Why find cancelation free formula for antipode?

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Outline

- Antipode and Generalized Chromatic Polynomials
- Hopf Algebra proof of Stanley's acyclic theorem
- Hopf monoid framework
- Toward cancelation free formula for antipode.

Combinatorial Hopf Algebra

$H = \bigoplus_{n \geq 1} H_n$ graded Hopf algebra with character $\zeta : H \to \mathbb{Q}$

$H_n = \mathbb{Q}\big[G : G \text{ iso class of graphs on } [n]\big]$

$G_1 \cdot G_2 = G_1 \cup G_2$

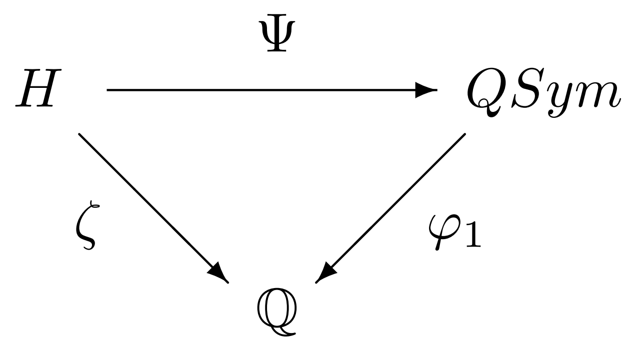
$\Delta(G) = \sum_{S \subseteq [n]} G|_S \otimes G|_{S^c}$

$$\zeta(G) = \begin{cases} 1 & \text{if } G \text{ has no edges} \\ o & \text{otherwise} \end{cases}$$

Combinatorial Hopf Algebra

[Aguiar-Bergeron-Sottile]

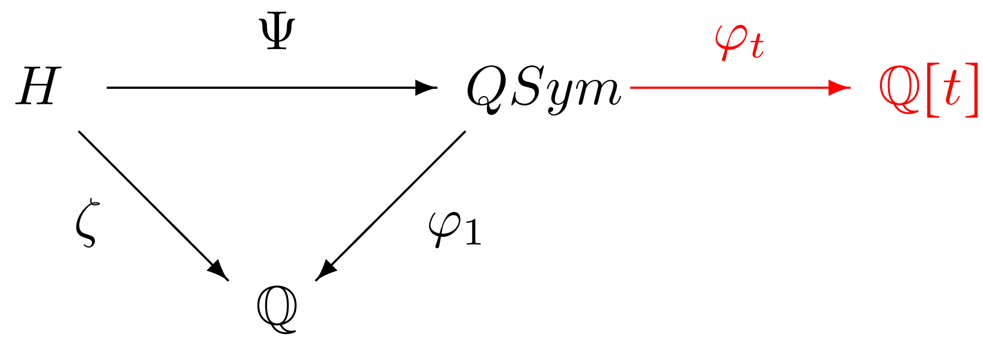
$H = \bigoplus_{n \geq 1} H_n$ graded Hopf algebra with character $\zeta \colon H \to \mathbb{Q}$



where $\varphi_1(f) = f(1, 0, 0, \cdots)$.

Combinatorial Hopf Algebra (ABS)

$H = \bigoplus_{n\geq 1} H_n$ graded Hopf algebra with character $\zeta\colon H \to \mathbb{Q}$



where $\varphi_1(f) = f(1,0,0,\cdots)$.

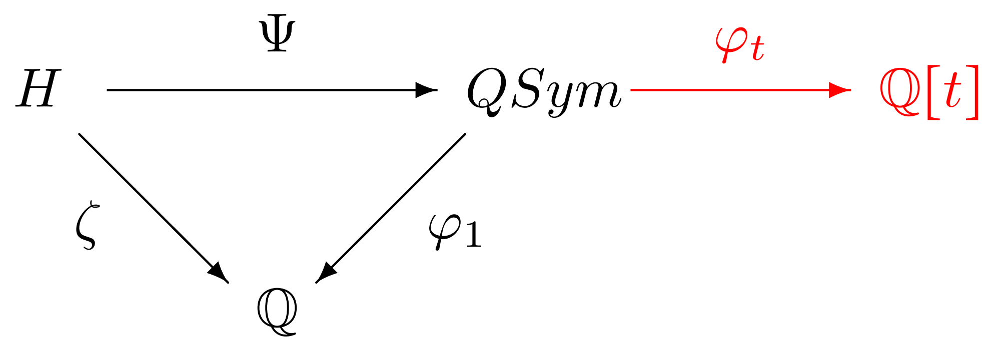
$\varphi_t(M_\alpha) = \binom{t}{\ell(\alpha)}$ is a Hopf morphism

$$\varphi_t\big(f(x_1,x_2,\ldots)\big)\Big|_{t=n} = f(\underbrace{1,\ldots,1}_{n},0,0,\ldots)\,.$$

Generalized Chromatic Polynomial

[Gringberg-Reiner]

$H = \bigoplus_{n\geq 1} H_n$ graded Hopf algebra with character $\zeta\colon H \to \mathbb{Q}$



Generalized Chromatic Polynomial

$$\chi_x(t) = \varphi_t \circ \Psi(x)$$

$$\varphi_t \circ \Psi\Big|_{t=1} = \left(\varphi_t\Big|_{t=1}\right) \circ \Psi = \varphi_1 \circ \Psi = \zeta.$$

Generalized Chromatic Polynomial

[Grinberg-Reiner]

$H = \bigoplus_{n\geq 1} H_n$ graded Hopf algebra with character $\zeta : H \to \mathbb{Q}$

$$H \xrightarrow{\quad\Psi\quad} QSym \xrightarrow{\quad\varphi_t\quad} \mathbb{Q}[t]$$

When $H_n = \mathbb{Q}\big[G : G \text{ iso class of graphs on } [n]\big]$

$$\chi_G(t) = \varphi_t \circ \Psi(G)$$

is the usual Chromatic polynomial.

Theorem [Stanley]

$$\boxed{\chi_G(-1) = \pm a(G)}$$

For a graph G; $a(G)$ is number of acyclic orientation.

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Antipode: $S\colon \mathbb{Q}[t] \to \mathbb{Q}[t]$

$$f(t) \mapsto f(-t)$$

$$\chi_G(-1) = S \circ \varphi_t \circ \Psi(G)\Big|_{t=1} = \varphi_t \circ \Psi \circ S(G)\Big|_{t=1} = \zeta \circ S(G)$$

$$\boxed{\chi_G(-1) = \zeta\big(S(G)\big)}$$

Theorem [Humpert-Martin]

Jeremy: *What can I do with a Hopf algebra I have just rediscovered*

Nantel: *Compute its antipode*

THEOREM

$$S(G) = \sum_F (-1)^{|F|} a(G/_F) G|_F$$

Corollary $\chi_G(-1) = \zeta\big(S(G)\big) = (-1)^{c(G)} a(G)$

So now I want better formulas for antipode

For $H = \bigoplus_{n\geq 0} H_n$ and $x \in H_n$ [Takeuchi]

$$S(x) = \sum_{\alpha \vDash n} (-1)^{\ell(\alpha)} m_\alpha \Delta_\alpha(x)$$

where

$m_\alpha : H_{\alpha_1} \otimes \cdots \otimes H_{\alpha_\ell} \to H_n$ and $\Delta_\alpha : H_n \to H_{\alpha_1} \otimes \cdots \otimes H_{\alpha_\ell}$

So now I want better formulas for antipode

For $H = \bigoplus_{n\geq 0} H_n$ and $x \in H_n$ (Takeuchi)

$$S(x) = \sum_{\alpha \models n} (-1)^{\ell(\alpha)} m_\alpha \Delta_\alpha(x)$$

MANY CANCELATIONS

ANSWER is FINER THAN GRADING

look at Humpert-Martin formula

$$S(G) = \sum_{F \models [n]} (-1)^{|F|} a(G/_F) G|_F$$

Hopf Monoids

[Aguiar-Mahajan]

For

$$H = \bigoplus_{I \subseteq \mathbb{N}} H[I]$$

Hopf Monoids

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$$H = \bigoplus_{I \subseteq \mathbb{N}} H[I]$$

Hopf Monoids

[Aguiar-Mahajan]

(Vector Spaces) species

A Functor $H : (\text{Sets}, \text{Bijs}) \to (\text{Vects}, \text{Transfs})$

$H[I]$ = Vector space

$m_{I,J}\colon H[I] \otimes H[J] \to H[I \uplus J]$ $\qquad$ $\Delta_{I,J}\colon H[I \uplus J] \to H[I] \otimes H[J]$

TAKEUCHI For $x \in H[I]$.

$$S(x) = \sum_{A \vDash I} (-1)^{\ell(A)} m_A \Delta_A(x)$$

Hopf Monoids

[Aguiar-Mahajan]

(Vector Spaces) species

A Functor $H : (\text{Sets}, \text{Bijs}) \to (\text{Vects}, \text{Transfs})$

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TAKEUCHI For $x \in H[I]$.

$$S(x) = \sum_{A \models I} (-1)^{\ell(A)} m_A \Delta_A(x)$$

STILL MANY CANCELATIONS

Example of Hopf Monoids

- $G[I] = \text{Span}\{G : G \text{ is a simple graph on } I\}$

$m_{I,J}\colon G[I]\otimes G[J]\to G[I\uplus J]$ $\qquad$ $\Delta_{I,J}\colon G[I\uplus J]\to G[I]\otimes G[J]$

$g\otimes g'\mapsto g\cup g'$ $\qquad$ $g\mapsto g|_I\otimes g|_J$

- $L[I] = \text{Span}\{\alpha : \alpha \text{ is a total order on } I\}$

$m_{I,J}(\alpha\otimes\beta) = \alpha\cdot\beta$ $\qquad$ $\Delta_{I,J}(\alpha) = \alpha|_I\otimes\alpha|_J$

- $\Pi[I] = \text{Span}\{A : A \text{ is a set partition on } I\}$

$m_{I,J}(A\otimes B) = A\cup B$ $\qquad$ $\Delta_{I,J}(A) = A|_I\otimes A|_J$

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$m_{I,J}: G[I] \otimes G[J] \to G[I \uplus J]$ $\qquad$ $\Delta_{I,J}: G[I \uplus J] \to G[I] \otimes G[J]$

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- $\Pi[I] = \mathrm{Span}\{A : A \text{ is a set partition on } I\}$

$m_{I,J}(A \otimes B) = A \cup B$ $\qquad$ $\Delta_{I,J}(A) = A|_I \otimes A|_J$

We can resolve Takeuchi formula with ad hoc sign reversing involution and get cancelation free formula

Linearizable Hopf Monoids

[Marberg]

- a (Set) species $\mathbf{h} : (\text{Sets}, \text{Bijs}) \to \text{Sets}$, is a basis for H if

  $$H[I] = \text{Span}\{x : x \in \mathbf{h}[I]\}$$

- A hopf monoid H is linearizable if $m_{I,J}$ and $\Delta_{I,J}$ are linear extension of

  $$m_{I,J} : \mathbf{h}[I] \otimes \mathbf{h}[J] \to \mathbf{h}[I \uplus J]$$

  $$\Delta_{I,J} : \mathbf{h}[I \uplus J] \to \mathbf{h}[I] \otimes \mathbf{h}[J] \cup \{0\}$$

Many Hopf monoids are linearizable

All our examples are... but not for all basis!!!

Functors from Hopf Monoids

[Aguiar-Mahajan]

$$\begin{array}{ccc} \text{Species} & \xrightarrow{K} & \text{Gr-Vect} \\ \mathrm{H} & \longmapsto & \bigoplus_{n\geq 0} H[n] \end{array}$$

$$\begin{array}{ccc} \text{Species} & \xrightarrow{\overline{K}} & \text{Gr-Vect} \\ \mathrm{H} & \longmapsto & \bigoplus_{n\geq 0} H[n]/_{\langle x-\sigma(x)\rangle} \end{array}$$

Both functors send Hopf monoids to graded Hopf algebras, but only $\overline{K}$ preserves antipode

$\overline{K}(L \times H) = K(H)$

[Aguiar-Mahajan]

Study antipode for $L \times H$ and H linearizable Hopf monoid.

For $(\alpha, x) \in (L \times \mathbf{h})[I]$, using Takeuchi

$$S_I(\alpha, x) = \sum_{(\beta,y)\in(\mathbf{l}\times\mathbf{h})[I]} \left(\sum_{A \in \mathcal{C}_{\alpha,x}^{\beta,y}} (-1)^{\ell(A)} \right) (\beta, y).$$

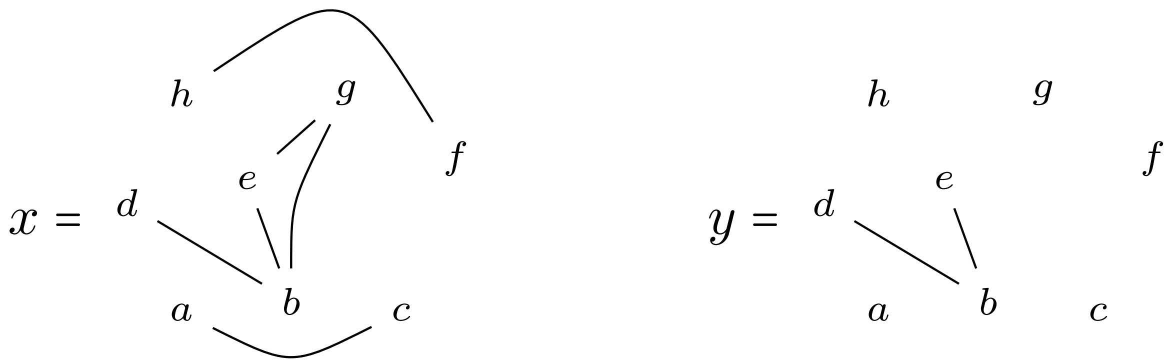
where

$$\mathcal{C}_{\alpha,x}^{\beta,y} = \left\{ A \vDash I : (\alpha_A, x_A) = (\beta, y) \right\}$$

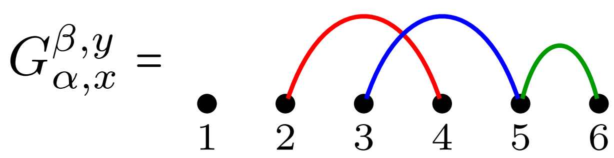
$\mathcal{C}_{\alpha,x}^{\beta,y}$

G: Hopf monoid of graph

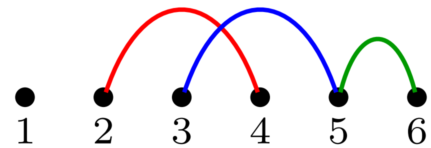
$I = \{a, b, c, d, e, f, g, h\}$; $\alpha = abcdefgh$; $\beta = abdefghc$

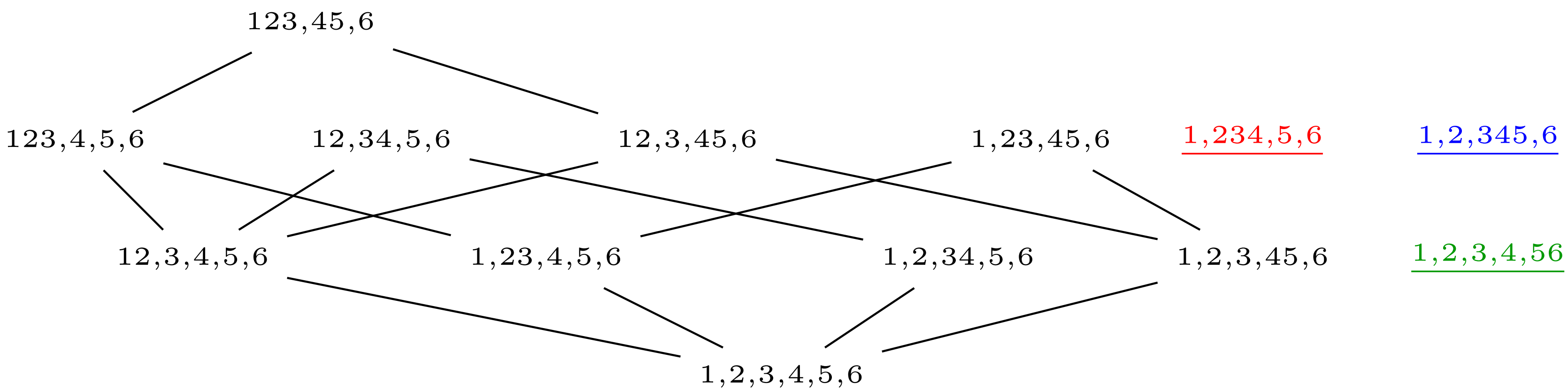


$\Lambda = (a, bde, f, g, h, c)$: minimum refinement of $\mathcal{C}_{\alpha,x}^{\beta,y}$



$$\mathcal{C}^{\beta,y}_{\alpha,x}$$

$G^{\beta,y}_{\alpha,x} =$ 



$\longrightarrow$ this coefficient is 0

THEOREM $\mathcal{C}_{\alpha,x}^{\beta,y}$

H: Any linearizable Hopf monoid with basis $\mathbf{h}$

(α, x) and (β, y) in $(L \times \mathbf{h})[I]$

- $\mathcal{C}_{\alpha,x}^{\beta,y}$ has a unique minimal refinement Λ
- $\mathcal{C}_{\alpha,x}^{\beta,y}$ is a lower ideal in $[\Lambda, I]$
- $[\Lambda, I] \smallsetminus \mathcal{C}_{\alpha,x}^{\beta,y}$ has minimums of the form

$$(\Lambda_1, \dots, \Lambda_{i-1}, \Lambda_i \cup \Lambda_{i+1} \cup \cdots \cup \Lambda_j, \Lambda_{j+1}, \dots, \Lambda_m)$$

- This define a graph $G_{\alpha,x}^{\beta,y}$ with no NESTING

  [no $(a, d), (b, c)$ with $a \le b < c \le d$]
- We have $\varphi\colon \mathcal{C}_{\alpha,x}^{\beta,y} \to \mathcal{C}_{\alpha,x}^{\beta,y}$ sign-reversing involution.

MERCI

GRACIAS