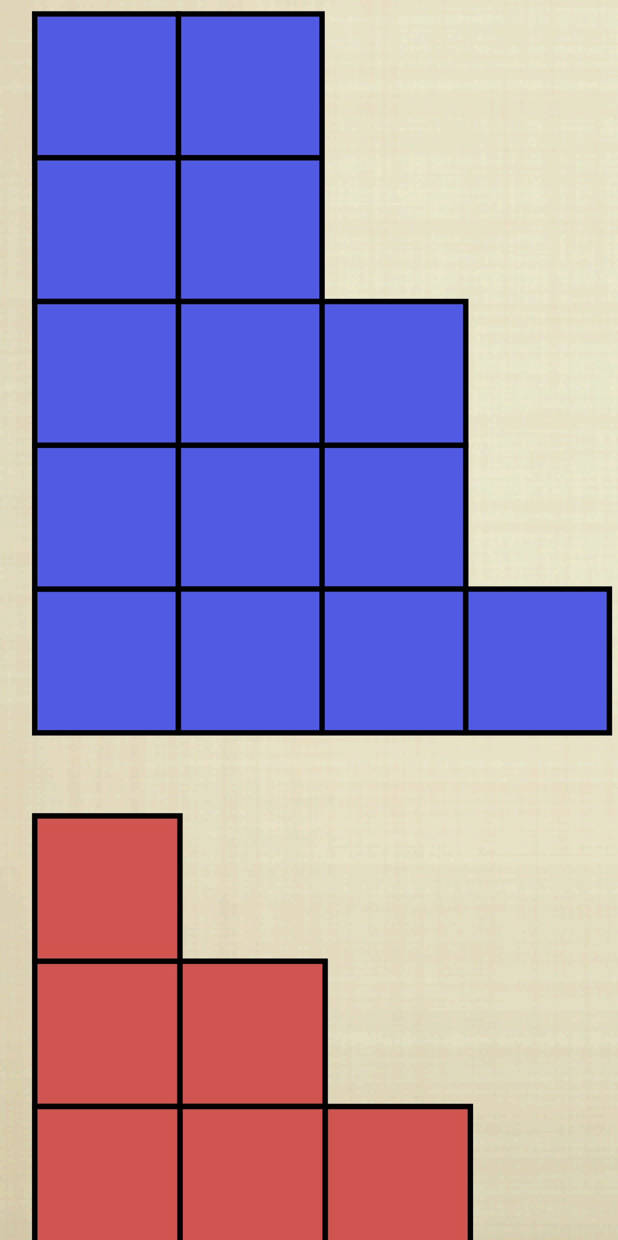

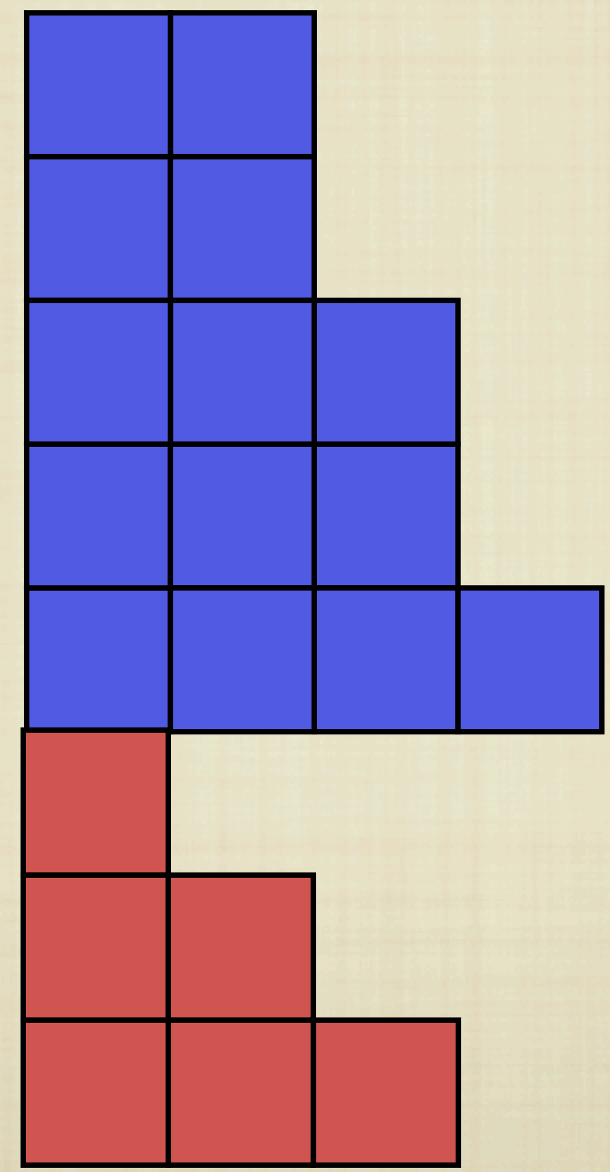

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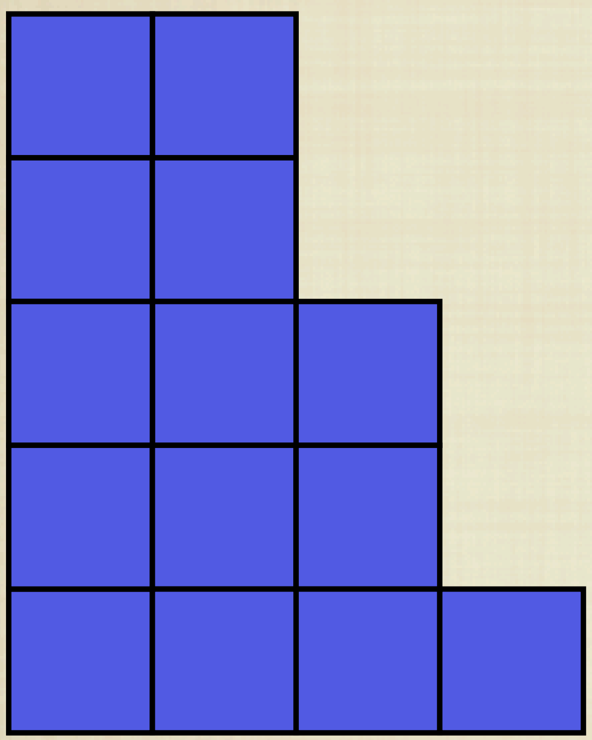
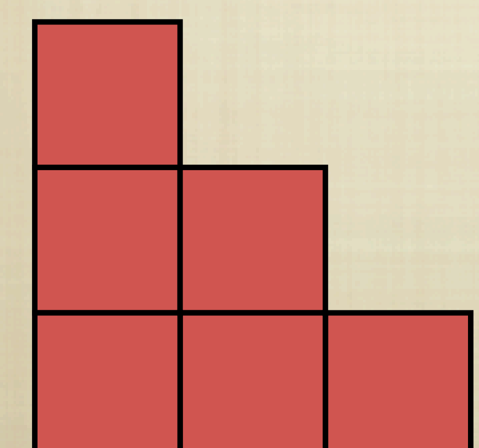
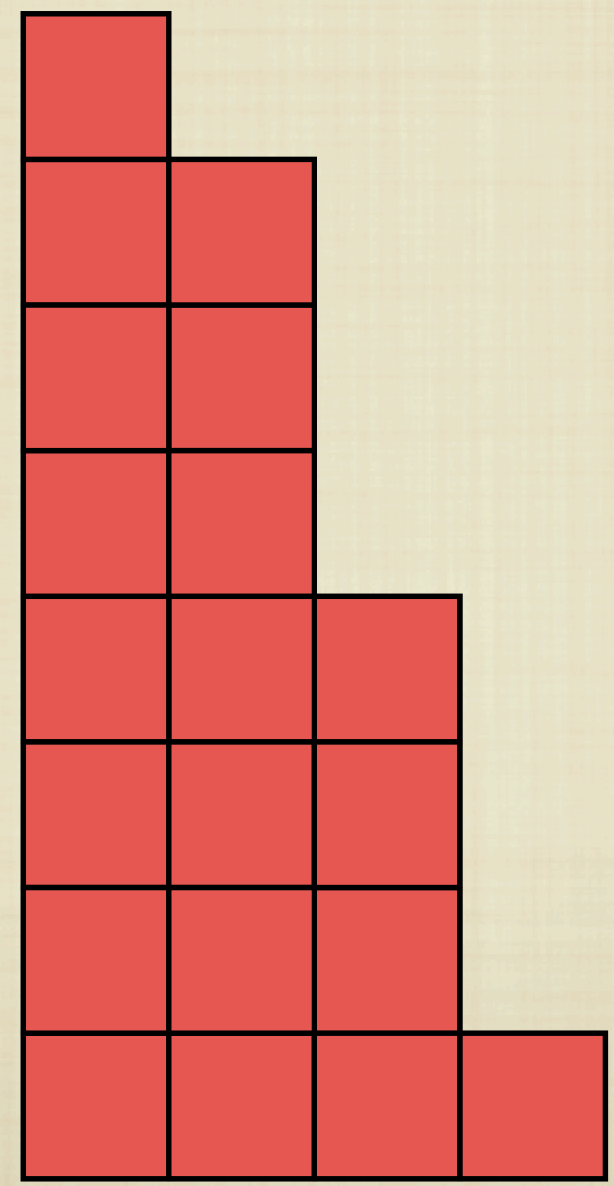


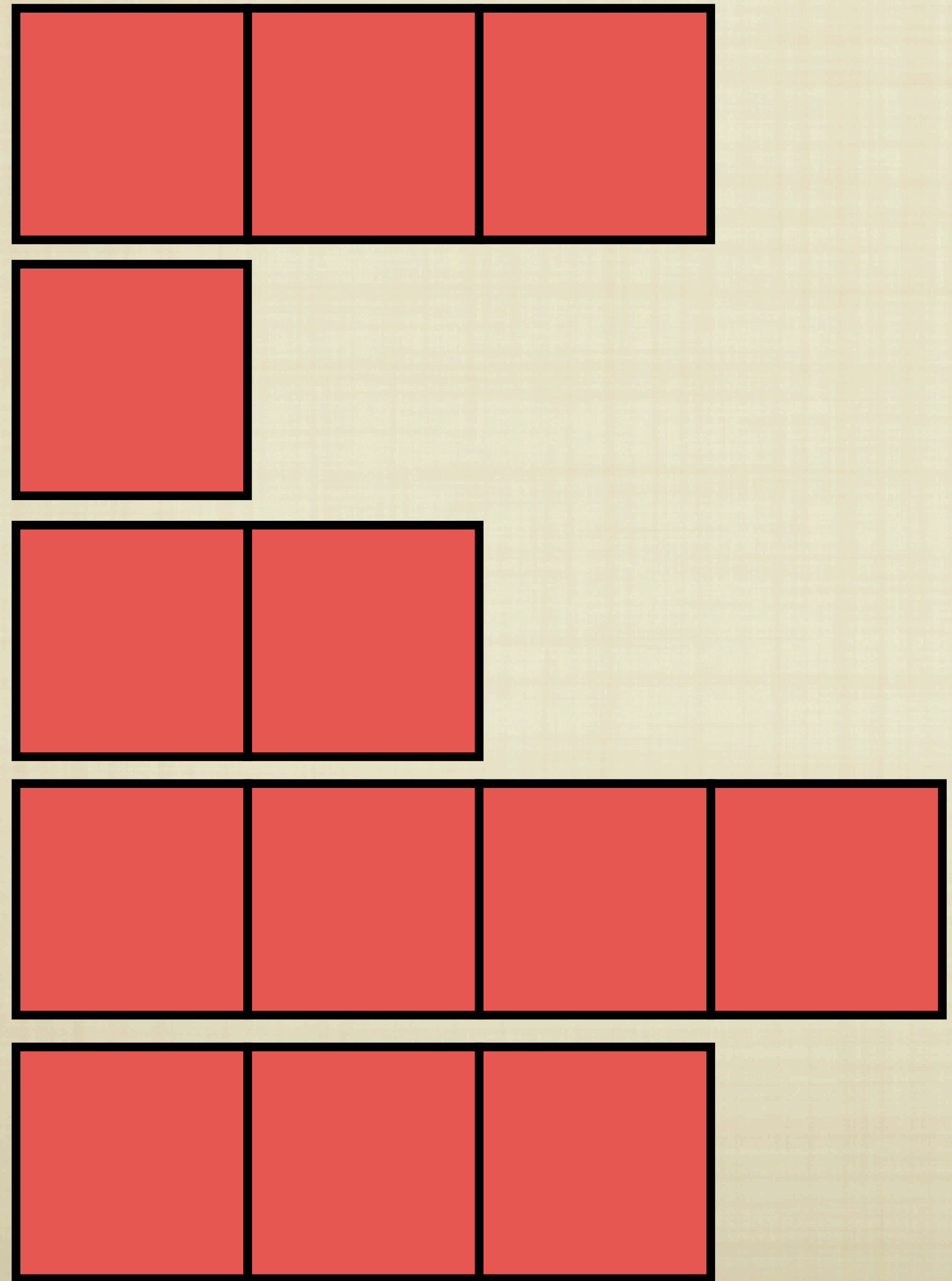
μ





Properties of this algebra

- Freely generated by building blocks of rows



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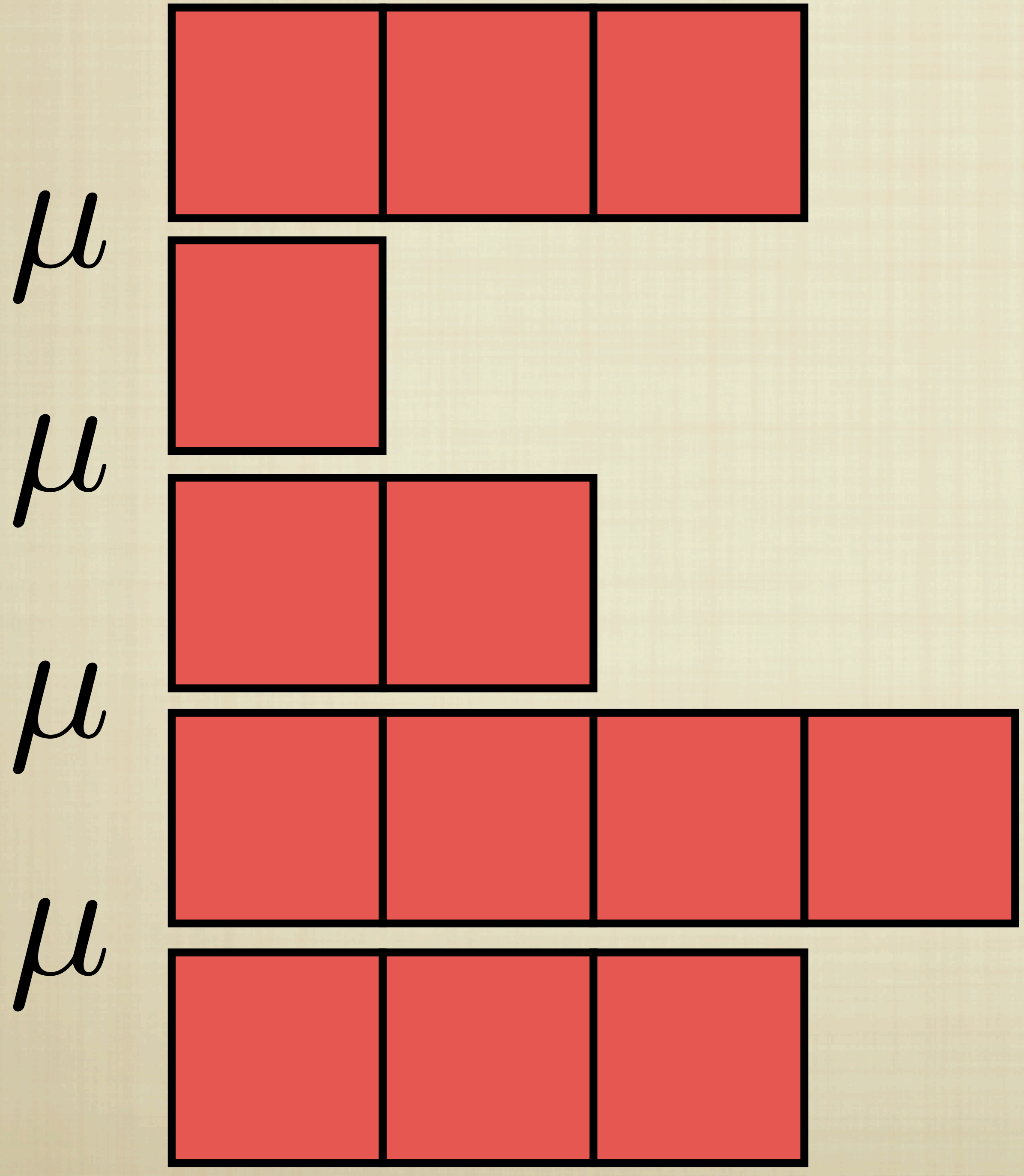

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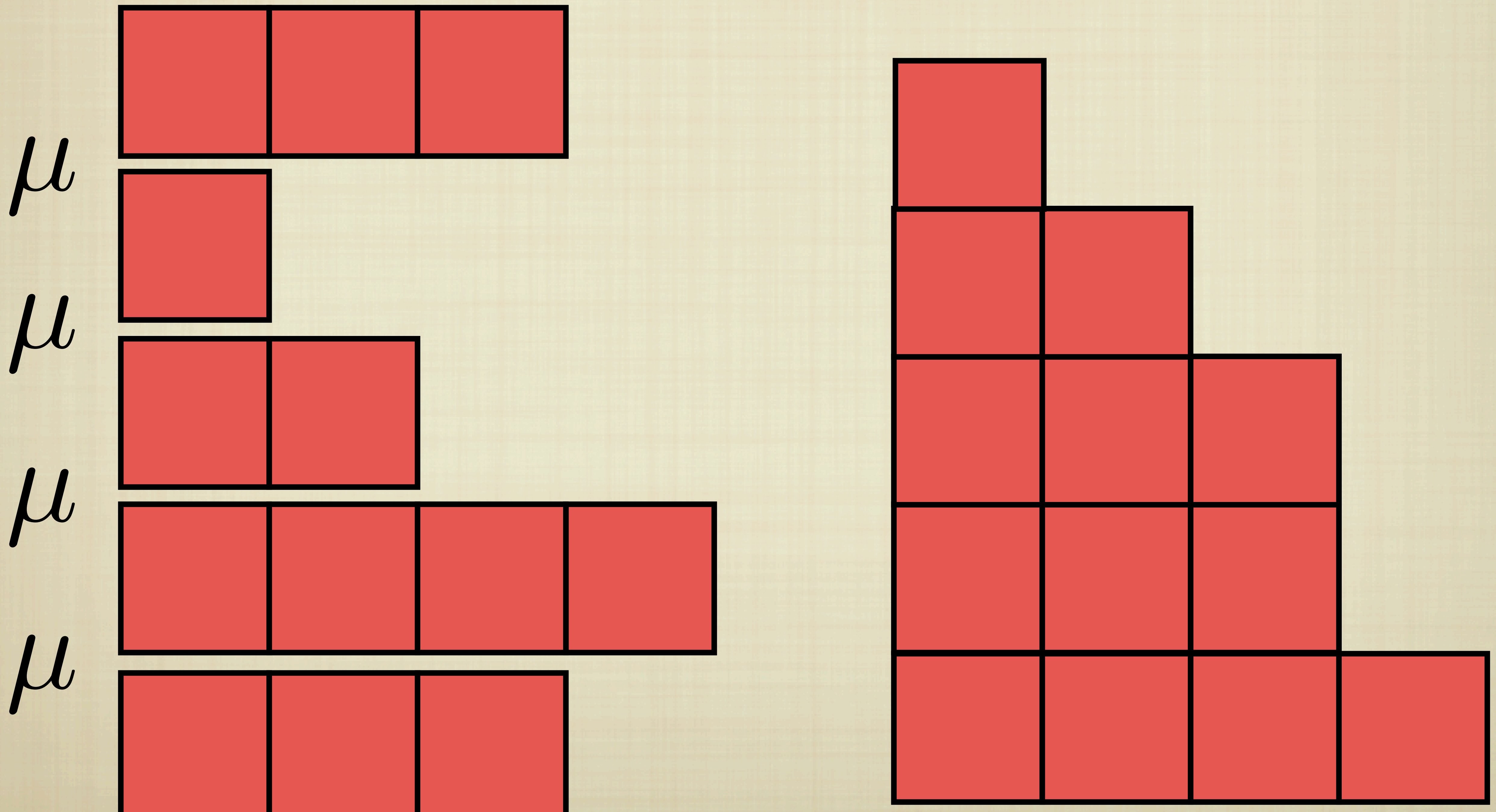

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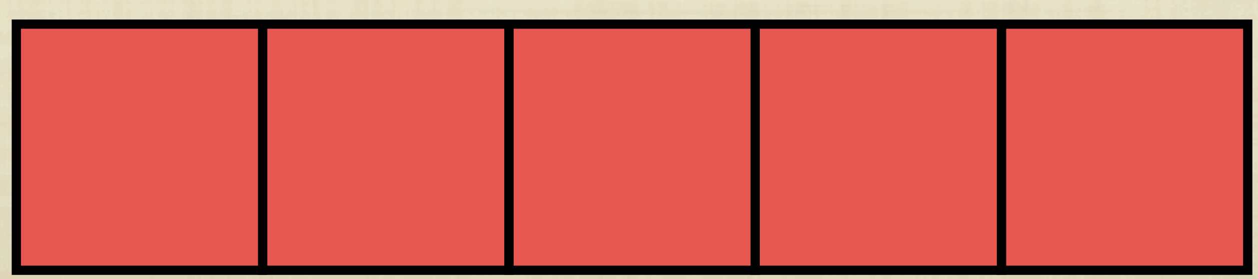
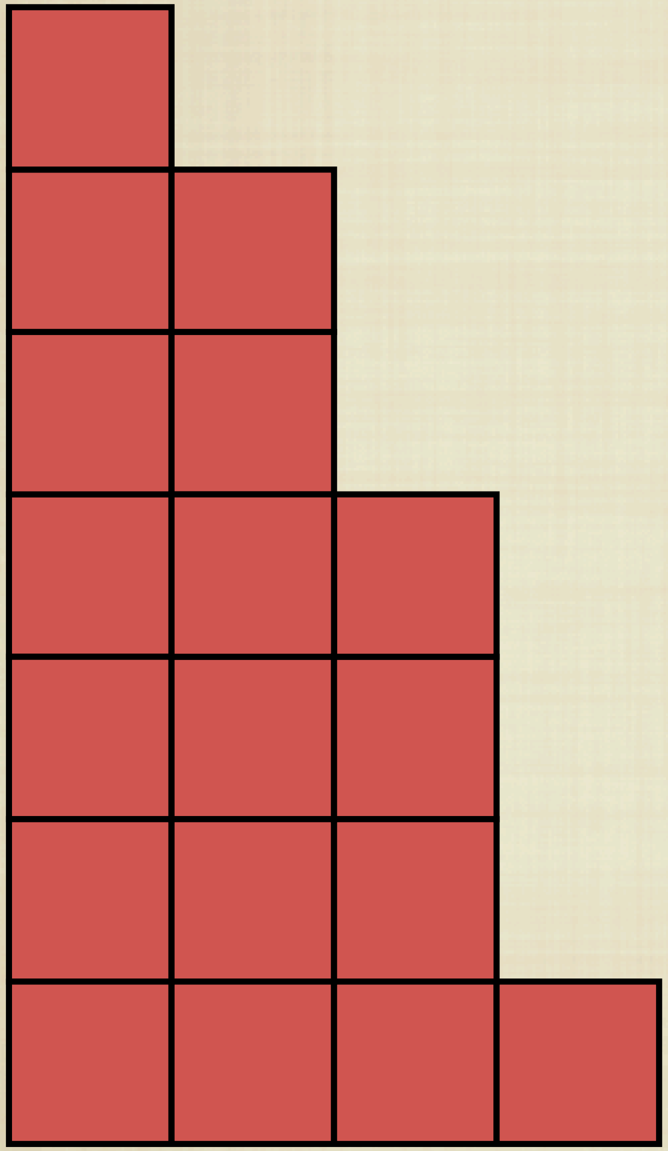
$$p_5 \leftrightarrow$$



From combinatorial object to algebra



partitions

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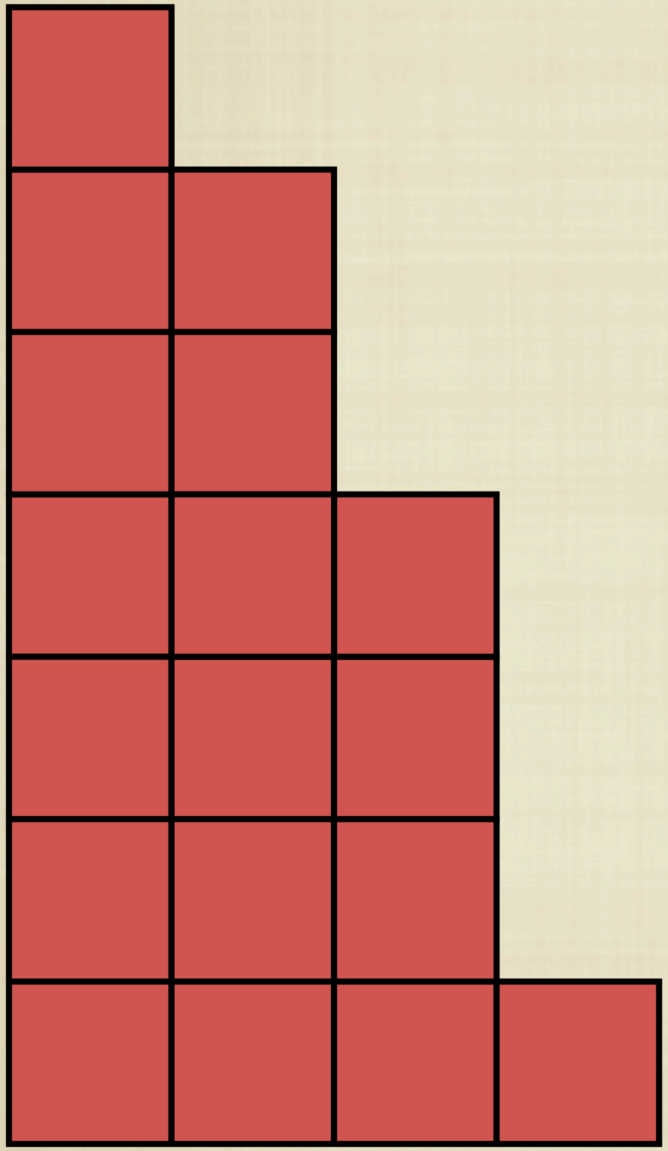

some commutative product

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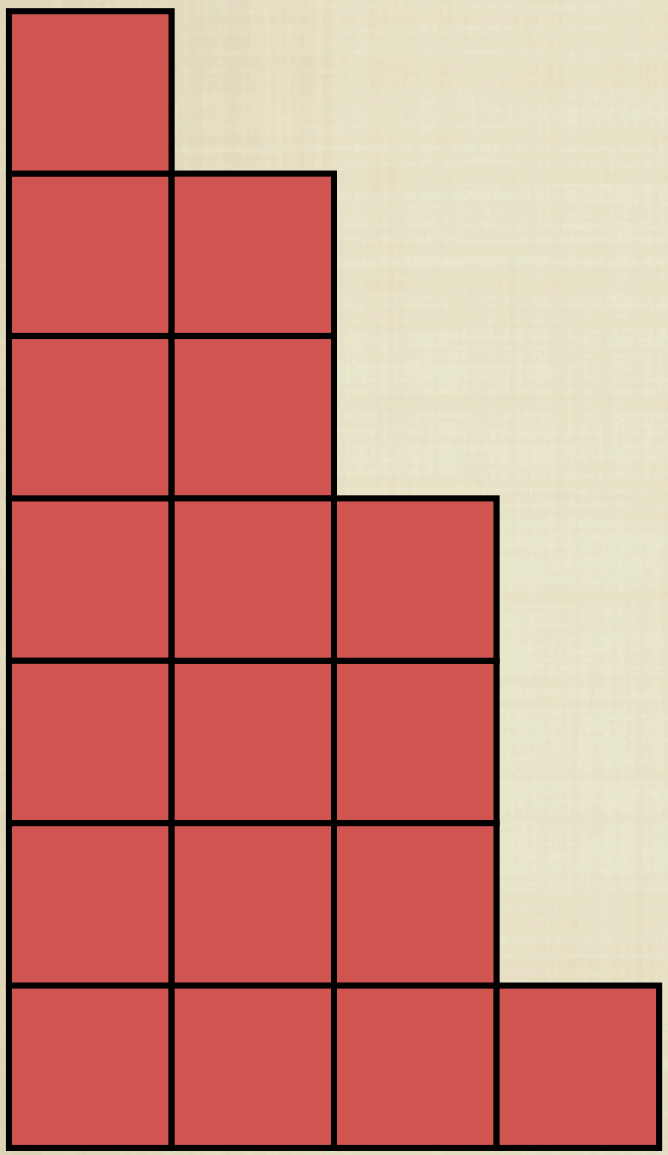


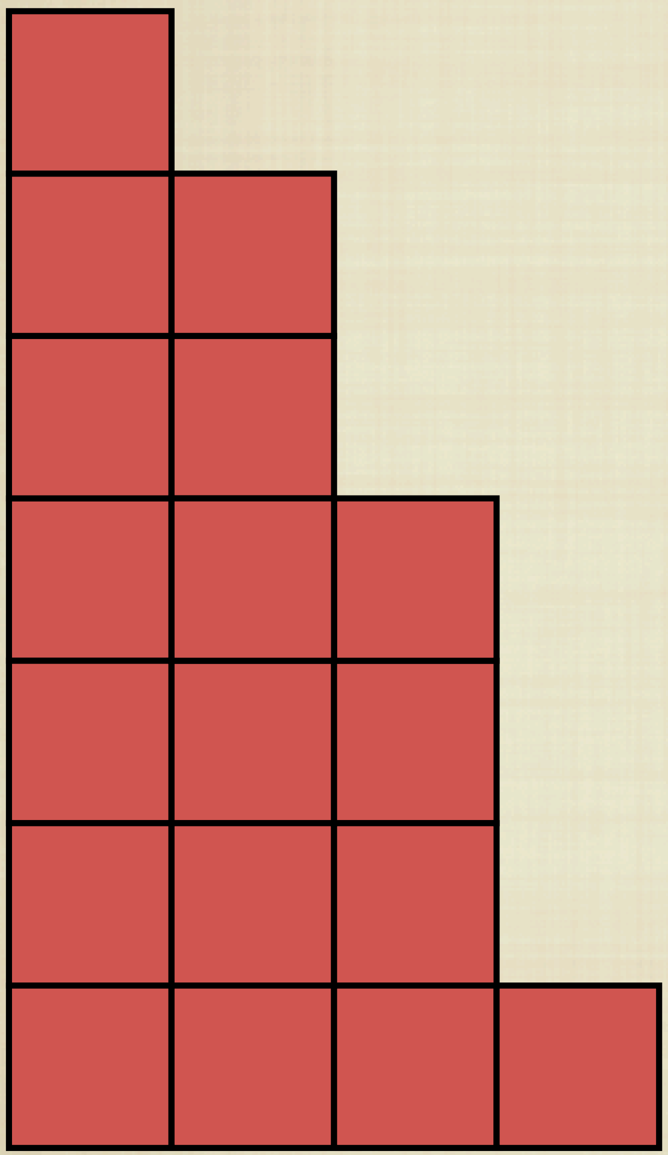
partitions

some commutative product

$$K[p_1, p_2, p_3, \ldots]$$

Algebra

From combinatorial object to algebra



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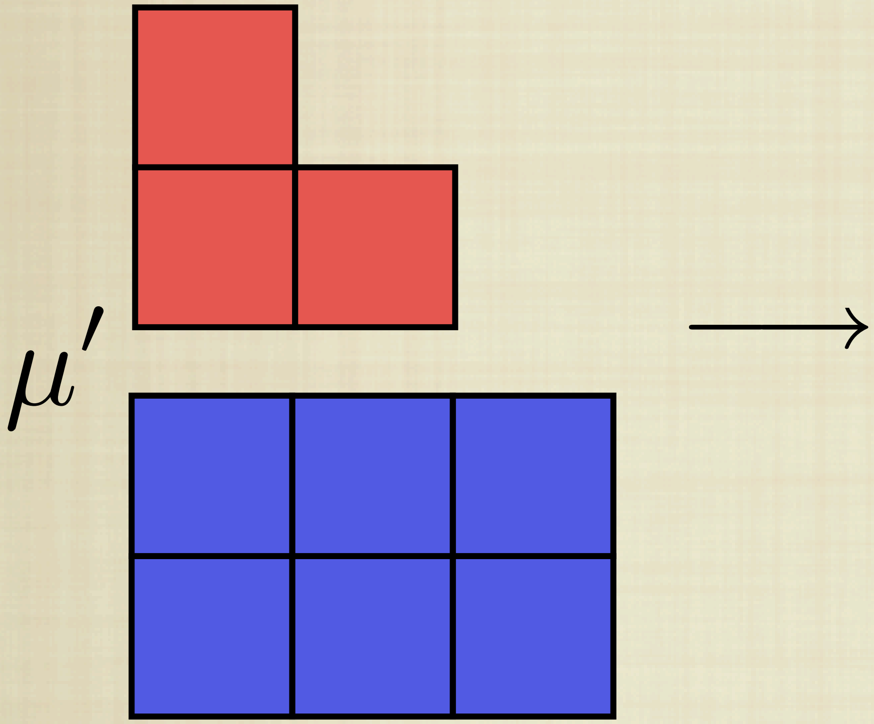
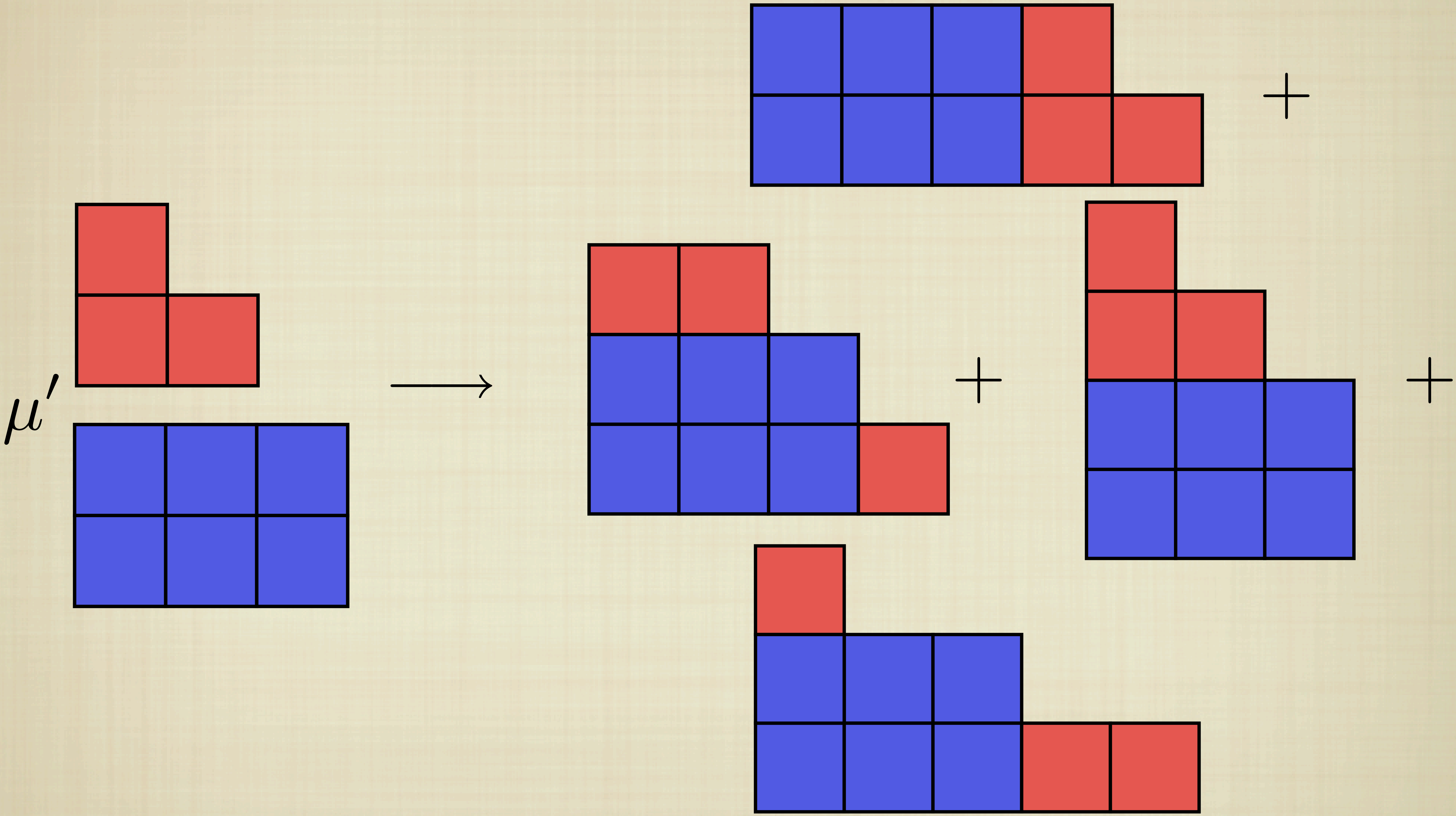
Algebra

A different commutative product



And yet the algebra which arises is isomorphic to $K[p_1, p_2, p_3, \ldots]$

A different commutative product



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- Has a Hopf algebra structure product + coproduct + antipode which all interact nicely with each other

What is a Hopf algebra?

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- Start with a Bialgebra

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Product

$$\mu : H \otimes H \to H$$

Coproduct

$$\Delta : H \to H \otimes H$$

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$$\Delta : H \to H \otimes H$$

with unit

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and an antipode map

$$S : H \to H$$

What is a Hopf algebra?

Start with a Bialgebra

$$\mu : H \bigotimes H \longrightarrow H$$

$$\Delta : H \longrightarrow H \bigotimes H$$

$$\eta : K \longrightarrow H$$

$$\varepsilon : H \longrightarrow K$$

$$S : H \longrightarrow H$$

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "S \otimes id"] & & H \otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\nabla"] \arrow[dr, "\nabla"'] \arrow[rr, "\varepsilon"] & & K \arrow[rr, "\eta"] & & H \\
 & H \otimes H \arrow[rr, "id \otimes S"'] & & H \otimes H \arrow[ur, "\mu"'] &
\end{tikzcd}$$

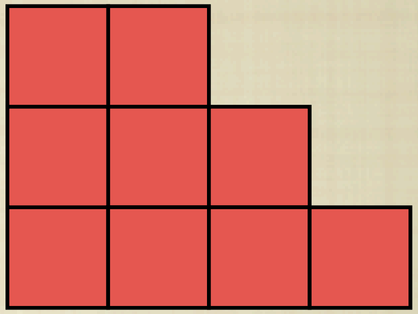
This diagram commutes

What is so good about a Hopf algebra?

- The graded kind associated with combinatorial objects have lots of structure
- There seems to be just “one” graded combinatorial hopf algebra for each type of combinatorial object
- Many of the combinatorial operations are reflected in the algebraic structure

CHA's in the mid-90s

The symmetric functions are commutative and generated by one element at each degree



Partitions

$$K[p_1, p_2, p_3, \ldots]$$

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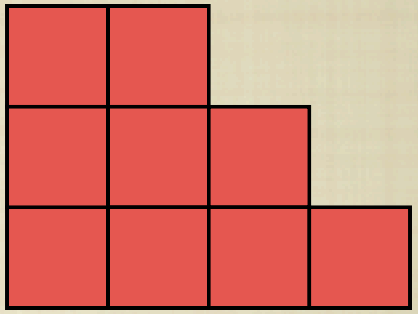

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Non-commutative symmetric functions will be non-commutative and generated by one element at each degree

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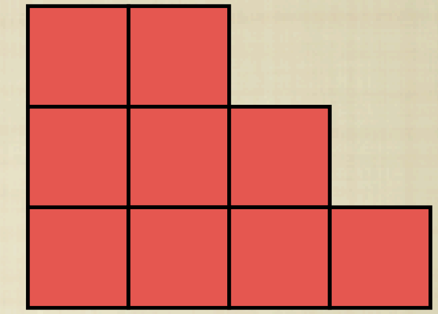

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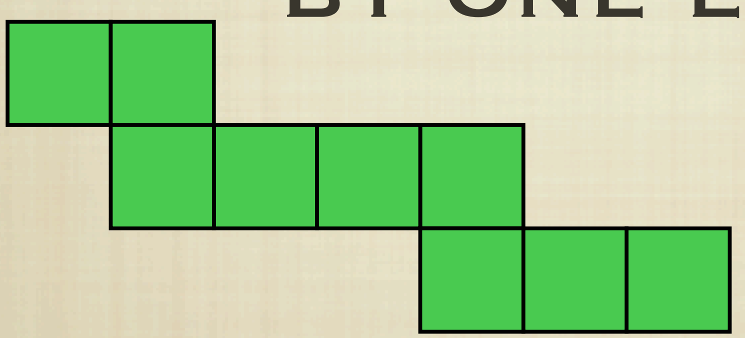

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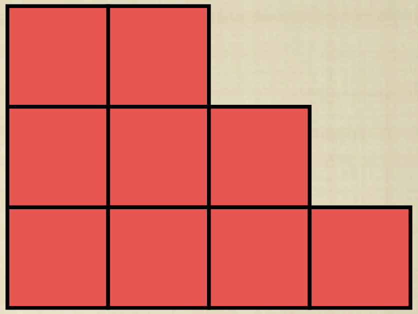

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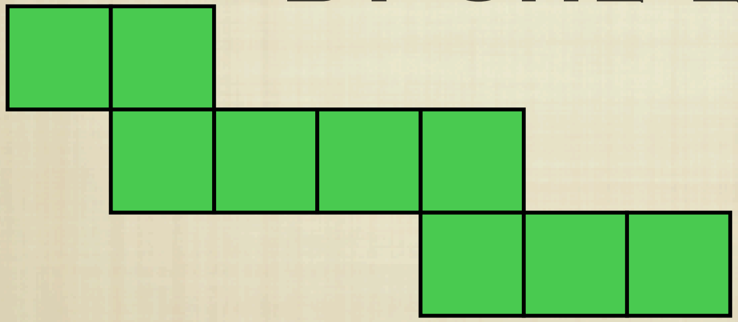

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Concatenation product

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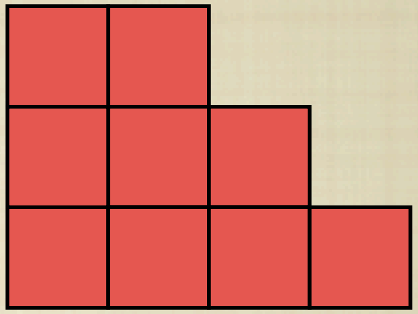

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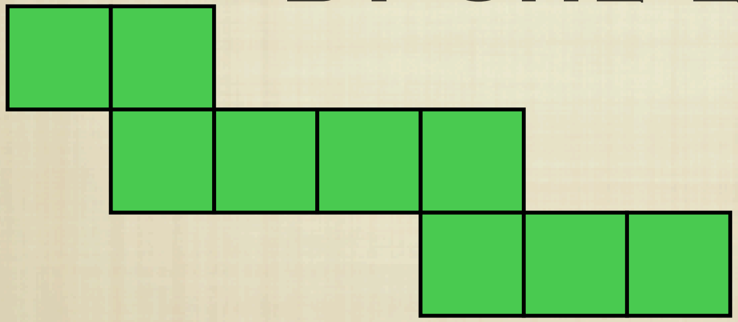

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$$K\langle p_1, p_2, p_3, \ldots\rangle$$

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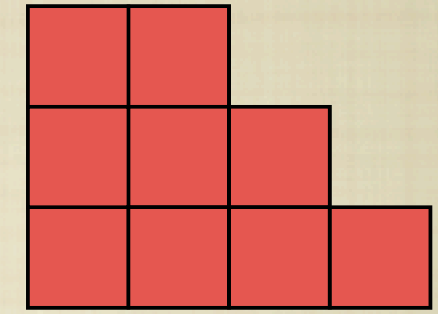

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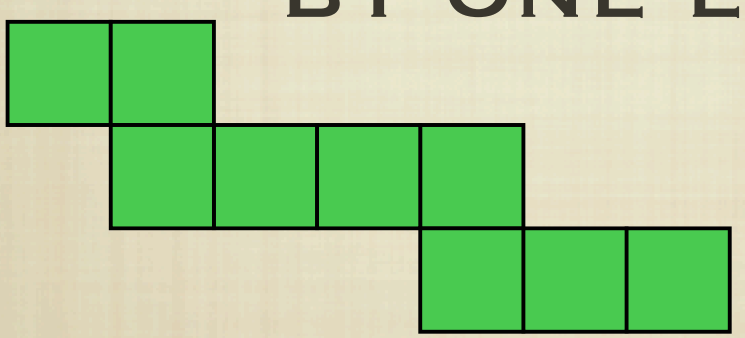

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I. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, and J.-Y. Thibon

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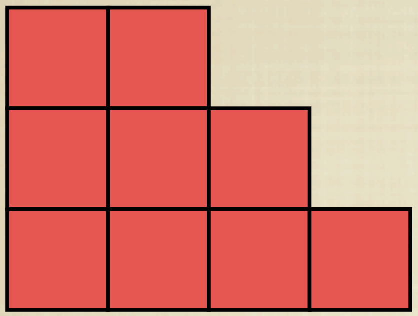

PARTITIONS

$K[p_1, p_2, p_3, \ldots]$

Sym

$NSym$

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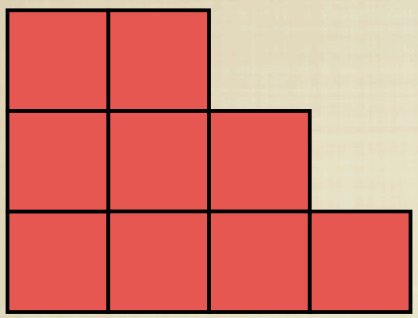

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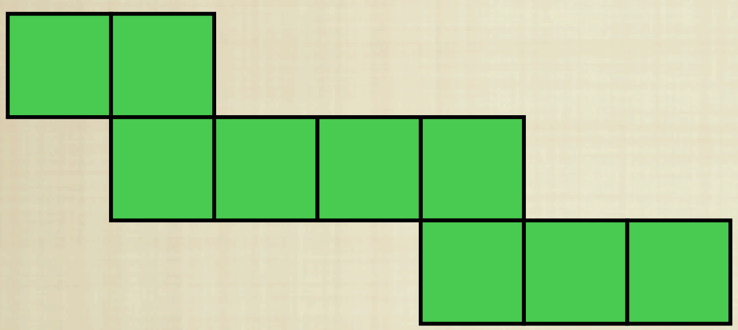

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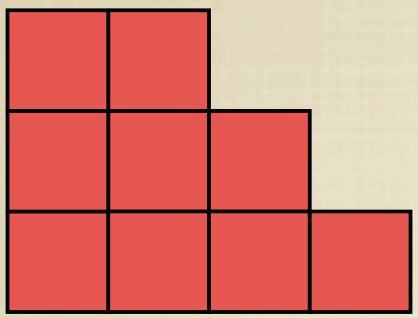

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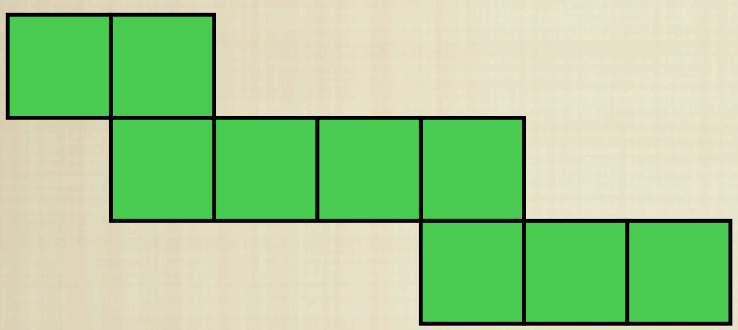

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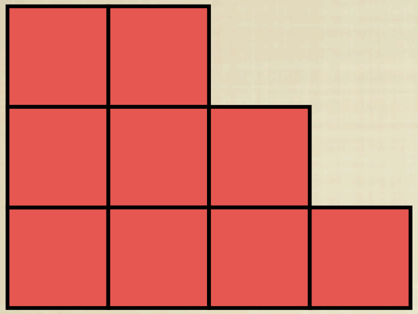

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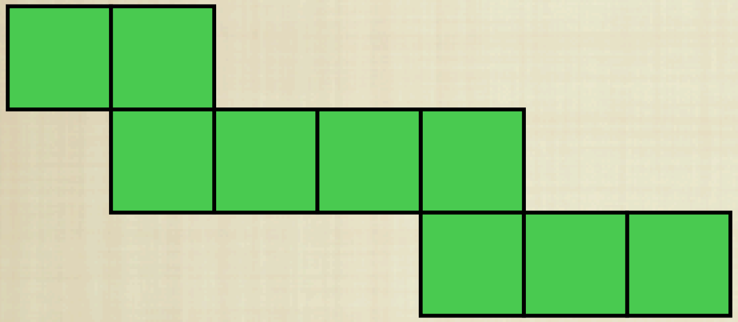

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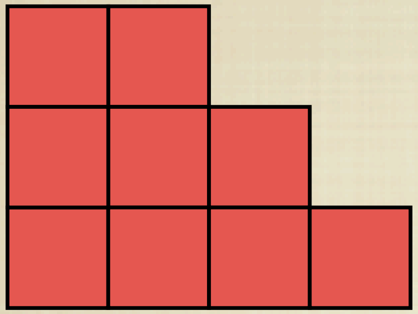
GKLLRT ('95)

CHA's in the mid-90s



PARTITIONS

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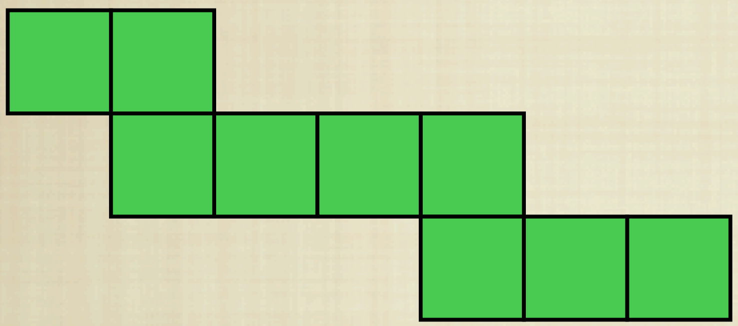

COMPOSITIONS

$$K\left\langle p_1, p_2, p_3, \ldots \right\rangle$$

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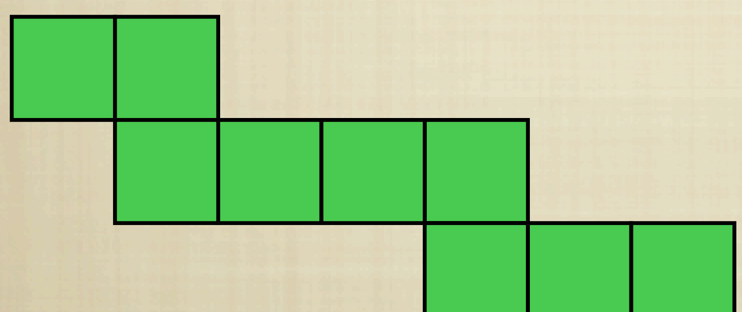
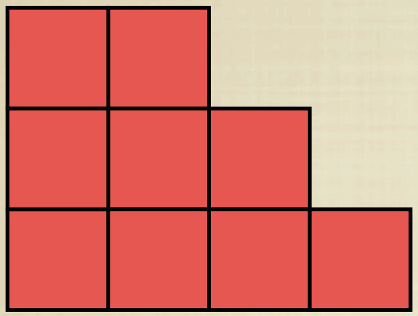


COMPOSITIONS

$QSym$

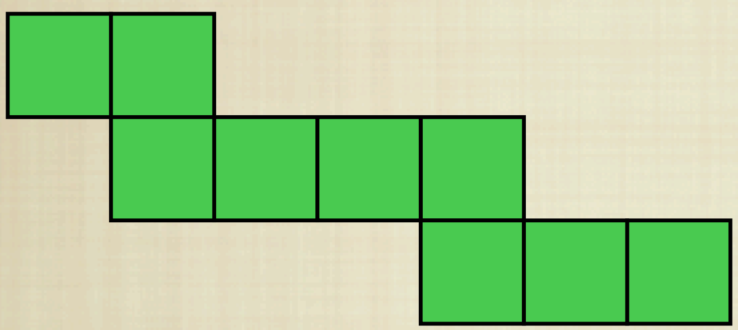
CHA's in the mid-90s



PARTITIONS

$$K[p_1, p_2, p_3, \dots]$$

Sym



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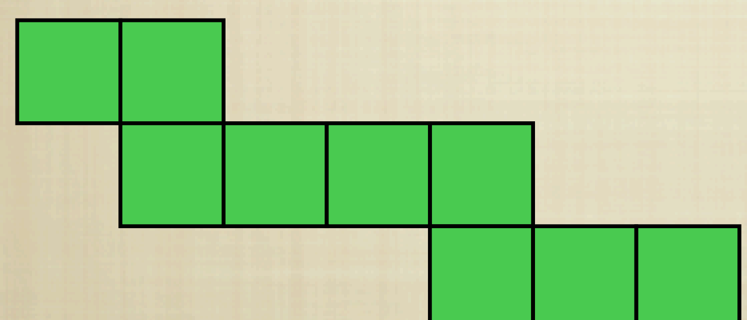
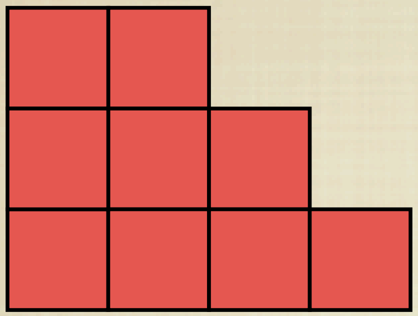


COMPOSITIONS

Commutative algebra
of Quasi-symmetric functions

$QSym$

CHA's in the mid-90s



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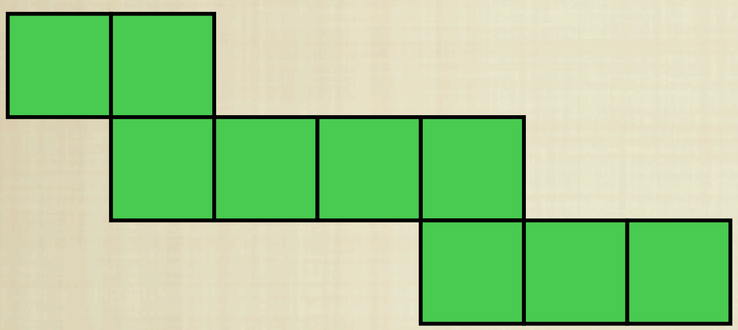

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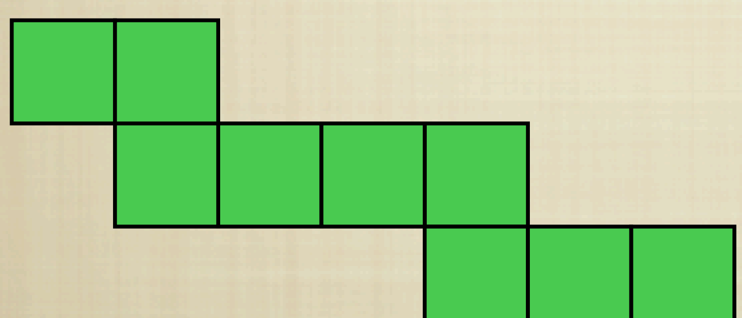


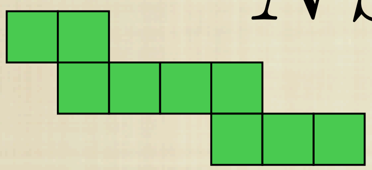
COMPOSITIONS

Commutative algebra of Quasi-symmetric functions

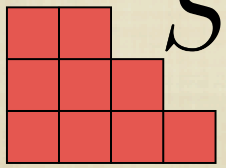
$QSym$

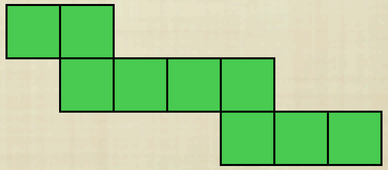
Gessel ('84)

CHA's in the mid-90s



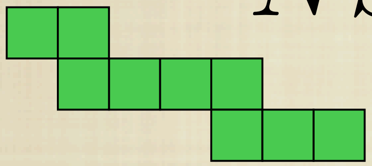
$$\begin{tikzcd}
NSym \arrow[d] & \\
Sym \arrow[r, hook] & QSym
\end{tikzcd}$$

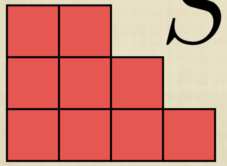


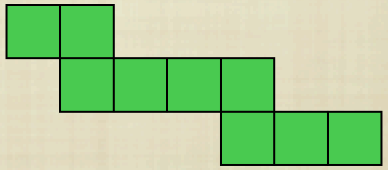


CHA's in the mid-90s

$$\begin{tikzcd} NSym \arrow[r, hook] \arrow[d] & \mathfrak{S}Sym \arrow[d] \\ Sym \arrow[r, hook] & QSym \end{tikzcd}$$







CHA's in the mid-90s

Malvenuto-Reutenauer ('95)

$$
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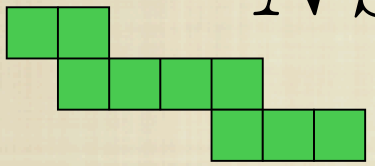
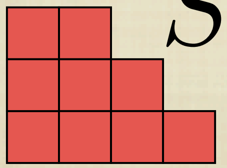
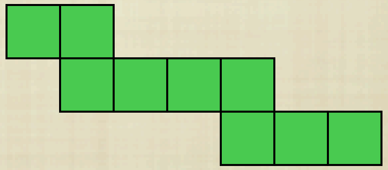
$$



315426798

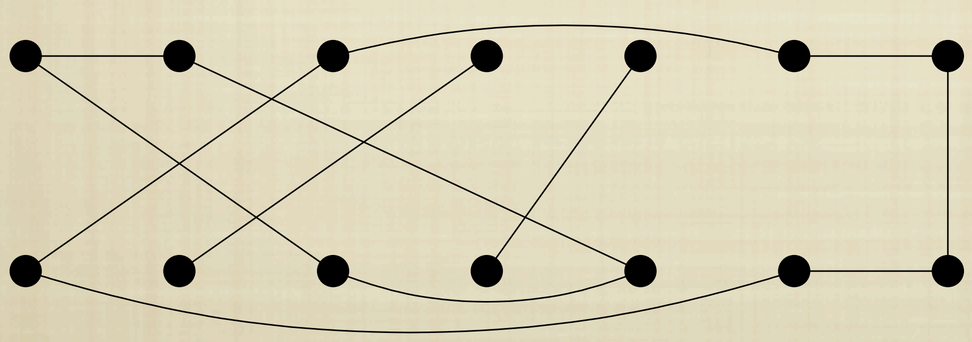
PERMUTATIONS





CHA's in the 90s+

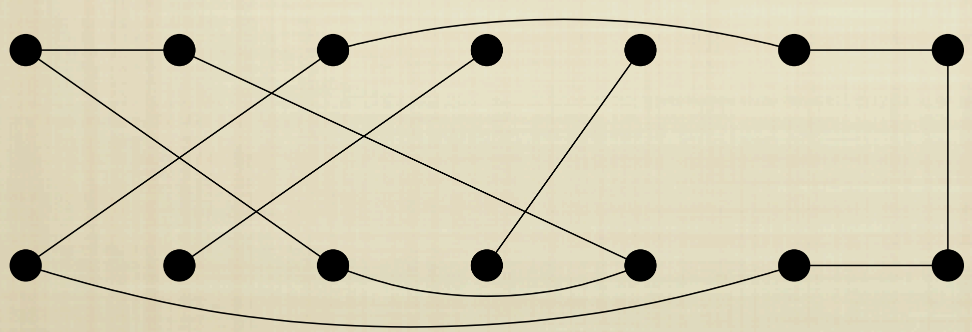
uniform block permutations



Aguiar-Orellana '05

CHA's in the 90s+

uniform block permutations



Aguiar-Orellana '05

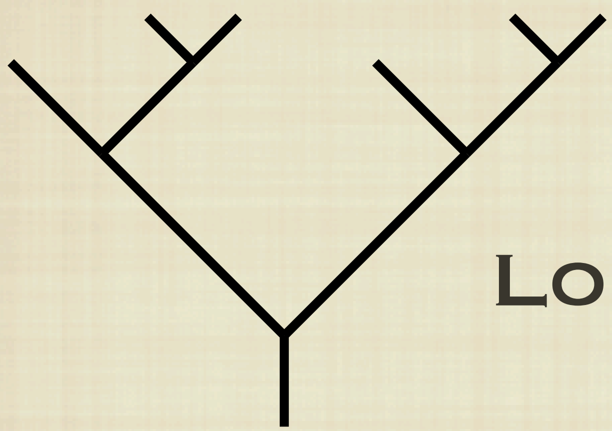
Tableaux

7				
6	8	11		
4	5	9	12	14
1	2	3	10	13

Poirier-Reutenauer '95

CHA's in the 90s+

binary trees



Connes-Kreimer '98

Grossman-Larson '89

Loday-Ronco '98

Tableaux

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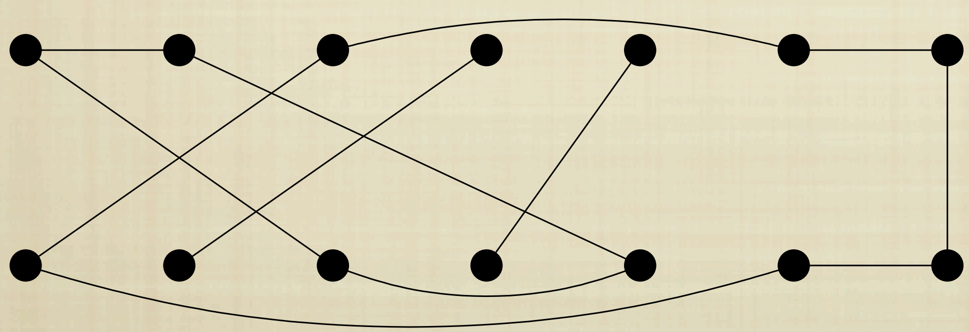
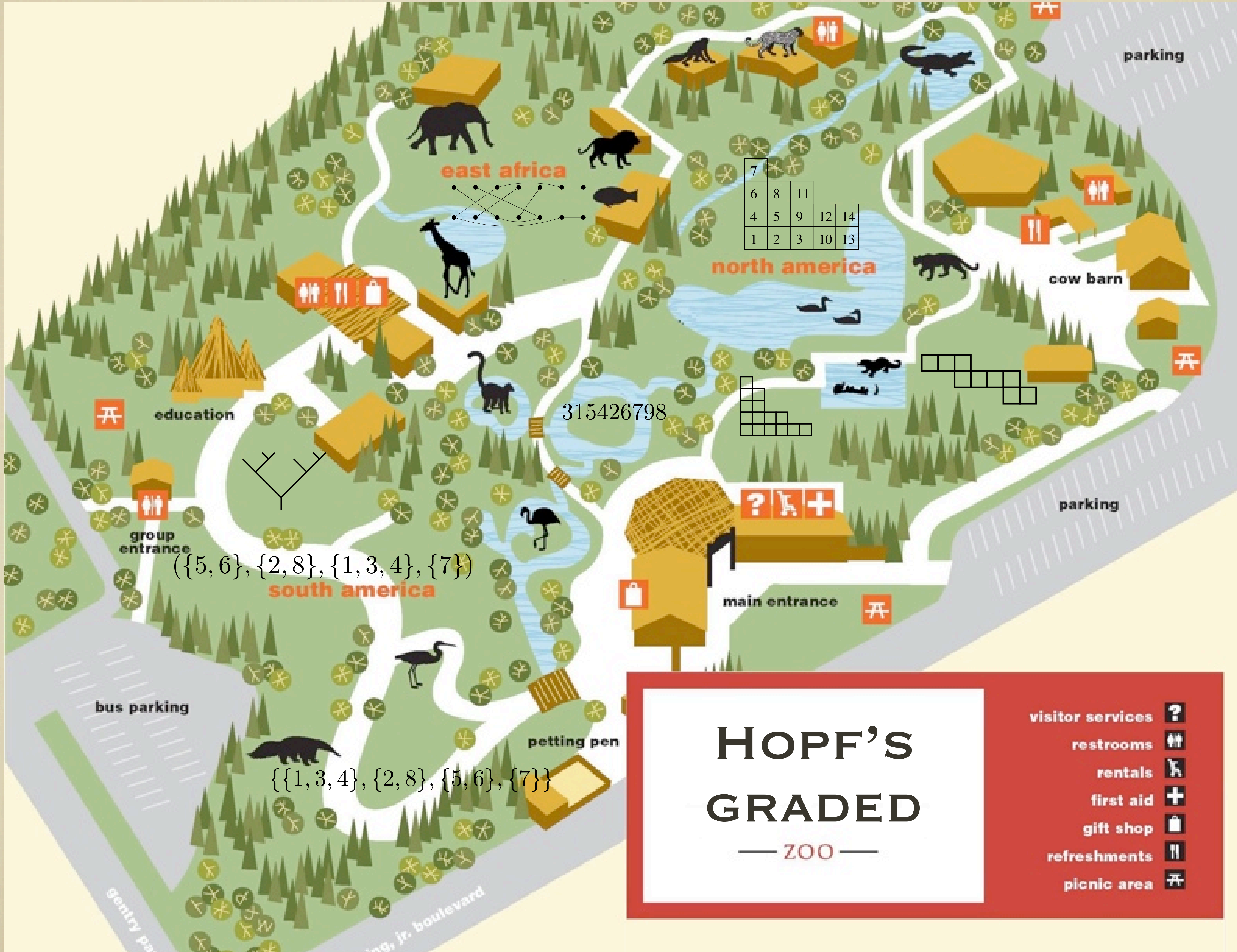


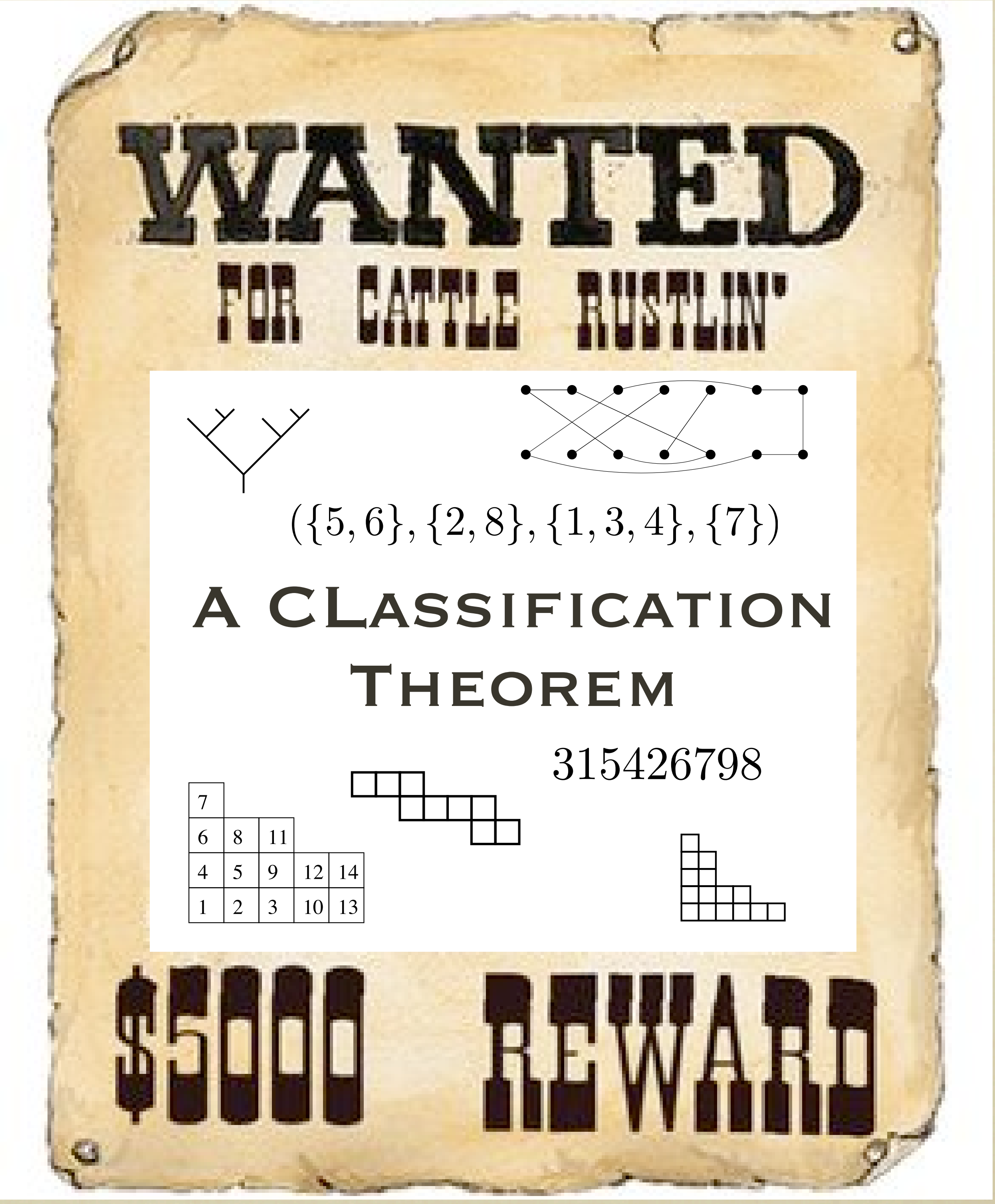
Aguiar-Orellana '05

A zoo of Hopf Algberas



A goal of research on graded Hopf algebras?

- One Goal of determining the Hopf algebras associated to combinatorial objects is to try and arrive at a classification theorem for Graded combinatorial Hopf algebras



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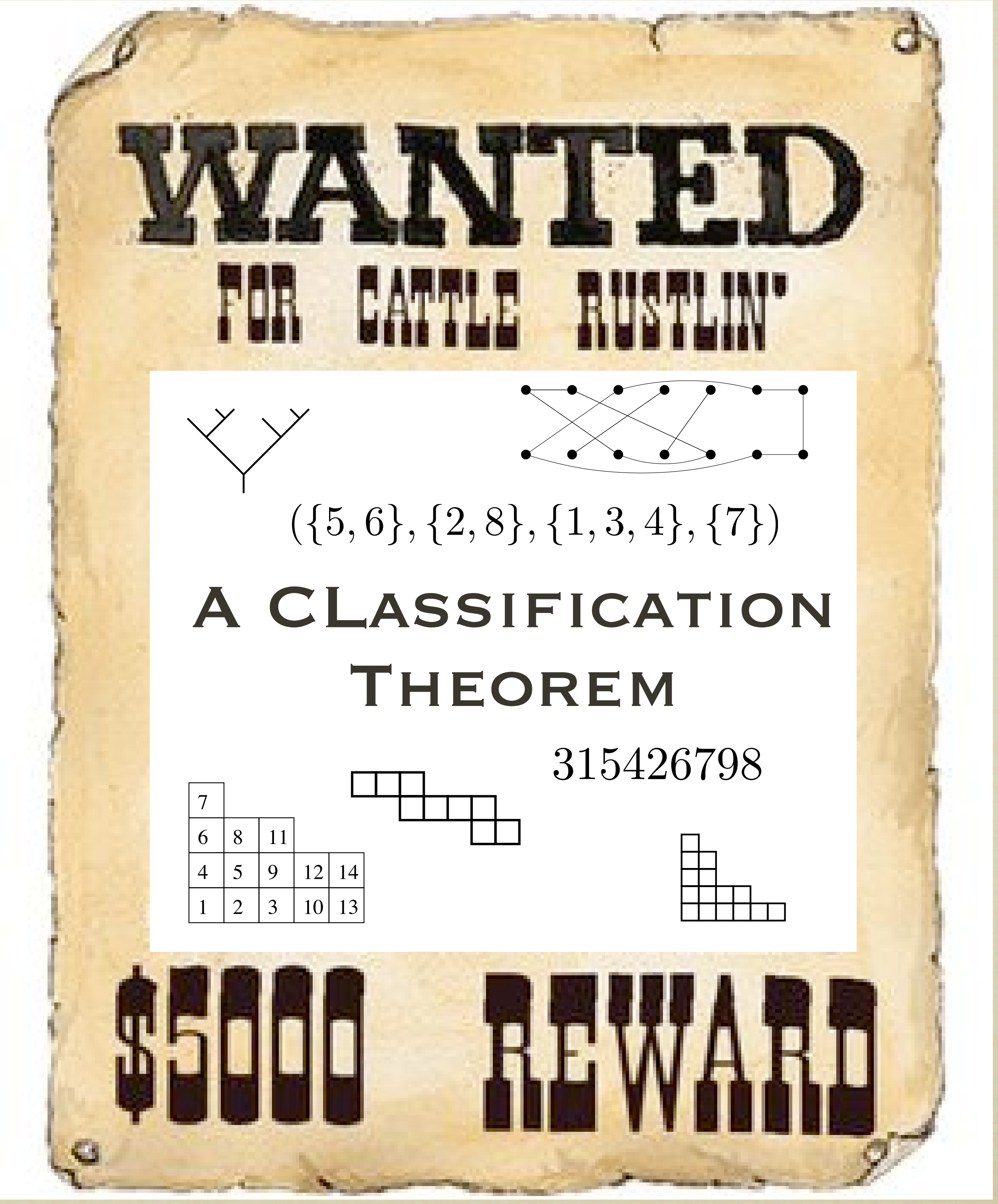

Marcelo Aguiar
Species $\longleftrightarrow$ CHAs

Hopf algebras of set partitions/compositions

$$(\{5,6\},\{2,8\},\{1,3,4\},\{7\})$$

$$\{\{1,3,4\},\{2,8\},\{5,6\},\{7\}\}$$

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Hopf algebras of set partitions/compositions

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Set composition definition:

$$(S_1, S_2, \ldots, S_k) : S_1 \uplus S_2 \uplus \cdots \uplus S_k = \{1, 2, \cdots, n\}$$

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Another type of non-commutative symmetric functions

Sym Invariants under the left action on the polynomial ring $\mathbb{Q}[X_n]$

$$\sigma(x_i) = x_{\sigma(i)}$$

$NSym$ Free algebra generated by one element at each degree

$NCSym$ Invariants under the left action on the non-comm poly ring $\mathbb{Q}\langle X_n \rangle$

Wolf 1936, Rosas-Sagan '03

Monomial $\longrightarrow$ set partition

For each monomial

$$x_{i_1} x_{i_2} \cdots x_{i_n}$$

Associate a set partition

$$\nabla(i_1, i_2, \ldots, i_n) = A \vdash [n]$$
$$r, s \in A_d \text{ whenever } i_r = i_s$$

$$m_A[X_n] = \sum_{\nabla(i_1, i_2, \ldots, i_n) = A} x_{i_1} x_{i_2} \cdots x_{i_n}$$

Example:

$$m_{\{\{1,3,5\},\{2,4\}\}}[X_n] = x_1x_2x_1x_2x_1 + x_2x_1x_2x_1x_2 + x_1x_3x_1x_3x_1 +$$

$$x_3x_1x_3x_1x_3 + x_2x_3x_2x_3x_2 + x_3x_2x_3x_2x_3 + x_1x_4x_1x_4x_1 + \cdots$$

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This is a non commutative polynomial
for a given n

Consider the elements m_A
to be the object by letting the
number of variables $n \to \infty$
in $m_A[X_n]$

Combinatorial rule for product

$$m_{\{\{1,3\},\{2,4\}\}} \cdot m_{\{\{1,2,4\},\{3\}\}} =$$

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$$m_{\{\{1,3\},\{2,4,5,6,8\},\{7\}\}} + m_{\{\{1,3,7\},\{2,4\},\{5,6,8\}\}} +$$

$$m_{\{\{1,3\},\{2,4,7\},\{5,6,8\}\}} + m_{\{\{1,3,7\},\{2,4,5,6,8\}\}} +$$

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$$m_{\{\{1,3,5,6,8\},\{2,4,7\}\}}$$

Coproduct

Inspiration:

$$m_A[X_n, Y_m]$$

$$\Delta(m_{\{\{1\},\{2,4,5,6\},\{3,7\}\}}) =$$

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Definition of $NCSym$

$$NCSym = \bigoplus_{n \geq 0} \mathcal{L}\{m_A : A \vdash [n]\}$$

Non-commutative
Co-commutative
Hopf algebra of set partitions

Analogy between Sym and NCSym

$$\begin{aligned} S(V^*) &= \text{symmetric tensor algebra} \\ &\simeq \mathbb{Q}[X_n] \end{aligned}$$

$$\begin{aligned} T(V^*) &= \text{tensor algebra} \\ &\simeq \mathbb{Q}\left\langle X_n \right\rangle \end{aligned}$$

Sym is to $S(V^*)$ as $NCSym$ is to $T(V^*)$

Properties of NCSym

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- Non-commutative and co-commutative

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- Has bases analogous to power, elementary, homogeneous, monomial symmetric functions in the algebra of Sym (Rosas-Sagan '03, Bergeron-Brlek)

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- Has bases analogous to power, elementary, homogeneous, monomial symmetric functions in the algebra of Sym (Rosas-Sagan '03, Bergeron-Brlek)
- The dimension of the part of degree n are the bell numbers

  1, 1, 2, 5, 15, 52, 203, 877, 4140, ...

Some questions to ask

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- what is a fundamental basis of this space (analogue of Schur basis)?

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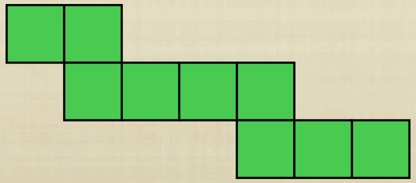
- What is the connection with representation theory?
- What is the structure of this algebra?
- What is the relationship with the 'other' non-commutative symmetric functions? (NSym)



compositions

$$\{\{1,3,4\},\{2,8\},\{5,6\},\{7\}\}$$

set partitions

The connection between NSym and NCSym

NCSym has graded dimensions

1, 1, 2, 5, 15, 52, 203, 877, 4140, ...

NSym has graded dimensions

1, 1, 2, 4, 8, 16, 32, 64, 128, ...

The connection between NSym and NCSym

NCSym has graded dimensions

1, 1, 2, 5, 15, 52, 203, 877, 4140, ...

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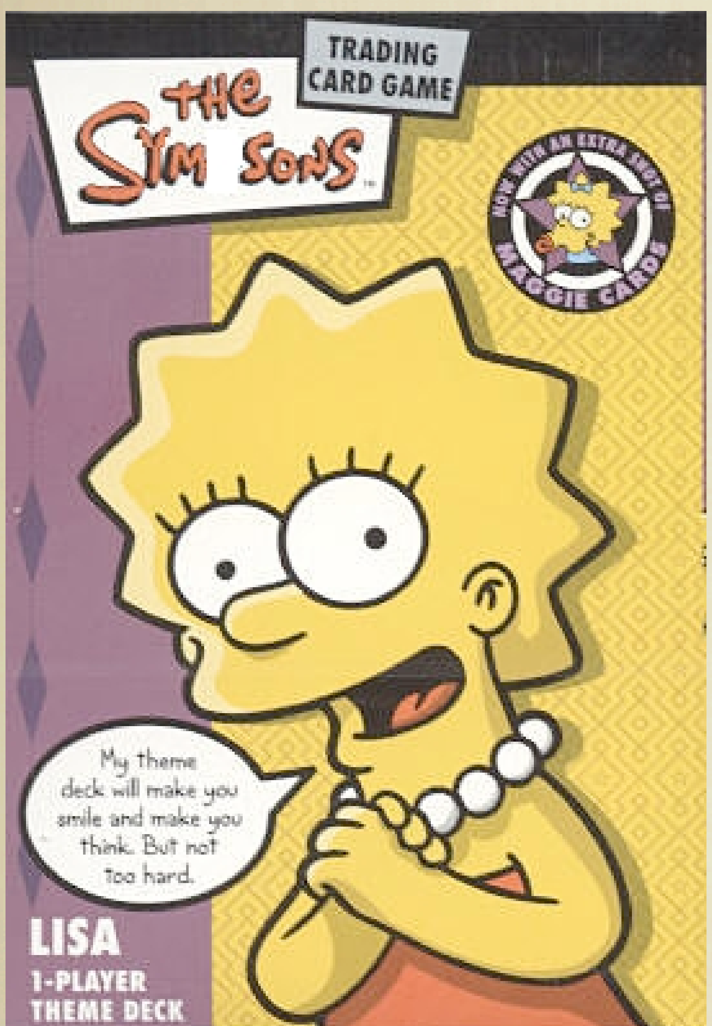
1, 1, 2, 4, 8, 16, 32, 64, 128, ...

There exists a Hopf morphism

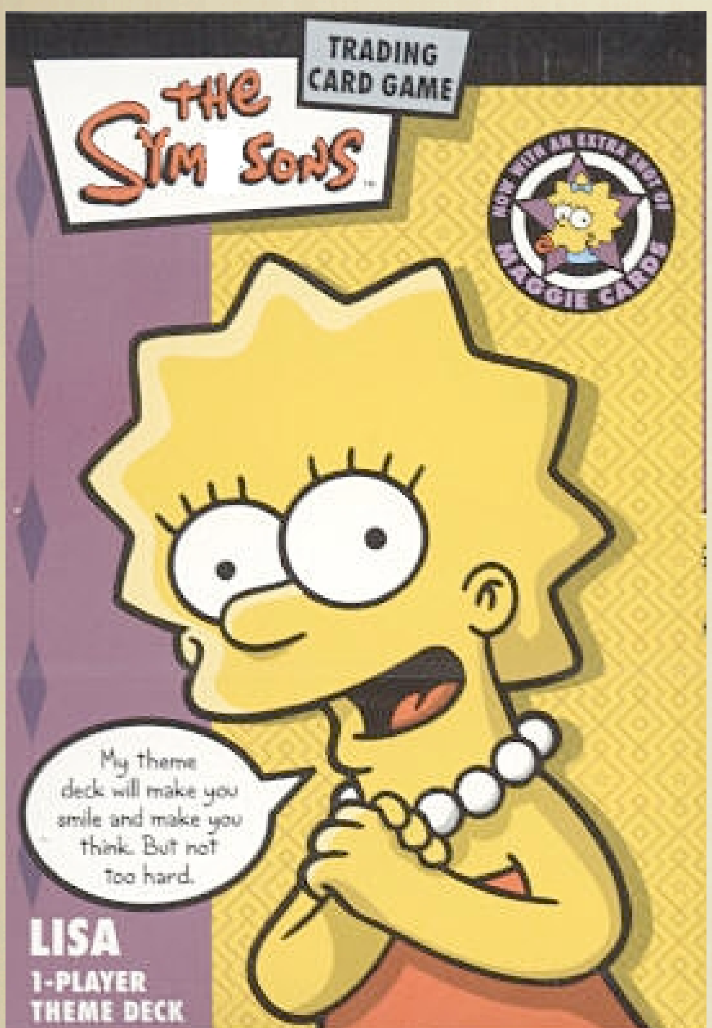
$$NSym \hookrightarrow NCSym$$

Families of morphisms



$$\begin{tikzcd}
NSym \arrow[rr, hook] \arrow[dd] \arrow[dr, hook] & & \overset{315426798}{\mathfrak{S}Sym} \arrow[dd] \\
& \underset{\{\{1,3,4\},\{2,8\},\{5,6\},\{7\}\}}{NCSym} \arrow[dd] & \\
Sym \arrow[rr] \arrow[dr, dash] & & QSym \\
& Sym &
\end{tikzcd}$$

Families of morphisms



$$
\begin{tikzcd}
NSym \arrow[rr, hook] \arrow[dd] \arrow[dr, hook] & & \mathfrak{S}Sym^{315426798} \arrow[dd] \arrow[dr, hook] & \\
& \underset{\{\{1,3,4\},\{2,8\},\{5,6\},\{7\}\}}{NCSym} \arrow[rr, hook] \arrow[dd] & & \underset{(\{5,6\},\{2,8\},\{1,3,4\},\{7\})}{NCQSym} \arrow[dd] \\
Sym \arrow[rr, hook] \arrow[dr, dash] & & QSym \arrow[dr, dash] & \\
& Sym \arrow[rr, hook] & & QSym
\end{tikzcd}
$$

One last open question

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