Immaculate basis and representation

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(with **Chris Berg, Franco Saliola, Luis Serrano and Mike Zabrocki**)

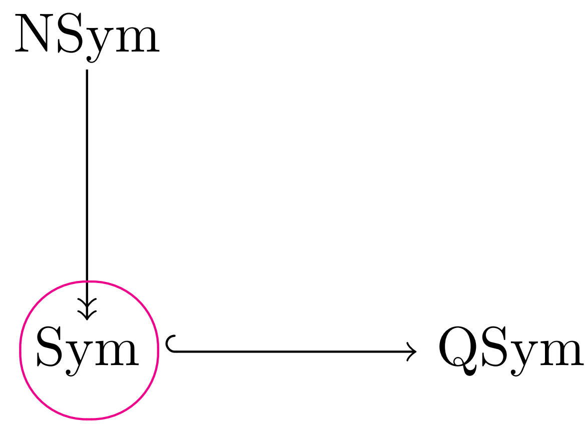
Outline

- NSym, Sym, QSym Combinatorial Hopf Algebras.
- Immaculate Basis Created from emptyness.
- Nice properties and posets is this basis interesting?
- Hall-Littlewood if time.

Sym, NSym, QSym

$$\begin{array}{ccc} \mathrm{NSym} & & \\ \big\downarrow\!\!\big\downarrow & & \\ \mathrm{Sym} & \hookrightarrow & \mathrm{QSym} \end{array}$$

Sym, NSym, QSym



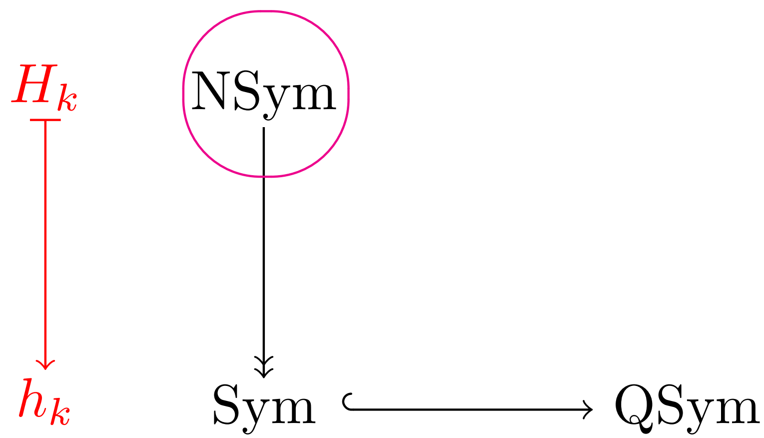
Sym: Symmetric functions

Basis: e_λ (Elementary); m_λ (monomials); p_λ (Power sums);

$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$ (Homogeneous) $\displaystyle\sum_{k\geq 0} h_k t^k = \prod_{i\geq 1} \frac{1}{1-x_i t}$

$s_\lambda = \det(h_{\lambda_i+j-i})$ (Schur)

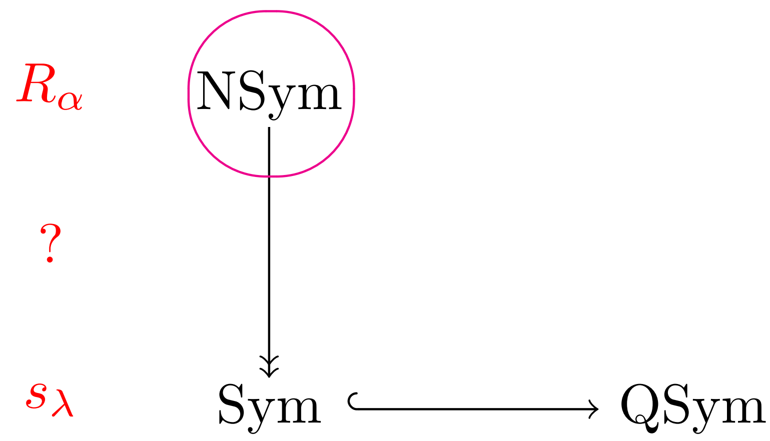
Sym, NSym, QSym



Sym: Symmetric functions $= \mathbb{Z}[h_1, h_2, \ldots]$

NSym: noncommutative symmetric functions $= \mathbb{Z}\langle H_1, H_2, \ldots \rangle$

Sym, NSym, QSym



Sym: Symmetric functions $= K_0\left(\bigoplus S_n\right)$: Irreducible $\leftrightarrow$ Schur

NSym: noncommutative symmetric functions $= \mathbb{Z}\langle H_1, H_2, \ldots\rangle$

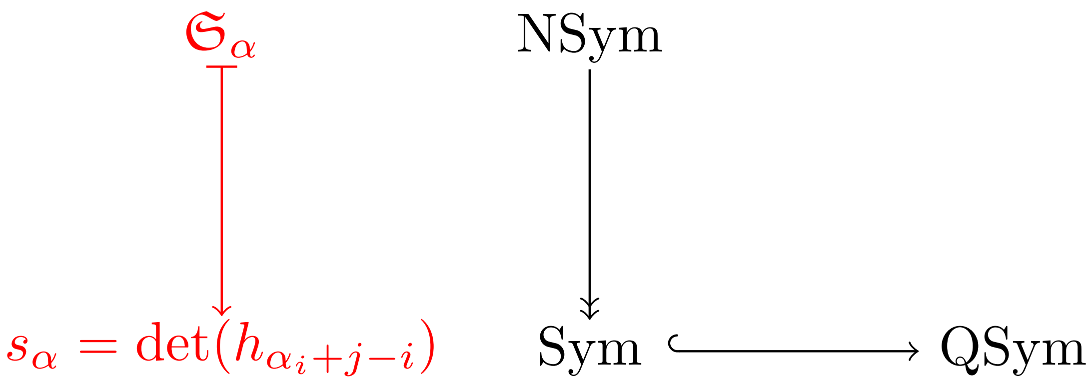
a quest for Schur function in NSym

$\text{NSym} = K_0\left(\bigoplus H_n(0)\right)$

[Grothendick group of representation of Hecke algebra at q=0]

Irreducible $\leftrightarrow$ noncommutative Ribon

Sym, NSym, QSym



Sym: Symmetric functions

NSym: noncommutative symmetric functions $= \mathbb{Z}\langle H_1, H_2, \ldots \rangle$

a quest for Schur function in NSym

Immaculate noncommutative symmetric functions [B-B-S-S-Z]

$$\mathfrak{S}_\alpha := \sum_{\sigma \in S_m} (-1)^\sigma H_{\alpha_1+\sigma_1-1} H_{\alpha_2+\sigma_2-2} \cdots H_{\alpha_m+\sigma_m-m}$$

Sym, NSym, QSym

$$\begin{array}{ccc} \text{NSym} & & \\ \downarrow\!\!\downarrow & & \\ \text{Sym} & \hookrightarrow & \text{QSym} \quad \mathfrak{S}^*_\alpha \end{array}$$

Sym: Symmetric functions

NSym: noncommutative symmetric functions $= \mathbb{Z}\langle H_1, H_2, \ldots\rangle$

QSym: Quasisymmetric functions $= NSym^*$

Immaculate noncommutative symmetric functions

$$\mathfrak{S}_\alpha := \sum_{\sigma \in S_m} (-1)^\sigma H_{\alpha_1+\sigma_1-1} H_{\alpha_2+\sigma_2-2} \cdots H_{\alpha_m+\sigma_m-m}$$

where we use the convention that $H_0 = 1$ and $H_{-m} = 0$ for $m > 0$.

Immaculate noncommutative symmetric functions

$$\mathfrak{S}_\alpha := \sum_{\sigma \in S_m} (-1)^\sigma H_{\alpha_1+\sigma_1-1} H_{\alpha_2+\sigma_2-2} \cdots H_{\alpha_m+\sigma_m-m}$$

We have $\mathfrak{S}_\alpha = H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_\ell}$ + higher lex term

Hence

$\{\mathfrak{S}_\alpha\}$ is a basis of NSym

Immaculate noncommutative symmetric functions

$$\mathfrak{S}_\alpha := \sum_{\sigma\in S_m} (-1)^\sigma H_{\alpha_1+\sigma_1-1} H_{\alpha_2+\sigma_2-2} \cdots H_{\alpha_m+\sigma_m-m}$$

$\{\mathfrak{S}_\alpha\}$ is a basis of NSym

Creation operator $\mathbb{B}_m : \mathrm{NSym}_n \to \mathrm{NSym}_{m+n}$

$$\mathbb{B}_m := \sum_{i\geq 0} (-1)^i H_{m+i} F_{1^i}^{\perp},$$

$F_\alpha = R_\alpha^* \in \mathrm{QSym} = \mathrm{NSym}^*$ and $\langle F_{1^i}^{\perp}\varphi, F_\alpha\rangle = \langle \varphi, F_{1^i} F_\alpha\rangle$

$$\mathfrak{S}_\alpha = \mathbb{B}_{\alpha_1}\mathbb{B}_{\alpha_2}\cdots\mathbb{B}_{\alpha_\ell} 1$$

immaculately concieved?

Immaculate noncommutative symmetric functions

$$\mathfrak{S}_\alpha := \sum_{\sigma \in S_m} (-1)^\sigma H_{\alpha_1+\sigma_1-1} H_{\alpha_2+\sigma_2-2} \cdots H_{\alpha_m+\sigma_m-m}$$

$\{\mathfrak{S}_\alpha\}$ is a basis of NSym

$$\mathfrak{S}_\alpha = \mathbb{B}_{\alpha_1} \mathbb{B}_{\alpha_2} \cdots \mathbb{B}_{\alpha_\ell} 1$$

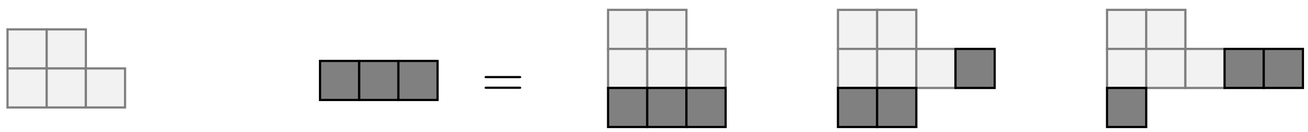
THEOREM Pieri Rule

$$\mathfrak{S}_\alpha H_s = \sum_{\alpha \subset_s \beta} \mathfrak{S}_\beta$$

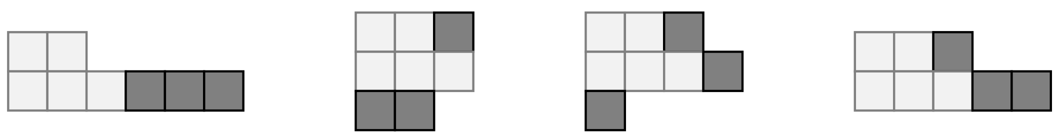
$\alpha \subset_s \beta$: $\quad |\beta| = |\alpha| + s, \quad \alpha_j \leq \beta_j, \quad \ell(\beta) \leq \ell(\alpha) + 1.$

Example of Pieri Rule

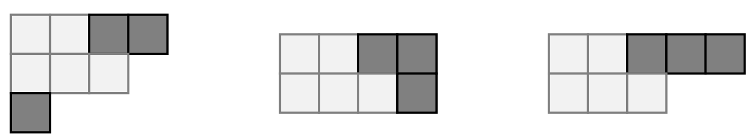
$$\mathfrak{S}_\alpha H_s = \sum_{\alpha \subset_s \beta} \mathfrak{S}_\beta$$



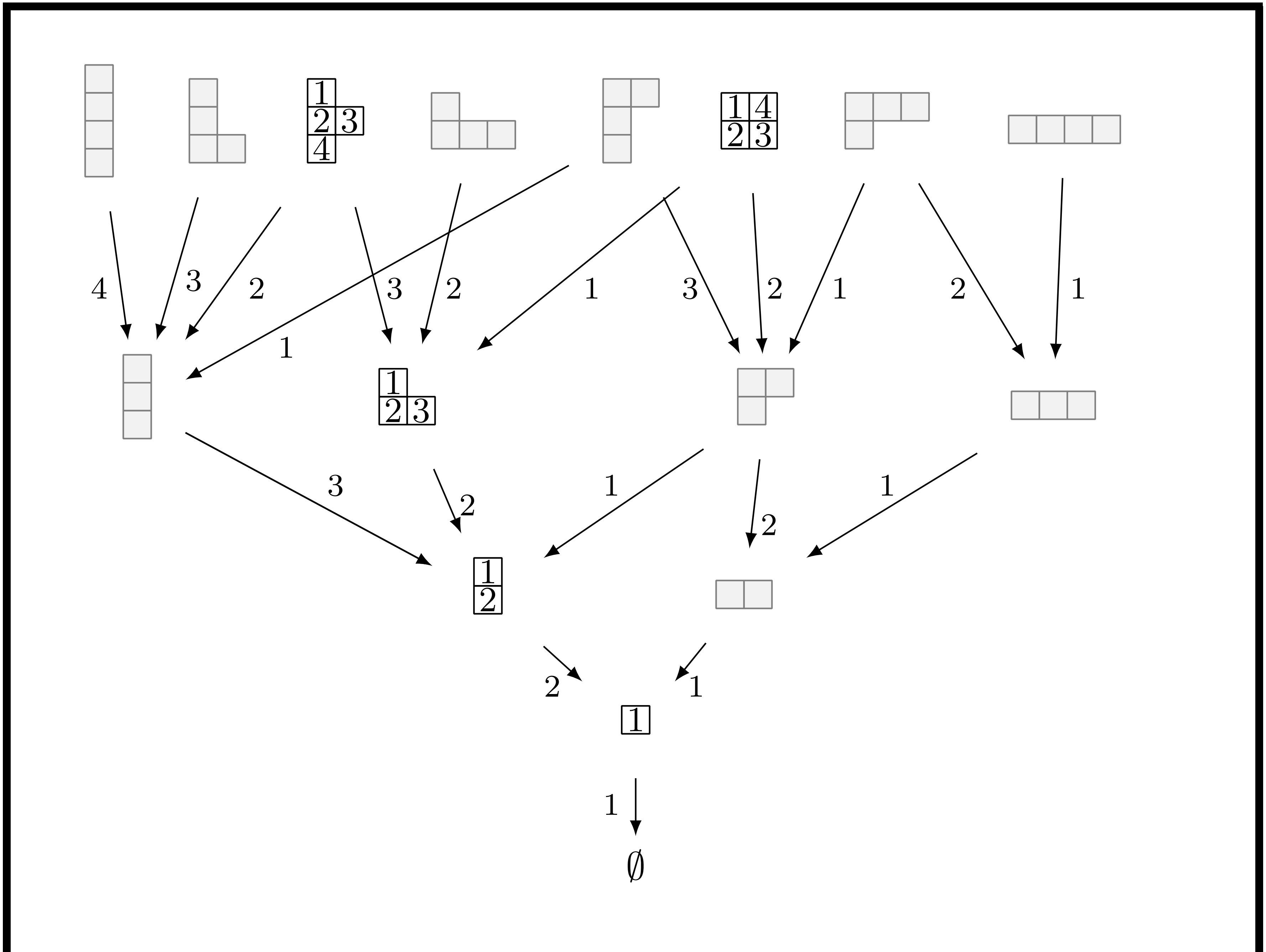
$$\mathfrak{S}_{23} \quad * \quad H_3 = \quad \mathfrak{S}_{233} \quad + \mathfrak{S}_{242} \quad + \mathfrak{S}_{251}$$



$$+ \mathfrak{S}_{26} \quad + \mathfrak{S}_{332} \quad + \mathfrak{S}_{341} \quad + \mathfrak{S}_{35}$$



$$+ \mathfrak{S}_{431} \quad + \mathfrak{S}_{44} \quad + \mathfrak{S}_{53}$$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Why Immaculate? Is it interesting?

THEOREM Pieri Rule

$$\mathfrak{S}_\alpha H_s = \sum_{\alpha \subset_s \beta} \mathfrak{S}_\beta$$

THEOREM Noncommutative Kotska

$$H_\beta = \sum_{\alpha \geq_\ell \beta} K_{\alpha,\beta} \mathfrak{S}_\alpha$$

THEOREM Noncommutative Littlewood-Richardson ($c^\beta_{\alpha,\lambda} \geq 0$)

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_\beta c^\beta_{\alpha,\lambda} \mathfrak{S}_\beta$$

And much more...

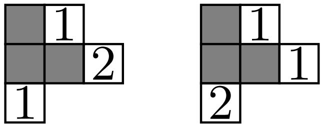
Noncommutative Littlewood-Richardson $c^{\beta}_{\alpha,\lambda}$

THEOREM Noncommutative Littlewood-Richardson ($c^{\beta}_{\alpha,\lambda} \geq 0$)

$$\mathfrak{S}_{\alpha}\mathfrak{S}_{\lambda} = \sum_{\beta} c^{\beta}_{\alpha,\lambda}\mathfrak{S}_{\beta}$$

$c^{\beta}_{\alpha,\lambda}$ is the number of immaculate tableaux (row strict, only the first column strict), of shape α/β and content λ and Yamanouchi.

For example $C^{(2,3,1)}_{(1,2),(2,1)} = 2$:



Noncommutative Littlewood-Richardson $c^{\beta}_{\alpha,\lambda}$

THEOREM Noncommutative Littlewood-Richardson ($c^{\beta}_{\alpha,\lambda} \geq 0$)

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_\beta c^{\beta}_{\alpha,\lambda} \mathfrak{S}_\beta$$

$c^{\beta}_{\alpha,\lambda}$ is the number of immaculate tableaux (row strict, only the first column strict), of shape α/β and content λ and Yamanouchi.

THEOREM for any ν such that $\ell(\nu) \leq \ell(\alpha)$

$$c^{\beta}_{\alpha,\lambda} = c^{\beta+\nu}_{\alpha+\nu,\lambda}$$

Noncommutative Littlewood-Richardson $c_{\alpha,\lambda}^{\beta}$

THEOREM Noncommutative Littlewood-Richardson ($c_{\alpha,\lambda}^{\beta} \geq 0$)

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_\beta c_{\alpha,\lambda}^{\beta} \mathfrak{S}_\beta$$

$c_{\alpha,\lambda}^{\beta}$ is the number of integral points in a certain polytope in $\binom{N}{2}$-dimension where $N = \max\{\ell(\alpha), \ell(\lambda), \ell(\beta)\}$.

$$\begin{array}{ccccc} & & x_{00} & & \\ & x_{10} & & x_{11} & \\ x_{20} & & x_{21} & & x_{22} \end{array}$$

Dual Immaculate and representation

THEOREM Noncommutative Kotska

$$H_\beta = \sum_{\alpha \geq_\ell \beta} K_{\alpha,\beta} \mathfrak{S}_\alpha$$

Dually:

$$\mathfrak{S}^*_\alpha = \sum_{\alpha \geq_\ell \beta} K_{\alpha,\beta} M_\beta$$

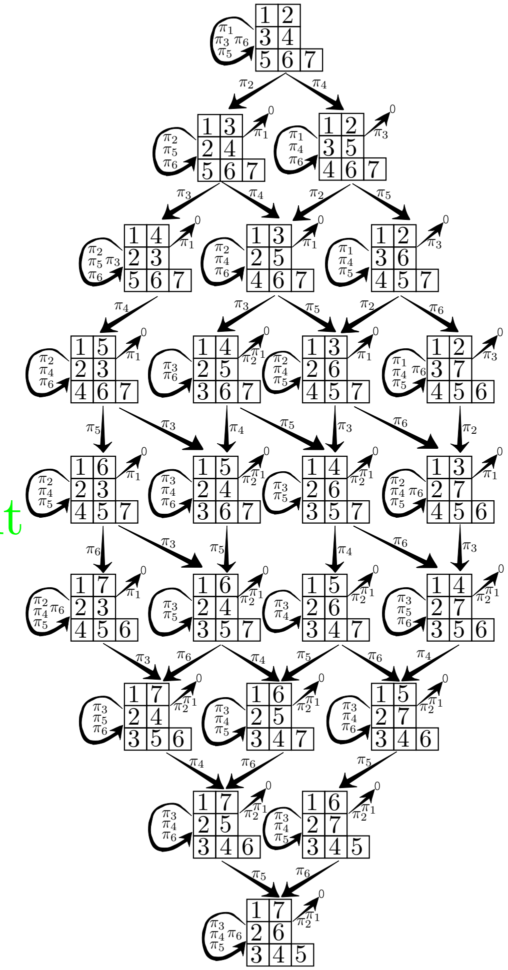
Naturally grouping terms leads to

$$\mathfrak{S}^*_\alpha = \sum_T F_{D(T)}$$

T runs over Standard Immaculate of shape α and $D(T)$ is the descent set of T.

Dual Immaculate and representation

$$\mathfrak{S}^*_\alpha = \sum_T F_{D(T)}$$



Action of $H_n(0)$ à la Lauve

Find something with α: M^α

Find large indecomposible quotient

Should be related to

Immaculate Hall-Littlewood

Immaculate Hall-Littlewood

Define $\widetilde{\mathbb{B}}_m : NSym[q]_n \to NSym[q]_{n+m}$ by:

$$\widetilde{\mathbb{B}}_m := \sum_{i \geq 0} q^i \mathbb{B}_{m+i} H_i^{\perp} .$$

then let

$$\mathcal{Q}_\alpha := \widetilde{\mathbb{B}}_{\alpha_1} \cdots \widetilde{\mathbb{B}}_{\alpha_m}(1) \ .$$

These are lifting to NSym of Hall-Littlewood symmetric

THEOREM

Under $NSym \twoheadrightarrow Sym$, we have $\mathcal{Q}_\alpha \mapsto Q'_\alpha$

This is the first lift of Hall-Littlewood to NSym. All many previous lift where Hall-Littlewood *like*!

Immaculate Hall-Littlewood

$$\mathcal{Q}'_{1111} = \mathfrak{S}_{1111} + q\mathfrak{S}_{112} + (q+q^2)\mathfrak{S}_{121} + q^3\mathfrak{S}_{13} + (q+q^2+q^3)\mathfrak{S}_{211} + (q^2+q^3+q^4)\mathfrak{S}_{22} + (q^3+q^4+q^5)\mathfrak{S}_{31} + q^6\mathfrak{S}_4 \,.$$

Under $NSym \twoheadrightarrow Sym$,

$$Q'_{1111} = s_{1111} + (q+q^2+q^3)s_{211} + (q^2+q^4)s_{22} + (q^3+q^4+q^5)s_{31} + q^6 s_4 \,.$$

For any partition λ, $\mathfrak{S}_\lambda \mapsto s_\lambda$.

$\mathfrak{S}_{112} \mapsto 0$, $\quad \mathfrak{S}_{121} \mapsto 0 \quad$ and $\quad \mathfrak{S}_{13} \mapsto -s_{22}$.

Immaculate Hall-Littlewood

CONJECTURE

$$\mathcal{Q}'_\lambda = \sum_T q^{st(T)} \mathfrak{S}_{shape(T)}$$

for some statistic st, over all immaculate tableau of content λ.