NCSym and supercharacters table

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(with **N. Thiem**)

(... and many others)

alghero 1988

Outline

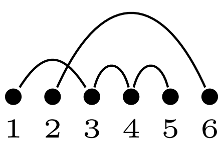
- Symmetric functions in noncommutative variables $NCSym$ a Hopf algebra
- Good Supercharacter theory of $U_n(q)$ $SC(q)$ another Hopf algebra
- Isomorphism $NCSym \cong SC(q)$.
- Supercharacter-basis
- P-basis
- Transition matrices positive LU decomposition of character table
- Some applications and enumerations

Symmetric functions in noncommutative variables

Wolf, Rosas-Sagan, Bergeron-Reutenauer-Rosas-Zabrocki...

- monomial symmetric functions m_λ basis (sum of orbit of a word)

$$x_1x_2x_1x_1x_1x_2$$

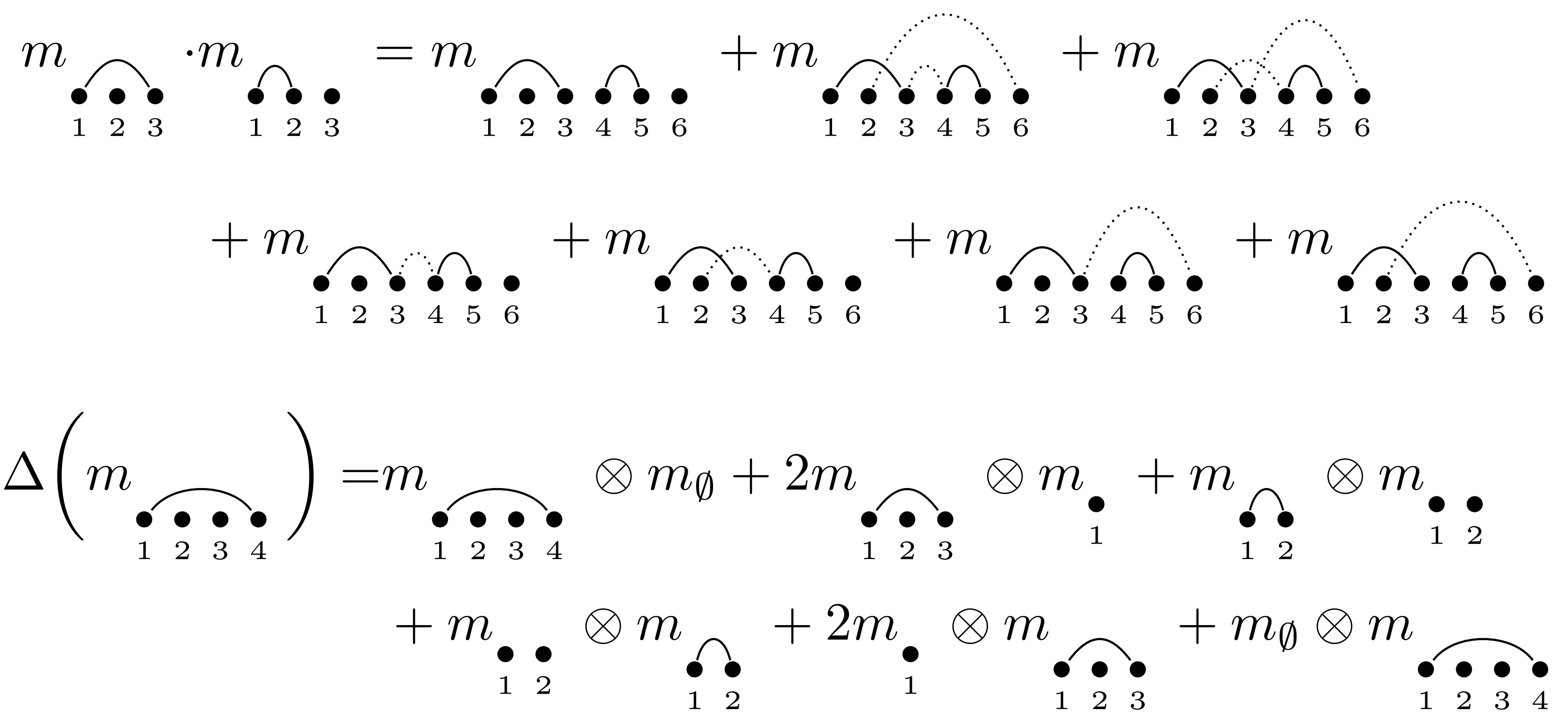
m


$$= x_1x_2x_1x_1x_1x_2 + \cdots + x_ix_jx_ix_ix_ix_j + \cdots$$

Symmetric functions in noncommutative variables

Wolf, Rosas-Sagan, Bergeron-Reutenauer-Rosas-Zabrocki...

- monomial symmetric functions m_λ basis (sum of orbit of a word)
- indexed by set partitions Hopf algebra structure is nice:



Symmetric functions in noncommutative variables

Wolf, Rosas-Sagan, Bergeron-Reutenauer-Rosas-Zabrocki...

- monomial symmetric functions m_λ basis (sum of orbit of a word)
- indexed by set partitions Hopf algebra structure is nice.
- Very nice quotient $R = \mathbb{Q}\langle x_1, x_2, \ldots, x_n \rangle$

$$R \big/ R{\cdot}NCSym^+_{(n)}$$

involving *suffix code*

- An open questions (conjecture??)

$$\dim \left(R \big/ \langle NCSym^+_{(n)} \rangle \right) < \infty ?$$

- Like *Sym*, we have that *NCSym* has representation theoretic side (Frobenious)

Supercharacter theory of $U_n(q)$

lumping W I L D conjugacy classes and characters together to get a more tame theory André, Diaconis-Isaac.

• Super classes Make a partition of the conjugacy classes. (with the identity in its own class)

• Supercharacters group together irreducibles to get

– Constant over superclases

– Form a subalgebra of the class functions of dimension the number of superclasses

– *integral valued* (this is non-standard).

Supercharacter theory of $U_n(q)$

lumping conjugacy classes and characters together to get a more tame theory André, Diaconis-Isaac.

- Unipotent upper triangular matrices over finite Fields $\mathbf{F}_q$: $U_n(q)$.
- Superclasses in $U_n(q)$:

$$A \cong B \quad \leftrightarrow \quad (A - I) = DM(B - I)N$$

superclass representative has at most one 1 in each row and column (strictly above the diagonal). INTEGRAL!

Supercharacter theory of $U_n(q)$

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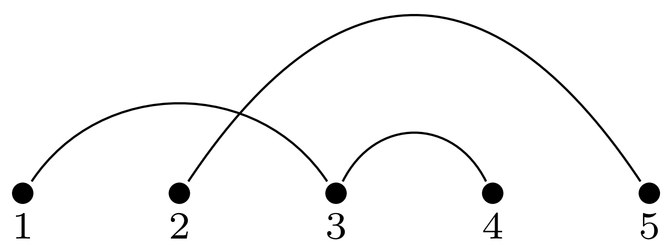
$$A \cong B \quad \leftrightarrow \quad (A-I) = DM(B-I)N$$

$$U = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 1 \\ & & 1 & 1 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \quad \Rightarrow$$



Supercharacter theory of $U_n(q)$

lumping conjugacy classes and characters together to get a more tame theory André, Diaconis-Isaac.

- Unipotent upper triangular matrices over finite Fields $\mathbf{F}_q$: $U_n(q)$.
- Superclasses in $U_n(q)$ λ
- Supercharacters χ^λ Hopf algebra structure see Adv. Math 28 authors:

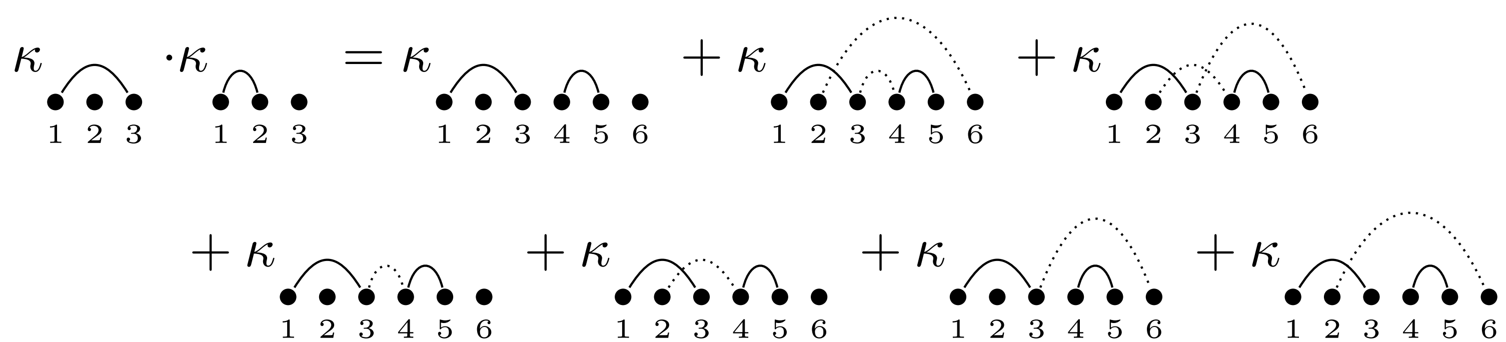
$$\Delta(\chi) = \sum_{A+B=[n]} \mathrm{Res}\,^{U_n(q)}_{U_{|A|}(q)\times U_{|B|}(q)} \chi$$

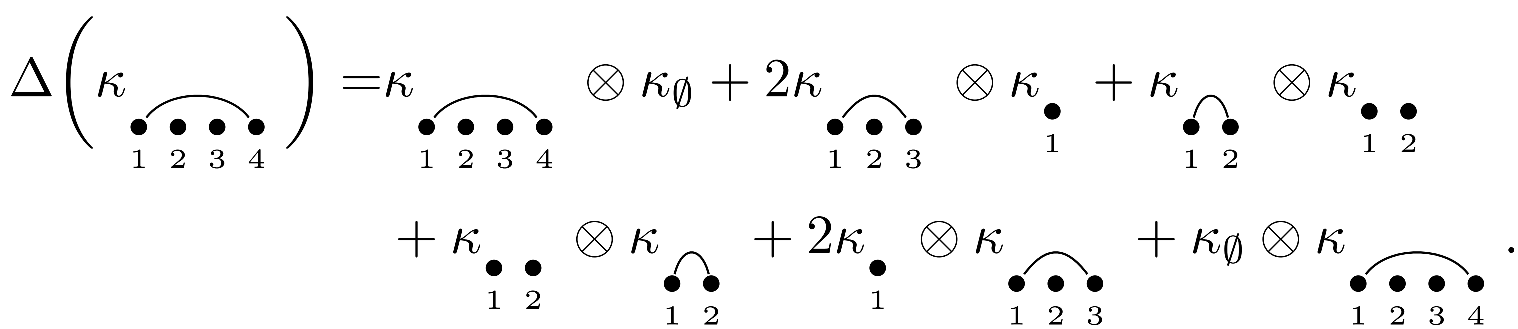
$$\chi \cdot \psi = \mathrm{Inf}\,^{U_{n+m}(q)}_{U_n(q)\times U_m(q)} \chi \otimes \psi = (\chi \otimes \psi) \circ \pi$$

where $\pi : U_{n+m}(q) \to U_n(q) \times U_m(q)$.

Supercharacter theory of $U_n(q)$

- Superclass functions κ_λ basis Hopf algebra structure is nice:





Isomorphism

Advances Math 28 authors paper

• the Hopf algebra of symmetric functions in noncommutative variables is isomorphic to the Hopf algebra of supperclass functions.

$$m_\lambda \longleftrightarrow \kappa_\lambda$$

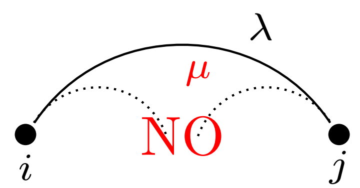
• Where is q?

– There is no q in the definition of $\kappa_\lambda = m_\lambda$ or in the Hopf structure of in term of this basis

– q appear in the definition of supercharacters:

Supercharacter basis

$$\chi^\lambda(u_\mu) = \begin{cases} \dfrac{t^{|\lambda|-|\lambda\cap\mu|} q^{\dim(\lambda)-|\lambda|}(-1)^{|\lambda\cap\mu|}}{q^{\mathrm{nst}_\mu^\lambda}} & \\ 0 & \text{otherwise,} \end{cases}$$

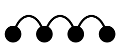
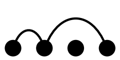
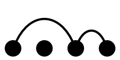
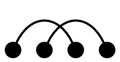


where

$$\dim(\lambda) = \sum_{i \frown j \in \lambda} j - i; \qquad t = q - 1;$$

$$\mathrm{nst}_\mu^\lambda = \#\{i < j < k < l \mid i \frown l \in \lambda, j \frown k \in \mu\}.$$

$$\chi^\lambda = \sum_\mu \chi^\lambda(u_\mu) m_\mu$$

	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	t	-1	t	t	-1	-1	t	-1	t	t	t	-1	t	t	t
	t	t	-1	t	-1	t	-1	-1	t	t	t	t	t	t	-1
	t	t	t	-1	t	-1	-1	-1	t	t	t	-1	t	t	t
	t^2	$-t$	$-t$	t^2	1	$-t$	$-t$	1	t^2	t^2	$-t$	t^2	t^2	t^2	$-t$
	t^2	$-t$	t^2	$-t$	$-t$	1	$-t$	1	t^2	t^2	$-t$	$-t$	t^2	t^2	t^2
	t^2	t^2	$-t$	$-t$	$-t$	$-t$	1	1	t^2	t^2	t^2	$-t$	t^2	t^2	$-t$
	t^3	$-t^2$	$-t^2$	$-t^2$	t	t	t	-1	t^3	t^3	$-t^2$	$-t^2$	t^3	t^3	$-t^2$
	tq	0	0	tq	0	0	0	0	$-q$	tq	0	$-q$	$-q$	tq	0
	tq	tq	0	0	0	0	0	0	tq	$-q$	$-q$	0	$-q$	tq	0
	t^2q	$-tq$	0	0	0	0	0	0	t^2q	$-tq$	q	0	$-tq$	t^2q	0
	t^2q	0	0	$-tq$	0	0	0	0	$-tq$	t^2q	0	q	$-tq$	t^2q	0
	t^2q^2	0	0	0	0	0	0	0	$-tq^2$	$-tq^2$	0	0	q^2	t^2q^2	0
	tq^2	0	tq	0	0	0	0	0	0	0	0	0	0	$-q^2$	$-q$
	t^2q^2	0	$-tq$	0	0	0	0	0	0	0	0	0	0	$-tq^2$	q

New (?) P-basis

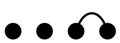
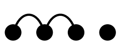
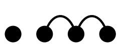
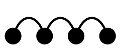
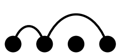
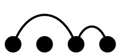
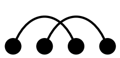
$$p_\lambda = \sum_{\mu \supseteq \lambda} m_\mu$$

- It is the basis q_λ of [B-Zabrocki].
- $p_\lambda p_\mu = p_{\lambda \cup \uparrow \mu}$, $\qquad \Delta(p_\lambda) = \sum_{\lambda = \mu \cup \nu} p_{\mathrm{st}(\mu)} \otimes p_{\mathrm{st}(\nu)}$.
- This is not the power sum basis of Sagan-Rosas: $p_\lambda = \sum_{\mu \geq \lambda} m_\mu$.

New q P-basis

$$p_\lambda(q) = \sum_{\mu \supseteq \lambda} \frac{1}{q^{\mathrm{nst}^{\lambda}_{\mu-\lambda}}} m_\mu$$

- $p_\lambda(q) p_\mu(q) = p_{\lambda \cup\!\uparrow \mu}(q)$
- $\Delta(p_\lambda(q))$ is a sligthly more complicated expression (see [Baker-Jarvis, B and T])

	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	0	1	0	0	1	1	0	1	0	0	1	0	0	0	0
	0	0	1	0	1	0	1	1	0	0	0	0	0	0	1
	0	0	0	1	0	1	1	1	0	0	0	1	0	0	0
	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0
	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0
	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0
	0	0	0	0	0	0	0	0	0	1	1	0	1	0	0
	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	1	q^{-1}
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Transition Matrices

$$p_\lambda(q) = \sum_{\mu \supseteq \lambda} \frac{1}{q^{\mathrm{nst}^{\lambda}_{\mu-\lambda}}} m_\mu$$

$$m_\mu = \sum_{\lambda \supseteq \mu} \frac{(-1)^{|\lambda-\mu|}}{q^{\mathrm{nst}^{\lambda}_{\lambda-\mu}}} p_\lambda(q).$$

Transition Matrices

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$$m_\mu = \sum_{\lambda \supseteq \mu} \frac{(-1)^{|\lambda-\mu|}}{q^{\mathrm{nst}^{\lambda}_{\lambda-\mu}}} p_\lambda(q).$$

$$\chi^\lambda = \sum_{\nu} c^{\lambda}_{\nu} p_\nu(q).$$

where c^{λ}_{ν}

$$\frac{(-1)^{|\nu|} q^{\dim(\lambda)} (q-1)^{|\lambda-\nu|}}{q^{|\lambda|+\mathrm{snst}^{\lambda}_{\nu}+\mathrm{nst}^{\nu}_{\nu}}} \Bigg(\prod_{i\frown j \in \nu\cap\lambda} (q-1) q^{\mathrm{nst}^{\lambda}_{i\frown j}} + q^{\mathrm{nst}^{\nu}_{i\frown j}} \Bigg)$$

$$\Bigg(\prod_{\substack{i\frown j \in \nu-\lambda \\ i\frown j \notin \mathrm{cflt}(\lambda)}} q^{\mathrm{nst}^{\lambda}_{i\frown j}} - q^{\mathrm{nst}^{\nu}_{i\frown j}} \Bigg).$$

Transition Matrices

$$\chi^\lambda = \sum_\nu c_\nu^\lambda \, p_\nu(q).$$

where c_ν^λ

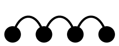
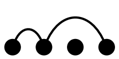
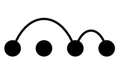
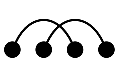
$$\frac{(-1)^{|\nu|} q^{\dim(\lambda)} (q-1)^{|\lambda-\nu|}}{q^{|\lambda|+\mathrm{snst}_\nu^\lambda+\mathrm{nst}_\nu^\nu}} \left(\prod_{i \frown j \in \nu \cap \lambda} (q-1) q^{\mathrm{nst}_{i\frown j}^\lambda} + q^{\mathrm{nst}_{i \frown j}^\nu} \right)$$

$$\left(\prod_{\substack{i\frown j \in \nu - \lambda \\ i \frown j \notin \mathrm{cflt}(\lambda)}} q^{\mathrm{nst}_{i\frown j}^\lambda} - q^{\mathrm{nst}_{i\frown j}^\nu} \right).$$

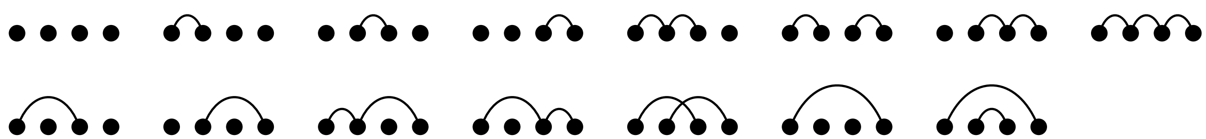
and ...

$$\mathrm{cflt}(\mu) = \{ j \frown k \mid \exists i \frown l \in \mu \text{ with } i = j < k < l \text{ or } i < j < k = l \}$$

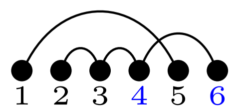
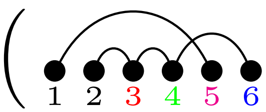
$$\mathrm{snst}_\mu^\lambda = \#\{ i < j < k < l \mid i \frown l \in \lambda, j \frown k \in \mu - \mathrm{cflt}(\lambda) \}$$

	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	t	$-q$	0	0	0	0	0	0	0	0	0	0	0	0	0
	t	0	$-q$	0	0	0	0	0	0	0	0	0	0	0	0
	t	0	0	$-q$	0	0	0	0	0	0	0	0	0	0	0
	t^2	$-tq$	$-tq$	0	q^2	0	0	0	0	0	0	0	0	0	0
	t^2	$-tq$	0	$-tq$	0	q^2	0	0	0	0	0	0	0	0	0
	t^2	0	$-tq$	$-tq$	0	0	q^2	0	0	0	0	0	0	0	0
	t^3	$-t^2q$	$-t^2q$	$-t^2q$	tq^2	tq^2	tq^2	$-q^3$	0	0	0	0	0	0	0
	tq	0	0	0	0	0	0	0	$-q^2$	0	0	0	0	0	0
	tq	0	0	0	0	0	0	0	0	$-q^2$	0	0	0	0	0
	t^2q	$-tq^2$	0	0	0	0	0	0	0	$-tq^2$	q^3	0	0	0	0
	t^2q	0	0	$-tq^2$	0	0	0	0	$-tq^2$	0	0	q^3	0	0	0
	t^2q^2	0	0	0	0	0	0	0	$-tq^3$	$-tq^3$	0	0	q^4	0	0
	tq^2	0	$-t^2q$	0	0	0	0	0	0	0	0	0	0	$-q^3$	0
	t^2q^2	0	$t(q^3-q^2+q)$	0	0	0	0	0	0	0	0	0	0	$-tq^3$	q^3

Transition Matrices



To obtain triangularity we need a special total order:

dimv $\Bigg($  $\Bigg) = (4, 2, 1, 1)$;   rnode $\Bigg($  $\Bigg) = (6, 5, 4, 3)$.

$\lambda \geq \mu$ if

(a) $\mathrm{dimv}(\lambda) >_{\mathrm{lex}} \mathrm{dimv}(\mu)$, or

(b) $\mathrm{dimv}(\lambda) = \mathrm{dimv}(\mu)$ and $\mathrm{rnode}(\lambda) \geq_{\mathrm{lex}} \mathrm{rnode}(\mu)$.

Some applications and enumerations

- New interesting basis of NCSym: $p_\lambda(q)$.

- Supercharacter table decomposition: LU-decomposition with entries $\mathbb{Z}[q]$ for L and $\mathbb{Z}[q^-1]$ for U.

- Det: $(-1)^{\sum_{\lambda\in\mathcal{P}_n}|\lambda|}\, q^{\sum_{\lambda\in\mathcal{P}_n}\dim(\lambda)-\mathrm{nst}_\lambda^\lambda}$.

- Statistic on set partitions: dim, nst, arcs, ...

$$\mathrm{dim}(n) = \sum_{\lambda\in\mathcal{P}_n} \dim(\lambda) \qquad \text{[Sloane A200580]}$$

$$\mathrm{arcs}(n) = \sum_{\lambda\in\mathcal{P}_n} |\lambda| \qquad \text{[Sloane A200660]}$$

$$\mathrm{nst}(n) = \sum_{\lambda\in\mathcal{P}_n} \mathrm{nst}_\lambda^\lambda \qquad \text{[Sloane A200673]}$$

Some applications and enumerations

Aitken's array (Sloane A011971)

$$b[1,1] = 1$$
$$b[n,1] = b[n-1,n-1]$$
$$b[n,k] = b[n,k-1] + b[n-1,k-1].$$

$$\text{arcs}(n) = \sum_{k=1}^{n-1} k \cdot b[n,k]$$

$$\text{dim}(n) = \sum_{k=1}^{n-1} k(n-k) \cdot b[n,k]$$

$$\text{nst}(n) = \sum_{j=1}^{n-3} \sum_{k=j+1}^{n-2} j(k-j) b[n,k,j].$$

Some applications and enumerations

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G R A Z I E   M I L L E