A MN Rule for Quantum Cohomology of the Flag Manifold

YORK
UNIVERSITÉ
UNIVERSITY

Nantel Bergeron

York Research Chair in Applied Algebra
www.math.yorku.ca/bergeron

E-Mail: **bergeron@yorku.ca**

[with C. Benedetti, L. Colmenarejo,
F. Saliola and F. Sottile]

CRM, Montreal

June 2017

People involved

Outline

- Warm up with classical case.

  hook product.

  Murnaghan-Nakayama in $H^*(F\ell_n)$.

- Quantum case.

  quantum hook product.

  Murnaghan-Nakayama in $qH^*(F\ell_n)$.

Schubert polynomials $\mathfrak{S}_w$

The ring $\mathbb{Z}[x_1, x_2, \dots]$ has a linear basis given by $\{\mathfrak{S}_w \,|\, w \in S_\infty\}$.

For $w \in S_n$:

$$\omega_0 = [n, \dots, 2, 1] \quad \rightsquigarrow \quad \mathfrak{S}_{\omega_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$$

$$w \neq \omega_0 \quad \rightsquigarrow \quad \mathfrak{S}_w = \partial_i \mathfrak{S}_{w s_i} \quad \text{for } w(i) < w(i+1)$$

$\partial_i P = \frac{P - s_i P}{x_i - x_{i+1}}$ [well defined]

Example

$$\mathfrak{S}_{54321} = x_1^4 x_2^3 x_3^2 x_4$$

$$\mathfrak{S}_{24531} = \partial_1 \mathfrak{S}_{42531} = \partial_1 \partial_2 \mathfrak{S}_{45231} = \partial_1 \partial_2 \partial_3 \mathfrak{S}_{45321} = \partial_1 \partial_2 \partial_3 \partial_1 \mathfrak{S}_{54321}$$

$$= x_1 x_2^2 x_3^2 x_4 + x_1^2 x_2^2 x_3 x_4 + x_1^2 x_2 x_3^2 x_4$$

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Stable (independent of n)

$$\begin{array}{rcl} S_n : & \hookrightarrow & S_{n+1} \\ w & \mapsto & w' = w(1) \ldots w(n)(n+1) \end{array}$$

$$\boxed{\mathfrak{S}_w = \mathfrak{S}_{w'}}$$

Schubert polynomials $\mathfrak{S}_w$

The ring $\mathbb{Z}[x_1, x_2, \ldots]$ has a linear basis given by $\{\mathfrak{S}_w \mid w \in S_\infty\}$.

Algebraic Geometry	$\leftrightarrow$	Algebraic Combinatorics
$H^*(F\ell_n)$		$\mathbb{Z}[x_1, \ldots, x_n]/\langle e_1, \ldots, e_n \rangle$

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$[X_w]$ Schubert classes	↔	$\mathfrak{S}_w \quad w \in S_n$

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Intersection		Multiplication

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Open Problem [Combinatorial]

$$\boxed{\mathfrak{S}_u \mathfrak{S}_v = \sum_w d_{u,v}^w \mathfrak{S}_w}$$

Monk's rule

$$\mathfrak{S}_{(k,k+1)}\mathfrak{S}_u = \sum_{\substack{i \le k < j \\ \ell(u)+1=\ell(u(i,j))}} \mathfrak{S}_{u(i,j)}$$

OBSERVATIONS

$$\mathfrak{S}_{(k,k+1)} = x_1 + x_2 + \ldots + x_k = h_1(x_1, x_2, \ldots, x_k)$$

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k-Bruhat order on S_∞:

$$u \lessdot_k u(i,j) \iff i \le k < j \text{ and } \ell(u)+1 = \ell(u(i,j))$$

Monk's rule

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OBSERVATIONS

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k-Bruhat order on S_∞:

$u \lessdot_k u(i,j) = (\alpha, \beta)u$

$$1246|357 \lessdot_4 1247356 \quad \rightsquigarrow \quad 1246357 \xrightarrow[(4,7)]{(6,7)} 1247356$$

$1246357 \cdot (4,7) = (6,7) \cdot 1246357$

Monk's rule

$$\mathfrak{S}_{(k,k+1)}\mathfrak{S}_u = \sum_{\substack{i\le k<j\\ \ell(u)+1=\ell(u(i,j))}} \mathfrak{S}_{u(i,j)}$$

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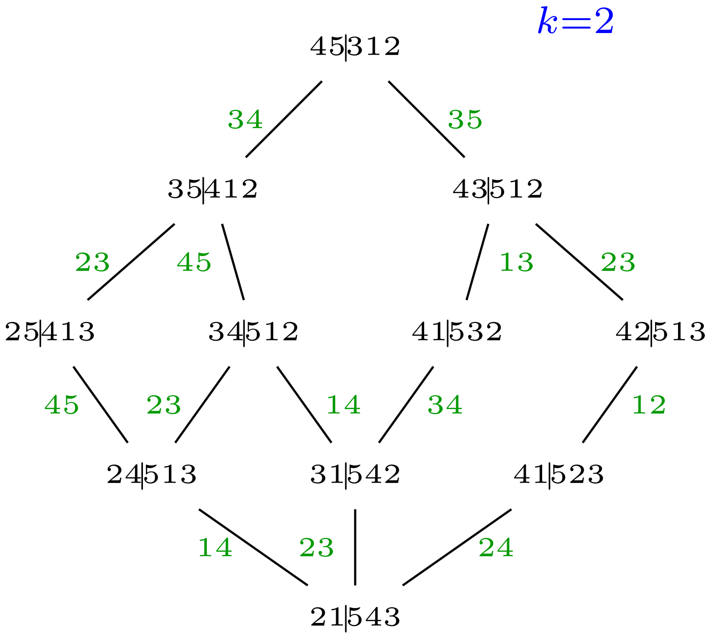


See weak reflection order

=Grassmannian order: [Bergeron, Sottile]

Reflection Order:

[Bagno, Biagioli, Novick, Woo, Tenner, Petersen...]

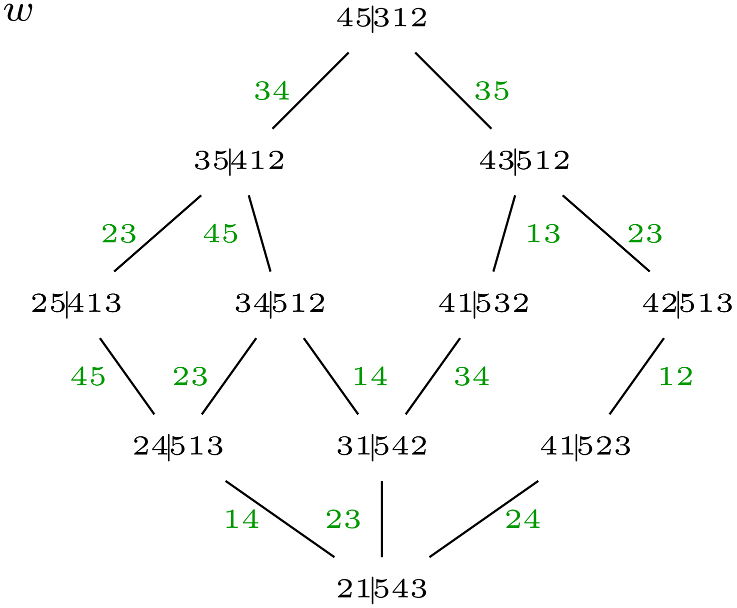
Monk's rule

$$\mathfrak{S}_{(k,k+1)}\mathfrak{S}_u = \sum_{\substack{i\le k<j \\ \ell(u)+1=\ell(u(i,j))}} \mathfrak{S}_{u(i,j)}$$

OBSERVATIONS

$$\mathfrak{S}_{(k,k+1)} = x_1 + x_2 + \ldots + x_k = h_1(x_1, x_2, \ldots, x_k)$$

$$(\mathfrak{S}_{(k,k+1)})^n \mathfrak{S}_u = \sum_w |Chs[u,w]_k| \cdot \mathfrak{S}_w$$



$$(\mathfrak{S}_{(2,3)})^4 \mathfrak{S}_{21543} = \mathbf{5}\,\mathfrak{S}_{45312} + \cdots$$

Monk's rule

$$\mathfrak{S}_{(k,k+1)}\mathfrak{S}_u = \sum_{\substack{i\leq k<j \\ \ell(u)+1=\ell(u(i,j))}} \mathfrak{S}_{u(i,j)}$$

OBSERVATIONS

$$\mathfrak{S}_{(k,k+1)} = x_1 + x_2 + \ldots + x_k = h_1(x_1, x_2, \ldots, x_k)$$

$$(\mathfrak{S}_{(k,k+1)})^n \mathfrak{S}_u = \sum_w |Chs[u,w]_k| \cdot \mathfrak{S}_w$$

$$(\mathfrak{S}_{(k,k+1)})^n = (h_1(x_1, \ldots, x_k))^n = \sum_{\lambda \vdash n} s_\lambda(x_1, \ldots, x_k)$$

$s_\lambda(x_1, \ldots, x_k) = \mathfrak{S}_{v(\lambda,k)}$ where $v(\lambda, k) \in S_\infty$

Open Problem
[combinatorial]
[Bergeron-Sottile]

$$\boxed{\mathfrak{S}_{v(\lambda,k)}\mathfrak{S}_u = \sum_w d^w_{u,v(\lambda,k)}\mathfrak{S}_w}$$

Hook case: $\mathfrak{S}_{v(b1^{a-1},k)}\mathfrak{S}_u$

$$\mathfrak{S}_{v(b1^{a-1},k)} = s_{a\updownarrow\underset{\overleftrightarrow{\;b\;}}{\llcorner}}(x_1, x_2, \ldots, x_k)$$

THEOREM

[Sottile 1996]

$$s_{a\updownarrow\underset{\overleftrightarrow{\;b\;}}{\llcorner}}(x_1, x_2, \ldots, x_k)\mathfrak{S}_u = \sum_{\substack{u \xrightarrow{\gamma}_k w(\gamma) \\ shape(\gamma)=b1^{(a-1)}}} \mathfrak{S}_{w(\gamma)}$$

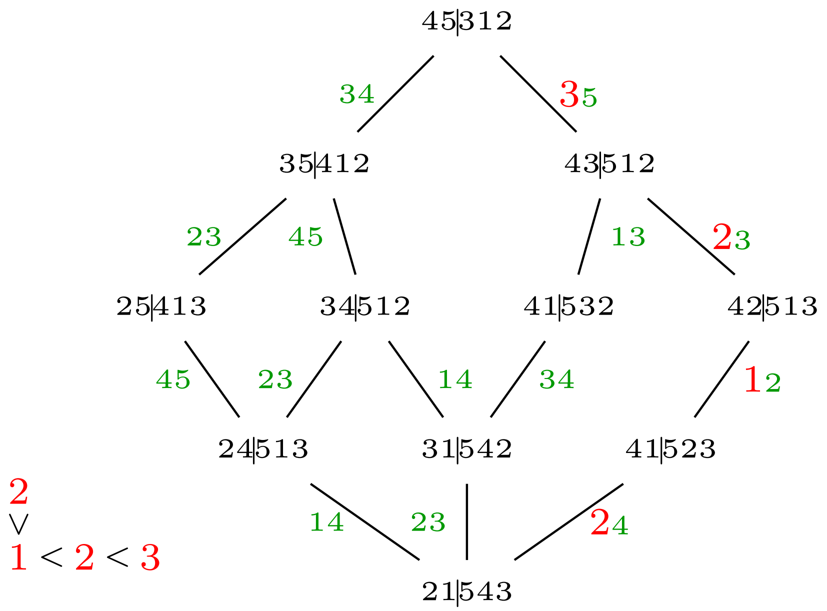
Hook case: $\mathfrak{S}_{v(b1^{a-1},k)}\mathfrak{S}_u$

$$s_{a\updownarrow\underset{\overleftrightarrow{\;b\;}}{\ydiagram{1,1,4}}}(x_1,x_2,\ldots,x_k)\mathfrak{S}_u = \sum_{\substack{u\xrightarrow{\gamma}_k w(\gamma)\\ shape(\gamma)=b1^{(a-1)}}} \mathfrak{S}_{w(\gamma)}$$

$$s_{\ydiagram{1,3}}(x_1,x_2)\mathfrak{S}_{21543} = \mathbf{1}\mathfrak{S}_{45312} + \cdots$$

$$s_{\ydiagram{4}}(x_1,x_2)\mathfrak{S}_{21543} = \mathbf{0}\mathfrak{S}_{45312} + \cdots$$

$$s_{\ydiagram{1,1,2}}(x_1,x_2)\mathfrak{S}_{21543} = \mathbf{0}\mathfrak{S}_{45312} + \cdots$$



Hook case: $\mathfrak{S}_{v(b1^{a-1},k)}\mathfrak{S}_u$

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$$s_{\ydiagram{1,3}}(x_1,x_2,x_3)\mathfrak{S}_{124356} = \mathbf{1}\mathfrak{S}_{236145} + \cdots$$

$$s_{\ydiagram{4}}(x_1,x_2,x_3)\mathfrak{S}_{124356} = \mathbf{0}\mathfrak{S}_{236145} + \cdots$$

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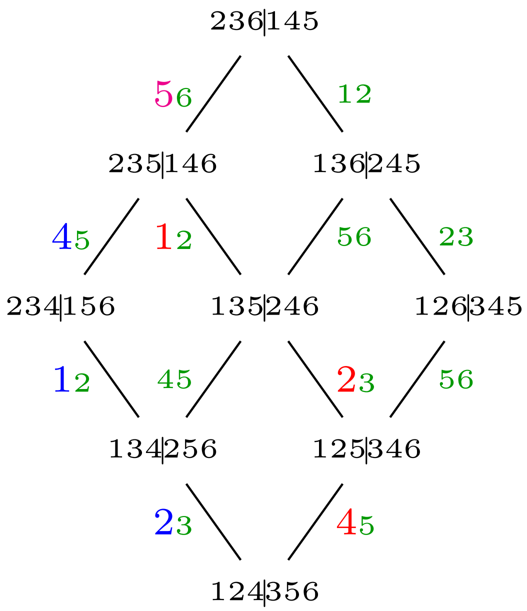
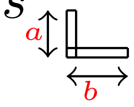


$$\begin{array}{c} 2 \\ \vee \\ 1 < 4 < 5 \end{array} \qquad \begin{array}{c} 4 \\ \vee \\ 2 \\ \vee \\ 1 < 5 \end{array}$$

Hook case: $\mathfrak{S}_{v(b1^{a-1},k)}\mathfrak{S}_u$

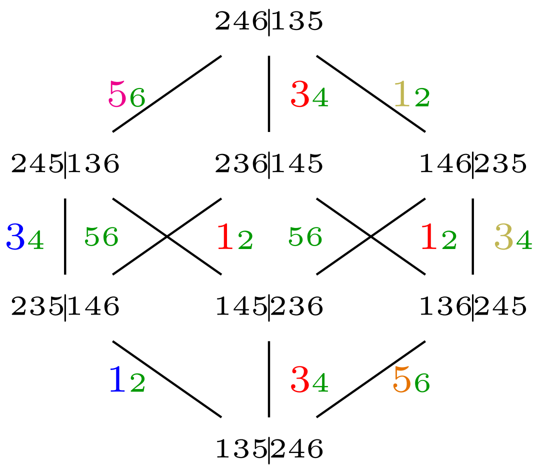


$$s(x_1,x_2,\ldots,x_k)\mathfrak{S}_u = \sum_{\substack{u \xrightarrow{\gamma}_k w(\gamma)\\ shape(\gamma)=b1^{(a-1)}}} \mathfrak{S}_{w(\gamma)}$$

$$s_{\ydiagram{1,2}}(x_1,x_2,x_3)\mathfrak{S}_{135246} = \mathbf{2}\mathfrak{S}_{246135} + \cdots$$

$$s_{\ydiagram{3}}(x_1,x_2,x_3)\mathfrak{S}_{135246} = \mathbf{1}\mathfrak{S}_{246135} + \cdots$$

$$s_{\ydiagram{1,1,1}}(x_1,x_2,x_3)\mathfrak{S}_{135246} = \mathbf{1}\mathfrak{S}_{246135} + \cdots$$



$1 < 3 < 5$

$\begin{array}{c}3\\ \vee\\ 1 < 5\end{array}$

$\begin{array}{c}5\\ \vee\\ 1 < 3\end{array}$

$\begin{array}{c}5\\ \vee\\ 3\\ \vee\\ 1\end{array}$

Murnaghan-Nakayama rule: $p_r(x_1,\ldots,x_k)\mathfrak{S}_u$

$$p_r = s_r - s_{(r-1)1} + s_{(r-2)1^2} - \cdots + (-1)^{r-1}s_{1^r}$$

THEOREM

[Morrison-Sottile 2016]

$$p_r(x_1,x_2,\ldots,x_k)\mathfrak{S}_u = \sum_{\substack{w:\ u\xrightarrow{\gamma}_k w\\ shape(\gamma)=b1^{(a-1)}\\ connected}} (-1)^{(a-1)}\mathfrak{S}_w$$

Murnaghan-Nakayama rule: $p_r(x_1,\dots,x_k)\mathfrak{S}_u$

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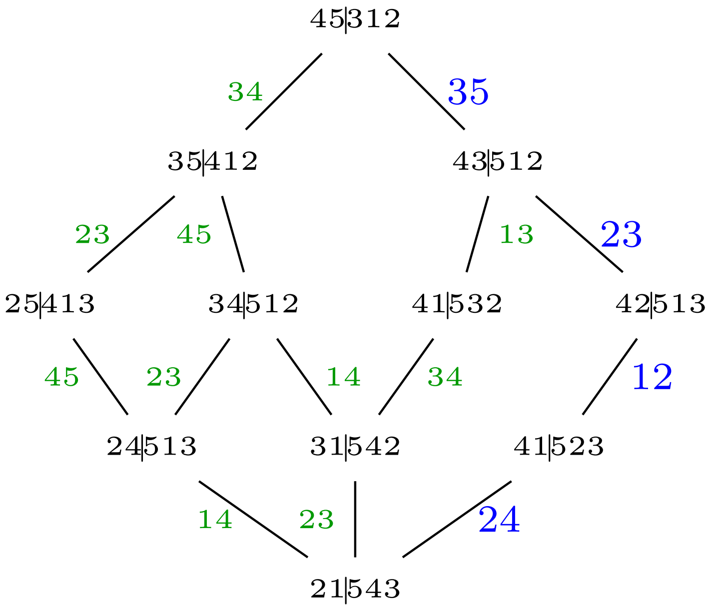
CONNECTED?

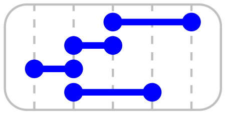
$s_{4}(x_1,x_2)\mathfrak{S}_{21543} = \mathbf{0}\mathfrak{S}_{45312} + \cdots$

$s_{31}(x_1,x_2)\mathfrak{S}_{21543} = \mathbf{1}\mathfrak{S}_{45312} + \cdots$

$s_{211}(x_1,x_2)\mathfrak{S}_{21543} = \mathbf{0}\mathfrak{S}_{45312} + \cdots$

$p_4(x_1,x_2)\mathfrak{S}_{21543} = (\mathbf{0}-\mathbf{1}+\mathbf{0}-\mathbf{0})\mathfrak{S}_{45312} + \cdots$



$\gamma :$ 

Murnaghan-Nakayama rule: $p_r(x_1,\ldots,x_k)\mathfrak{S}_u$

$$p_r = s_r - s_{(r-1)1} + s_{(r-2)1^2} - \cdots + (-1)^{r-1}s_{1^r}$$

$$p_r(x_1,x_2,\ldots,x_k)\mathfrak{S}_u = \sum_{\substack{w:\ u\xrightarrow{\gamma}_k w\\ shape(\gamma)=b1^{(a-1)}\\ connected}} (-1)^{(a-1)}\mathfrak{S}_w$$

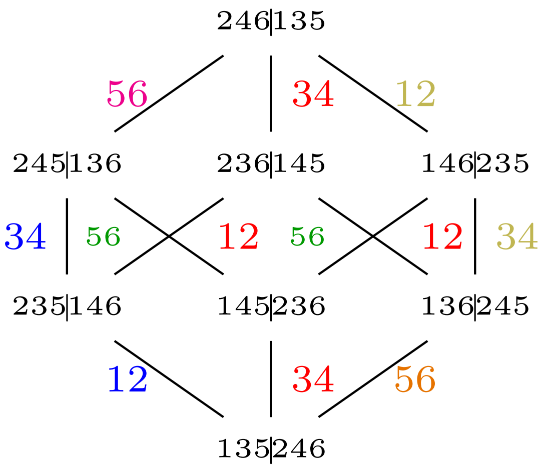
CONNECTED?

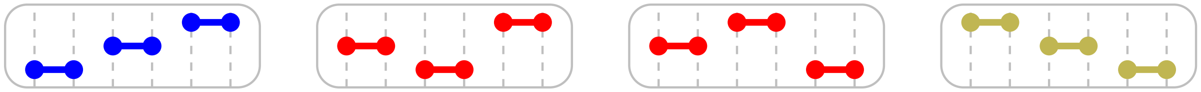
$s_{\ydiagram{3}}(x_1,x_2,x_3)\mathfrak{S}_{135246} = \mathbf{1}\mathfrak{S}_{246135} + \cdots$

$s_{\ydiagram{2,1}}(x_1,x_2,x_3)\mathfrak{S}_{135246} = \mathbf{2}\mathfrak{S}_{246135} + \cdots$

$s_{\ydiagram{1,1,1}}(x_1,x_2,x_3)\mathfrak{S}_{135246} = \mathbf{1}\mathfrak{S}_{246135} + \cdots$

$p_4(x_1,x_2)\mathfrak{S}_{135246} = (\mathbf{1}-\mathbf{2}+\mathbf{1})\mathfrak{S}_{246135} + \cdots$





Quantum World

Algebraic Geometry $\leftrightarrow$ Algebraic Combinatorics

$\mathbf{q}H^*(F\ell_n)$ $\mathbb{Z}[x_1,\ldots,x_n][q_1,\ldots,q_{n-1}]/\langle E_1,\ldots,E_n\rangle$

What are those E's?

$$det\begin{vmatrix} x_1-t & q_1 & 0 & \cdots & 0 & 0 \\ -1 & x_2-t & q_2 & \cdots & 0 & 0 \\ 0 & -1 & x_3-t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x_{n-1}-t & q_{n-1} \\ 0 & 0 & 0 & \cdots & -1 & x_n-t \end{vmatrix} = \sum_{i=0}^{n}(-1)^{n-i}E_i t^{n-i}$$

Quantum World

Algebraic Geometry ↔ Algebraic Combinatorics

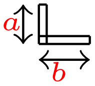
$\mathbf{q}H^*(F\ell_n)$ $\quad$ $\mathbb{Z}[x_1,\ldots,x_n][q_1,\ldots,q_{n-1}]/\langle E_1,\ldots,E_n\rangle$

$[X_w]$ Schubert classes ↔ $\mathfrak{S}^{\mathbf{q}}_w \quad w\in S_n$

Intersection $\quad$ Multiplication

Our Result

$$p^{\mathbf{q}}_r(x_1,\ldots,x_k)\mathfrak{S}^{\mathbf{q}}_u = \sum_{\substack{q^{\alpha}w:\ \exists\gamma:\ u\xrightarrow{\gamma}{}^{\mathbf{q}}_{k}\, q^{\alpha}w \\ \exists i:\ c^{i}(\gamma)\ \text{pure classic, connected} \\ \exists a,b:\ a+b=r,\ shape(c^{min}(\gamma))=a\updownarrow}} (-1)^{a-1}\mathbf{q}^{\alpha}\mathfrak{S}^{\mathbf{q}}_w$$



q-Monk's rule

THEOREM

[Fomin-Gelfand-Postnikov 1997]

$$\mathfrak{S}^{\mathbf{q}}_{(k,k+1)}\mathfrak{S}^{\mathbf{q}}_{u} = \sum_{\substack{i \le k < j \\ \ell(u(i,j))=\ell(u)+1}} \mathfrak{S}^{\mathbf{q}}_{u(i,j)} + \sum_{\substack{i \le k < j \\ \ell(u(i,j))=\ell(u)-2(j-i)+1}} q_{ij}\mathfrak{S}^{\mathbf{q}}_{u(i,j)}$$

$q_{ij} = q_i q_{i+1} \cdots q_{j-1}$

$\mathfrak{S}^{\mathbf{q}}_{(k,k+1)} = s^{\mathbf{q}}_1(x_1, \ldots, x_k) = E_1$

q-Monk's rule

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[Fomin-Gelfand-Postnikov 1997]

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$q_{ij} = q_i q_{i+1}\cdots q_{j-1}$

Quantum k-bruhat order on $S_n[q]$:

$S_n[q] := \left\{\mathbf{q}^\alpha u \ : \ u \in S_n, \quad \mathbf{q}^\alpha = q_1^{\alpha_1}\cdots q_{n-1}^{\alpha_{n-1}}\right\}$

$\mathbf{q}^\alpha u \lessdot_k^q \mathbf{q}^\alpha u(i,j)$ if $i\le k<j$ and $\ell(u(i,j))=\ell(u)+1$

$\mathbf{q}^\alpha u \lessdot_k^q q_{ij}\mathbf{q}^\alpha u(i,j)$ if $i\le k<j$ and $\ell(u(i,j))=\ell(u)-2(j-i)+1$

Quantum k-bruhat order on $S_n[q]$

$S_n[q] := \{\mathbf{q}^\alpha u \ : \ u \in S_n, \quad \mathbf{q}^\alpha = q_1^{\alpha_1} \cdots q_{n-1}^{\alpha_{n-1}}\}$

$u \lessdot_k^q u(i,j)$ if $i \leq k < j$ and $\ell(u(i,j)) = \ell(u)+1$

$u \lessdot_k^q q_{ij} u(i,j)$ if $i \leq k < j$ and $\ell(u(i,j)) = \ell(u) - 2(j-i)+1$

$$u = 3256|1748 \lessdot_3^q 3456|1728 = u(2,7) = (2,4)u \qquad (k=4)$$

$$3256|1748 \xrightarrow{24}{}_4^{\mathbf{q}} 3456|1728$$

we can swap 2 and 4 if

- 2 is to the left of k
- 4 is to the right of k
- **Nothing** in-between is in-between

$\boxed{\textbf{Classic cover}}$

Quantum k-bruhat order on $S_n[q]$

$$S_n[q] := \{\mathbf{q}^\alpha u \ : \ u \in S_n, \quad \mathbf{q}^\alpha = q_1^{\alpha_1} \cdots q_{n-1}^{\alpha_{n-1}}\}$$

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$u \lessdot_k^q q_{ij} u(i,j)$ if $i \le k < j$ and $\ell(u(i,j)) = \ell(u) - 2(j-i)+1$

$$u = 674|5312 \lessdot_3^q q_2q_3q_4 634|5712 = u(2,5) = (7,3)u \qquad (k=3)$$

$$674|5312 \xrightarrow{73}{}_k^{\mathbf{q}} q_2q_3q_4 634|5712$$

we can swap 7 and 3 if

- 7 is to the left of k
- 3 is to the right of k
- **Everything** in-between is in-between

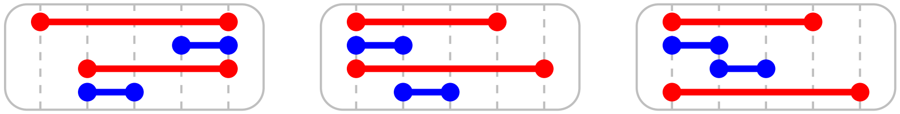
$\boxed{\text{Quantum cover}}$

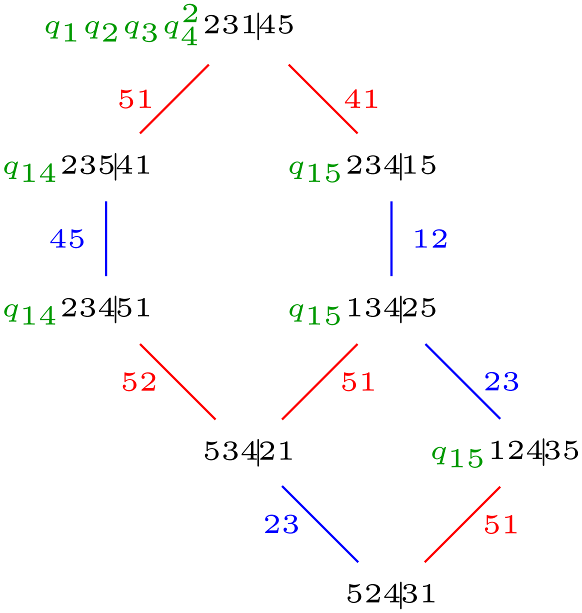
Quantum k-bruhat order on $S_n[q]$

$$S_n[q] := \{\mathbf{q}^\alpha u \ : \ u \in S_n, \quad \mathbf{q}^\alpha = q_1^{\alpha_1} \cdots q_{n-1}^{\alpha_{n-1}}\}$$

$u \lessdot_k^q u(i,j)$ if $i \le k < j$ and $\ell(u(i,j)) = \ell(u)+1$

$u \lessdot_k^q q_{ij} u(i,j)$ if $i \le k < j$ and $\ell(u(i,j)) = \ell(u) - 2(j-i)+1$





Cyclic symmetry of Quantum k-bruhat order

LEMMA

[B-B-C-S-S 2017]

$$\boxed{u \lessdot_k^q \mathbf{q}^\gamma (\alpha,\beta) u \quad\iff\quad cu \lessdot_k^q \frac{\mathbf{q}^\gamma}{\mathbf{q}(u,(\alpha,\beta)u)} c(\alpha,\beta)c^{-1}cu}$$

$c = [2,3,\ldots,n,1] \in S_n$

$$674|5312 \xrightarrow{73}{}_k^{\mathbf{q}} \quad q_2q_3q_4 \; 634|5712$$

$$715|6423 \xrightarrow{14}{}_k^{\mathbf{q}} \quad 745|6123$$

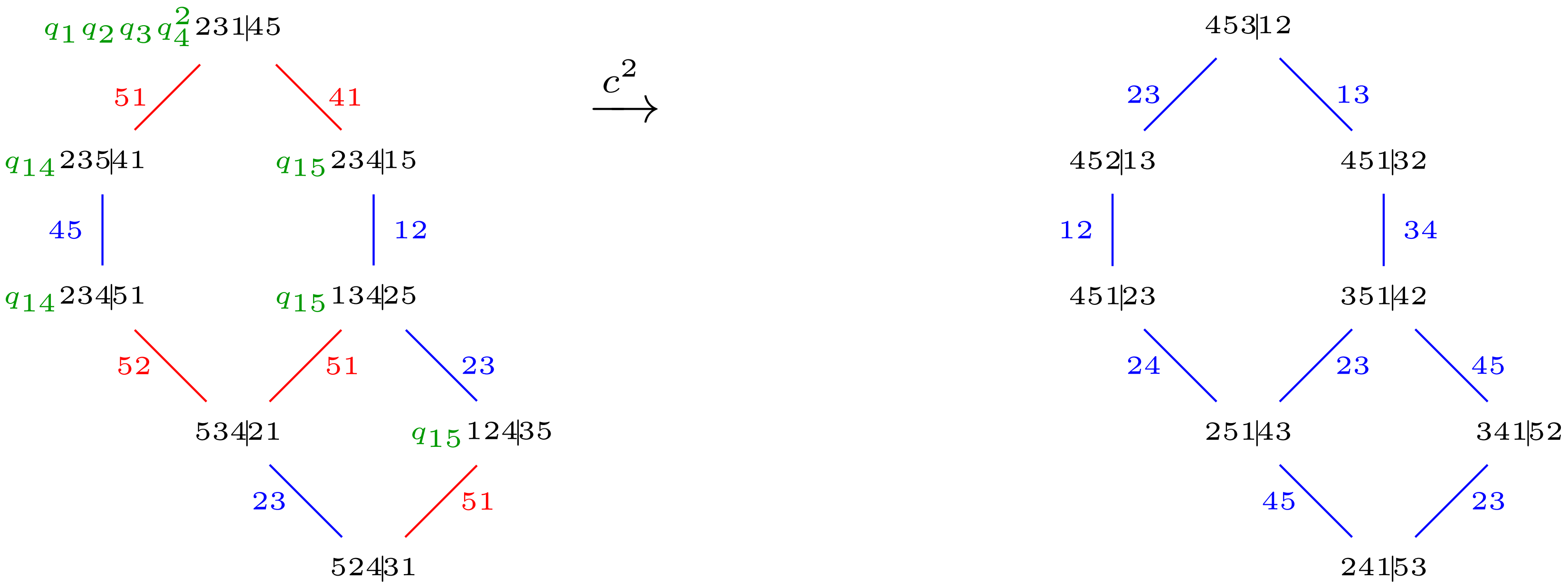
$$126|7534 \xrightarrow{25}{}_k^{\mathbf{q}} \quad 156|7234$$

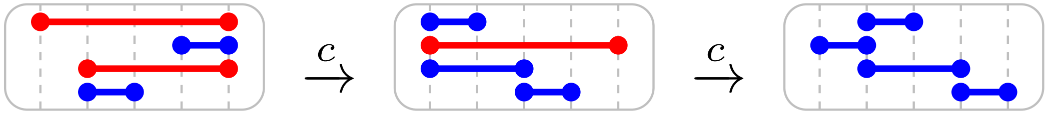
$$237|1645 \xrightarrow{36}{}_k^{\mathbf{q}} \quad 267|1345$$

$$341|2756 \xrightarrow{47}{}_k^{\mathbf{q}} \quad 371|2456$$

$$452|3167 \xrightarrow{51}{}_k^{\mathbf{q}} \quad q_2q_3q_4 \; 412|3567$$

Cyclic symmetry of Quantum k-bruhat order



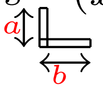
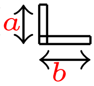


QUESTION: What intervals can be cycled to a pure classic?

Quantum Hook case...

THEOREM

[Mészáros-Panova-Postnikov 2014]

$s^{\mathbf{q}}$$(x_1,\ldots,x_k)\mathfrak{S}^{\mathbf{q}}_u \;=\; \sum_{D \text{ forest}} \binom{s(D)-1}{ht(D)-a} \sum_{D_L \in \mathcal{L}(D($$))} \mathfrak{S}^{\mathbf{q}}_{u\mathbf{u}_{D_L}}.$

Quantum Hook case...

THEOREM

[Mészáros-Panova-Postnikov 2014]

$$s^{\mathbf{q}}_{a\updownarrow\underset{\overleftrightarrow{b}}{\llcorner}}(x_1,\ldots,x_k)\mathfrak{S}^{\mathbf{q}}_u = \sum_{\boxed{D \text{ forest}}} \binom{s(D)-1}{ht(D)-a} \sum_{D_L \in \mathcal{L}(D(\boxed{a\updownarrow\underset{\overleftrightarrow{b}}{\llcorner}}))} \mathfrak{S}^{\mathbf{q}}_{u\mathbf{u}_{D_L}}.$$

THEOREM

[B-B-C-S-S 2017]

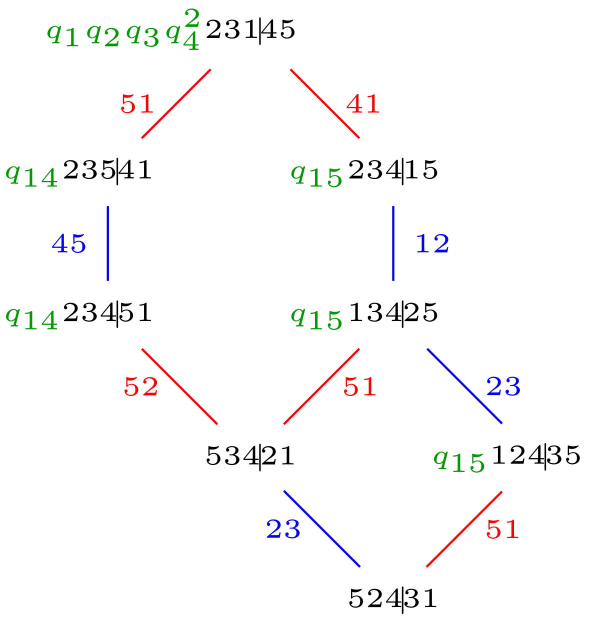
$$s^{\mathbf{q}}_{a\updownarrow\underset{\overleftrightarrow{b}}{\llcorner}}(x_1,\ldots,x_k)\mathfrak{S}^{\mathbf{q}}_u = \sum_{\substack{\gamma:\ u \xrightarrow{\gamma}{}^{\mathbf{q}}_{k} \mathbf{q}^{\alpha(\gamma)} w(\gamma) \\ \exists i:\ c^i(\gamma) \text{ pure classic} \\ shape(c^{min}(\gamma)) = a\updownarrow\underset{\overleftrightarrow{b}}{\llcorner}}} \mathbf{q}^{\alpha(\gamma)} \mathfrak{S}^{\mathbf{q}}_{w(\gamma)}$$

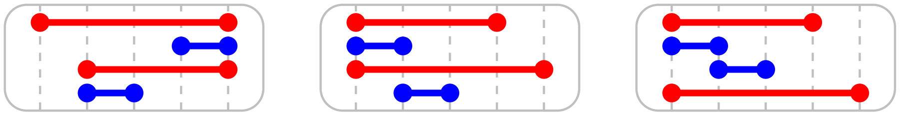
Quantum Hook case...

$$s^{\mathbf{q}}_{a\updownarrow\llcorner_{\overset{\leftrightarrow}{b}}}(x_1,\ldots,x_k)\mathfrak{S}^{\mathbf{q}}_u = \sum_{\gamma:\ u \xrightarrow{\gamma}{}^{\mathbf{q}}_{k}\, \mathbf{q}^{\alpha(\gamma)} w(\gamma)} \mathbf{q}^{\alpha(\gamma)}\mathfrak{S}^{\mathbf{q}}_{w(\gamma)}$$

$\exists i:\ c^i(\gamma)$ pure classic

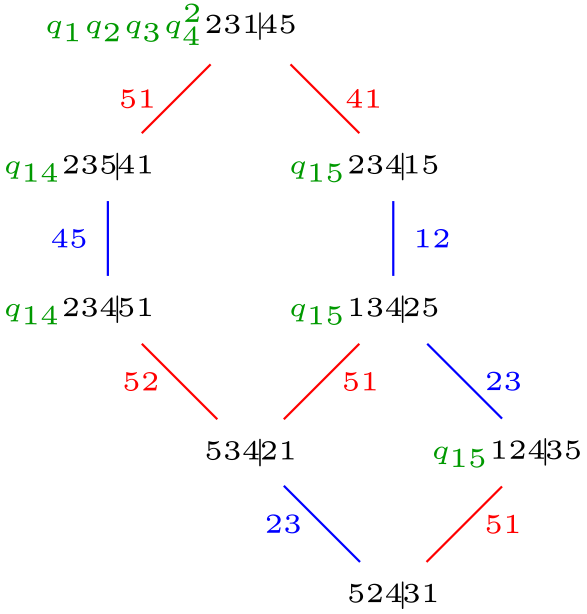
$shape(c^{min}(\gamma)) = a\updownarrow\llcorner_{\overset{\leftrightarrow}{b}}$

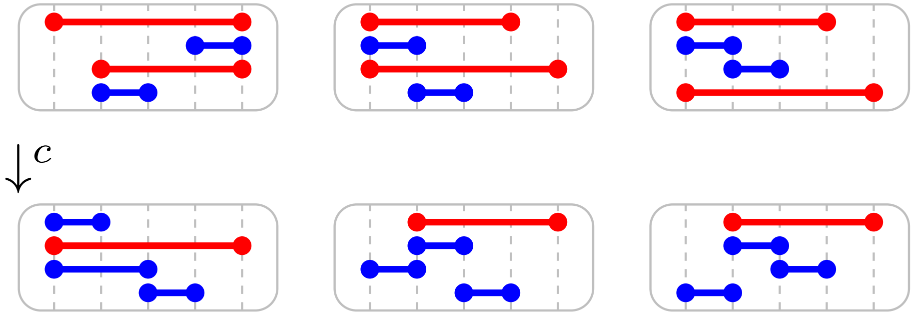




Quantum Hook case...

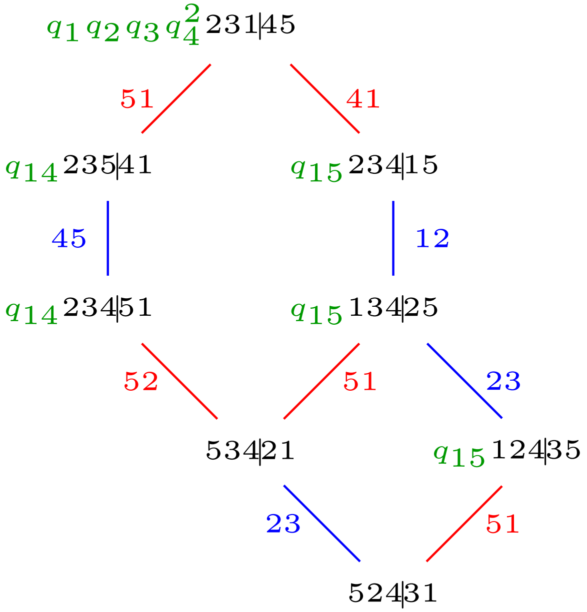
$$s^{\mathbf{q}}_{a\updownarrow\llcorner_{\underset{b}{\leftrightarrow}}}(x_1,\ldots,x_k)\mathfrak{S}^{\mathbf{q}}_u \;=\; \sum_{\substack{\gamma:\; u\xrightarrow{\gamma}{}^{\mathbf{q}}_{k}\,\mathbf{q}^{\alpha(\gamma)}w(\gamma)\\ \exists i:\; c^i(\gamma)\ \text{pure classic}\\ shape(c^{min}(\gamma))=a\updownarrow\llcorner_{\underset{b}{\leftrightarrow}}}} \mathbf{q}^{\alpha(\gamma)}\mathfrak{S}^{\mathbf{q}}_{w(\gamma)}$$

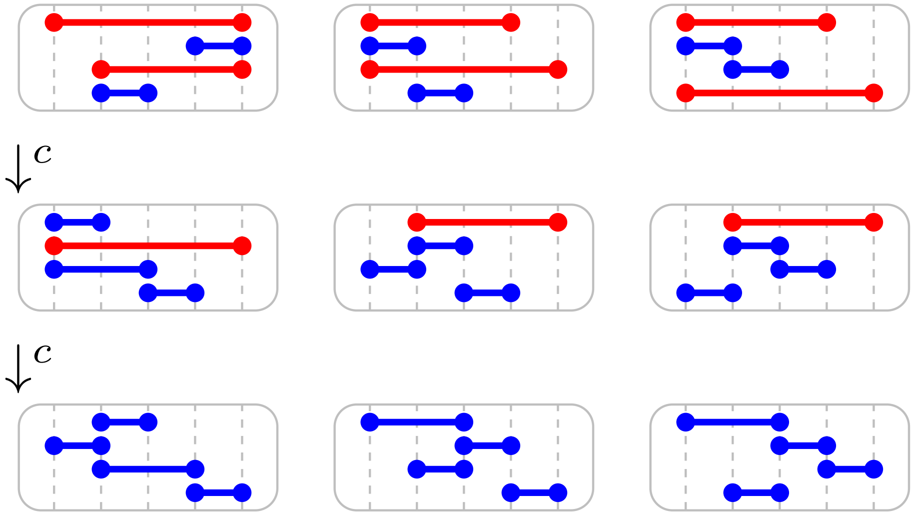




Quantum Hook case...

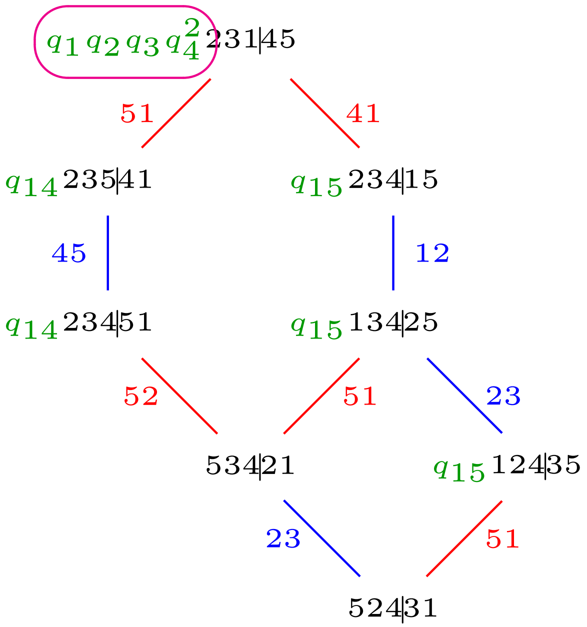
$$s^{\mathbf{q}}_{a\updownarrow\underset{\overset{\leftrightarrow}{b}}{\llcorner}}(x_1,\dots,x_k)\mathfrak{S}^{\mathbf{q}}_u = \sum_{\substack{\gamma:\ u\xrightarrow{\gamma}{}^{\mathbf{q}}_{k}\,\mathbf{q}^{\alpha(\gamma)}w(\gamma)\\ \exists i:\ c^i(\gamma)\ \text{pure classic}\\ shape(c^{min}(\gamma))=a\updownarrow\underset{\overset{\leftrightarrow}{b}}{\llcorner}}} \mathbf{q}^{\alpha(\gamma)}\mathfrak{S}^{\mathbf{q}}_{w(\gamma)}$$

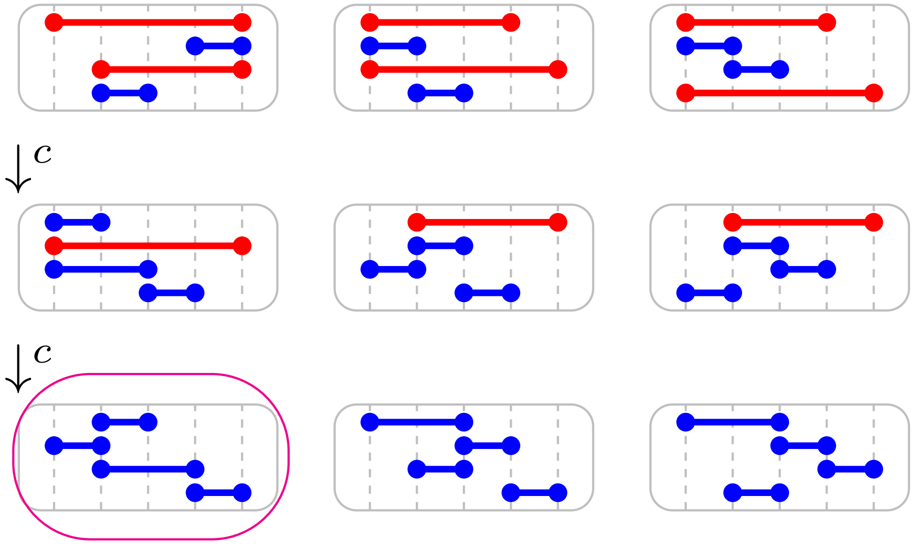




Quantum Hook case...

$$s^{\mathbf{q}}_{a\updownarrow\underset{\overset{\leftrightarrow}{b}}{\llcorner}}(x_1,\dots,x_k)\mathfrak{S}^{\mathbf{q}}_u = \sum_{\substack{\gamma:\ u\xrightarrow{\gamma}{}^{\mathbf{q}}_{k}\,\mathbf{q}^{\alpha(\gamma)}w(\gamma)\\ \exists i:\ c^i(\gamma)\ \text{pure classic}\\ shape(c^{min}(\gamma))=a\updownarrow\underset{\overset{\leftrightarrow}{b}}{\llcorner}}} \mathbf{q}^{\alpha(\gamma)}\mathfrak{S}^{\mathbf{q}}_{w(\gamma)}$$





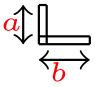
$$s^{\mathbf{q}}_{211}(x_1,x_2,x_3)\mathfrak{S}_{52431} = \mathbf{1}\,q_1q_2q_3q_4^2\mathfrak{S}_{23145} \quad + \quad \cdots$$

Quantum Murnaghan-Nakayama

THEOREM

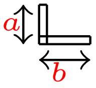
[B-B-C-S-S 2017]

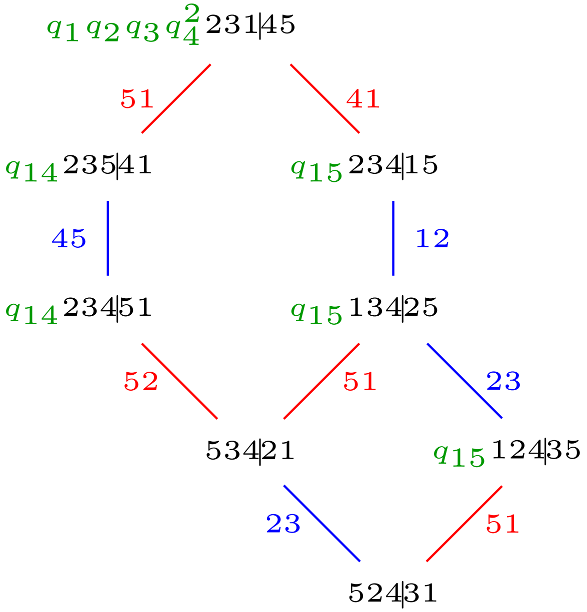
$$p_r^{\mathbf{q}}(x_1,\ldots,x_k)\mathfrak{S}_u^{\mathbf{q}} = \sum_{\substack{q^{\alpha} w:\ \exists\gamma:\ u \xrightarrow{\gamma}{}^{\mathbf{q}}_{k} q^{\alpha} w \\ \exists i:\ c^{i}(\gamma)\ \text{pure classic, connected} \\ \exists a,b:\ a+b=r,\ shape(c^{min}(\gamma))=a}} (-1)^{a-1}\mathbf{q}^{\alpha}\mathfrak{S}_w^{\mathbf{q}}$$

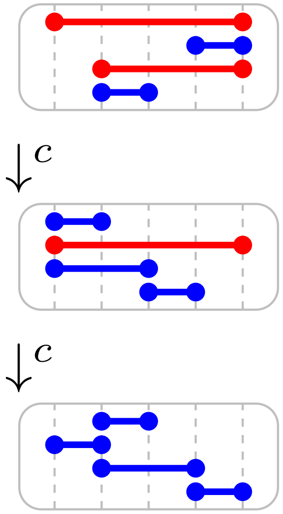


Quantum Murnaghan-Nakayama

$$p_r^{\mathbf{q}}(x_1,\dots,x_k)\mathfrak{S}_u^{\mathbf{q}} = \sum_{\substack{q^\alpha w:\ \exists\gamma:\ u\xrightarrow{\gamma}{}_k^{\mathbf{q}}\, q^\alpha w \\ \exists i:\ c^i(\gamma)\ \text{pure classic, connected} \\ \exists a,b:\ a+b=r,\ shape(c^{min}(\gamma))=}} (-1)^{a-1}\mathbf{q}^\alpha \mathfrak{S}_w^{\mathbf{q}}$$



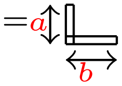




$$p_4^{\mathbf{q}}(x_1,x_2,x_3)\mathfrak{S}_{52431} = (-\mathbf{1})^{\mathbf{2}}\, q_1q_2q_3q_4^2\mathfrak{S}_{23145} \quad + \quad \cdots$$

Quantum Murnaghan-Nakayama

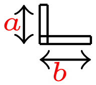
$$p_r^{\mathbf{q}}(x_1,\ldots,x_k)\mathfrak{S}_u^{\mathbf{q}} = \sum_{\substack{q^{\alpha} w:\ \exists\gamma:\ u \xrightarrow{\gamma}{}_{k}^{\mathbf{q}} q^{\alpha} w \\ \exists i:\ c^{i}(\gamma)\ \text{pure classic, connected} \\ \exists a,b:\ a+b=r,\ shape(c^{min}(\gamma))=}} (-1)^{a-1}\mathbf{q}^{\alpha}\mathfrak{S}_w^{\mathbf{q}}$$



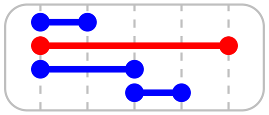
REMARKS:

This sum is over $q^{\alpha} w$ not the γ.

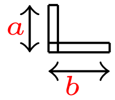
In each interval there is at MOST one chain γ that is connected, can be cycled to a pure classic and of shape

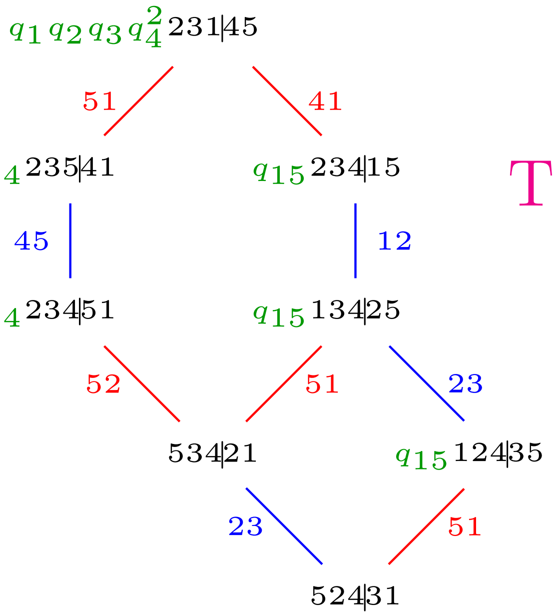


[From Bergeron-Sottile 1998]

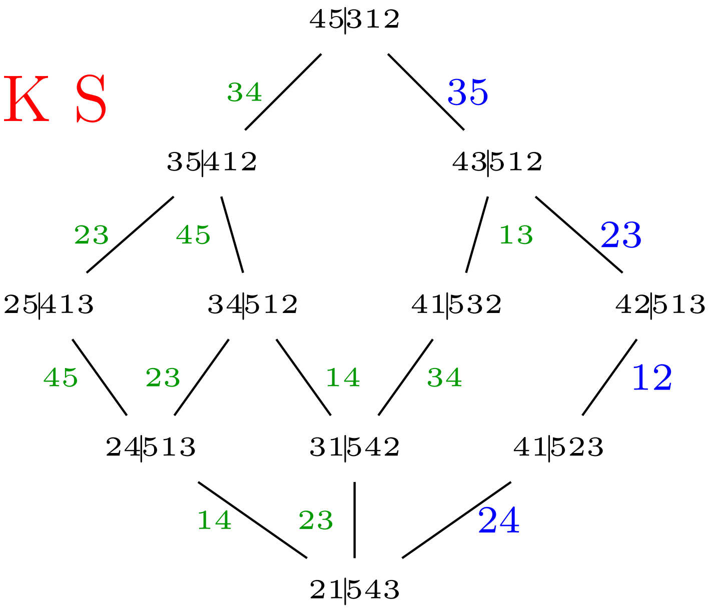


M E R C I





T H A N K S



G R A C I A S