QuasiSymmetric functions
Part I

Nantel Bergeron

York Research Chair in Applied Algebra
www.math.yorku.ca/bergeron

E-Mail: **bergeron@yorku.ca**

Outline

(part I) QSym

– definition

– basis

– P-partitions

– structure (Hopf)

– Relation to others Hopf algebras

– ABS Theory

– Representation Theory of $H_n(0)$

– much much more...

Outline

(part II) Positivity

– F-positivity

– Schur positivity

– A Question of Billera

Antipode

– formula

– Multiplicity free formula?

– A proof of Stanley's acyclic theorem

Quasisymmetric functions

$QSym$ vector space of bounded degree series such that

for any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, where $\alpha_i > 0$ and $\alpha_1 + \cdots + \alpha_\ell = n$

for any $i_1 < i_2 < \cdots < i_\ell$ we have $a = b$.

$$\varphi(x_1, x_2, \ldots) = \cdots + a x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell} + \cdots + b x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell} + \cdots$$

EXAMPLEs

$M_{(1)} = x_1 + x_2 + x_3 + \cdots + x_i + \cdots$

$M_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \cdots + x_{i_1}^2 x_{i_2} + \cdots$

$M_{(1,2)} = x_1 x_2^2 + x_1 x_3^2 + x_2 x_3^2 + \cdots + x_{i_1} x_{i_2}^2 + \cdots$

Quasisymmetric functions (Basis M)

$QSym$ vector space ... Bases?

Fix a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$

where $\alpha_i > 0$ and $\alpha_1 + \cdots + \alpha_\ell = n \geq 0$

We denote this $\boxed{\alpha \vDash n}$.

Quasisymmetric functions (Basis M)

$QSym$ vector space ... Bases?

Fix a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_\ell}$$

$\bigcup_{n \geq 0} \{M_\alpha : \alpha \vDash n\}$ is a graded basis of $QSym$

$$QSym = \bigoplus_{n \geq 0} QSym_n$$

Quasisymmetric functions (Basis F)

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \vDash n$ we have $D(\alpha) \subseteq [n-1] = \{1, 2, \ldots n-1\}$:

$$\begin{array}{rcl} D : \{\alpha : \alpha \vDash n\} & \leftrightarrow & \{S : S \subseteq [n-1]\} \\ \alpha & \mapsto & \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{\ell-1}\} \end{array}$$

EXAMPLE

$(2,1,2) \mapsto \{2,3\}$

$(5) \mapsto \emptyset$

$(1,1,2,1) \mapsto \{1,2,4\}$

Quasisymmetric functions (Basis F)

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \vDash n$ we have $D(\alpha) \subseteq [n-1]$:

$$F_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n \\ k \in D(\alpha) \implies i_k < i_{k+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

Quasisymmetric functions (Basis F)

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \vDash n$ we have $D(\alpha) \subseteq [n-1]$:

$$F_\alpha = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ k \in D(\alpha) \implies i_k < i_{k+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}$$
$$= \sum_{\beta \le \alpha} M_\beta$$

So $\bigcup_{n \ge 0} \{F_\alpha : \alpha \vDash n\}$ is a graded basis of $QSym$

Quasisymmetric functions (Basis F)

For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell) \vDash n$ we have $D(\alpha) \subseteq [n-1]$:

$$F_\alpha = \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ k \in D(\alpha) \Longrightarrow i_k < i_{k+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

$$= \sum_{\beta \le \alpha} M_\beta$$

So $\bigcup_{n \ge 0} \{F_\alpha : \alpha \vDash n\}$ is a graded basis of $QSym$

... many other interesting basis... QuasiSchur, Immaculate, etc

Quasisymmetric functions (P-partition excurtion)

For P finite poset and $\gamma \colon P \to \mathbf{N} = \{1, 2, \ldots\}$.

$$\mathcal{F}_{P,\gamma} = \left\{ f \colon P \to \mathbf{N} : i <_P j \implies \left(\substack{f(i) \leq f(j) \text{ and} \\ \gamma(i) > \gamma(j) \implies f(i) < f(j)} \right) \right\}$$

We then define

$$\overline{F}_{P,\gamma} = \sum_{f \in \mathcal{F}_{P,\gamma}} \prod_{i \in P} x_{f(i)}$$

EXAMPLE

$(P, \gamma) =$ 

$$\mathcal{F}_{P,\gamma} = \{ ? \}$$

Quasisymmetric functions (P-partition excursion)

For P finite poset and $\gamma : P \to \mathbf{N} = \{1, 2, \ldots\}$.

$$\mathcal{F}_{P,\gamma} = \left\{ f : P \to \mathbf{N} : i <_P j \implies \left({f(i) \le f(j) \text{ and} \atop \gamma(i) > \gamma(j) \implies f(i) < f(j)} \right) \right\}$$

We then define

$$\overline{F}_{P,\gamma} = \sum_{f \in \mathcal{F}_{P,\gamma}} \prod_{i \in P} x_{f(i)}$$

EXAMPLE

$$(P, \gamma) = \begin{array}{cl} 2 & \\ | & < \\ 3 & \\ | & \le \\ 1 & \end{array} \qquad \mathcal{F}_{P,\gamma} = \Big\{$$

Quasisymmetric functions (P-partition excursion)

For P finite poset and $\gamma : P \to \mathbf{N} = \{1, 2, \ldots\}$.

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$$\overline{F}_{P,\gamma} = \sum_{f \in \mathcal{F}_{P,\gamma}} \prod_{i \in P} x_{f(i)}$$

EXAMPLE

$$(P,\gamma) = \begin{array}{cl} 2 & \\ \vert & < \\ 3 & \\ \vert & \leq \\ 1 & \end{array} \qquad \mathcal{F}_{P,\gamma} = \left\{ \begin{array}{c} 2 \\ 1 \\ 1 \end{array}, \right.$$

Quasisymmetric functions (P-partition excursion)

For P finite poset and $\gamma\colon P \to \mathbf{N} = \{1, 2, \ldots\}$.

$$\mathcal{F}_{P,\gamma} = \left\{ f\colon P \to \mathbf{N} : i <_P j \implies \left(\substack{f(i)\le f(j) \text{ and} \\ \gamma(i)>\gamma(j) \implies f(i)<f(j)} \right) \right\}$$

We then define

$$\overline{F}_{P,\gamma} = \sum_{f \in \mathcal{F}_{P,\gamma}} \prod_{i \in P} x_{f(i)}$$

EXAMPLE

$$(P,\gamma) = \begin{array}{cl} 2 & \\ | & < \\ 3 & \\ | & \le \\ 1 & \end{array} \qquad \mathcal{F}_{P,\gamma} = \left\{ \begin{array}{c} 2 \\ 1 \\ 1 \end{array}, \begin{array}{c} 3 \\ 1 \\ 1 \end{array}, \begin{array}{c} 3 \\ 2 \\ 1 \end{array}, \begin{array}{c} 3 \\ 2 \\ 2 \end{array}, \begin{array}{c} 4 \\ 1 \\ 1 \end{array}, \ldots \right\}$$

$$\begin{aligned} \overline{F}_{P,\gamma} &= x_1x_1x_2 + x_1x_1x_3 + x_1x_2x_3 + x_2x_2x_3 + x_1x_1x_4 + \cdots \\ &= \sum_{i_1 \le i_2 < i_3} x_{i_1}x_{i_2}x_{i_3} = F_{(2,1)} \end{aligned}$$

Quasisymmetric functions (P-partition excursion)

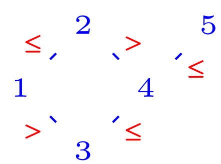
For P finite poset and $\gamma\colon P \to \mathbf{N} = \{1,2,\ldots\}$.

$$\mathcal{F}_{P,\gamma} = \left\{ f\colon P \to \mathbf{N} : i <_P j \implies \left(\substack{f(i)\le f(j) \text{ and} \\ \gamma(i)>\gamma(j) \implies f(i)<f(j)} \right) \right\}$$

We then define

$$\overline{F}_{P,\gamma} = \sum_{f\in\mathcal{F}_{P,\gamma}} \prod_{i\in P} x_{f(i)}$$

EXAMPLE

$(P,\gamma) =$ 

$$\mathcal{F}_{P,\gamma} = \left\{ \begin{smallmatrix}2&2\\1&1&1\end{smallmatrix},\ \begin{smallmatrix}2&2\\1&1&2\end{smallmatrix},\ \begin{smallmatrix}2&3\\1&1&1\end{smallmatrix},\ \begin{smallmatrix}2&3\\1&1&2\end{smallmatrix},\ \begin{smallmatrix}2&3\\1&2&2\end{smallmatrix}, \ldots \right\}$$

$$\overline{F}_{P,\gamma} = S_{(3,2)}$$

Quasisymmetric functions (P-partition excursion)

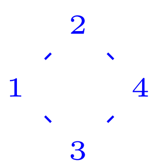
For P finite poset and $\gamma\colon P \to \mathbf{N} = \{1,2,\ldots\}$.

$$\mathcal{F}_{P,\gamma} = \left\{ f\colon P \to \mathbf{N} : i <_P j \implies \left(\substack{f(i)\leq f(j) \text{ and} \\ \gamma(i)>\gamma(j) \implies f(i)<f(j)} \right) \right\}$$

Stanley

$$\mathcal{F}_{P,\gamma} \cong \biguplus_{(L,\gamma)\in\mathcal{L}(P,\gamma)} \mathcal{F}_{L,\gamma}$$

EXAMPLE

$(P,\gamma) =$ 

$$\mathcal{F}_{P,\gamma} = \left\{ \begin{array}{|c|c|}\hline 2&2\\\hline 1&1\\\hline\end{array},\ \begin{array}{|c|c|}\hline 2&3\\\hline 1&1\\\hline\end{array},\ \begin{array}{|c|c|}\hline 2&3\\\hline 1&2\\\hline\end{array},\ \begin{array}{|c|c|}\hline 3&3\\\hline 1&1\\\hline\end{array},\ \begin{array}{|c|c|}\hline 3&3\\\hline 1&2\\\hline\end{array}, \ldots \right\}$$

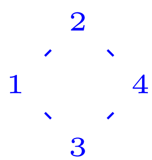
$$\mathcal{L}(P,\gamma) = \left\{ \begin{array}{c} 2\\ |\\ 4\\ |\\ 1\\ |\\ 3 \end{array},\ \begin{array}{c} 2\\ |\\ 1\\ |\\ 4\\ |\\ 3 \end{array} \right\} \quad \mathcal{F}_{L,\gamma} = \left\{ \begin{array}{c} 3\\ 2\\ 2\\ 1 \end{array}, \ldots \right\} \quad \mathcal{F}_{L',\gamma} = \left\{ \begin{array}{c} 2\\ 2\\ 1\\ 1 \end{array},\ \begin{array}{c} 3\\ 2\\ 1\\ 1 \end{array},\ \begin{array}{c} 3\\ 3\\ 1\\ 1 \end{array},\ \begin{array}{c} 3\\ 3\\ 2\\ 1 \end{array}, \ldots \right\}$$

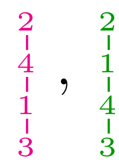
Quasisymmetric functions (POSITIVITY)

From last example we see:

$$S_\lambda = \sum_{(L,\gamma)\in\mathcal{L}(P,\gamma)} \overline{F}_{L,\gamma}$$

EXAMPLE

$(P,\gamma) =$ 

$\mathcal{L}(P,\gamma) = \{$  $\}$

$$S_{(2,2)} = F_{(1,2,1)} + F_{(2,2)}$$

$Sym \hookrightarrow QSym$ Useful to show positivity...

Quasisymmetric functions (multiplication)

Multiplication rule: $F_\alpha \cdot F_\beta$

– Find (P,γ) and (Q,ν) such that $F_\alpha = \overline{F}_{P,\gamma}$ and $F_\beta = \overline{F}_{Q,\nu}$

$$F_\alpha \cdot F_\beta = \overline{F}_{P,\gamma} \cdot \overline{F}_{Q,\nu} = \overline{F}_{P\cup Q,\gamma\cup\nu} = \sum_{(L,\gamma\cup\nu)\in\mathcal{L}(P\cup Q,\gamma\cup\nu)} \overline{F}_{L,\gamma\cup\nu}$$

EXAMPLE

$F_{(2)} \cdot F_{(1,1)}$: $\quad (P,\gamma) = \begin{smallmatrix} 2 \\ \cdot \\ 1 \end{smallmatrix} \quad (Q,\nu) = \begin{smallmatrix} 3 \\ \cdot \\ 4 \end{smallmatrix}$

$$\mathcal{L}\left(\begin{smallmatrix} 2 \\ \cdot \\ 1 \end{smallmatrix} \quad \begin{smallmatrix} 3 \\ \cdot \\ 4 \end{smallmatrix} \right) = \left\{ \begin{smallmatrix} 3 \\ | \\ 4 \\ | \\ 2 \\ | \\ 1 \end{smallmatrix}, \begin{smallmatrix} 3 \\ | \\ 2 \\ | \\ 4 \\ | \\ 1 \end{smallmatrix}, \begin{smallmatrix} 2 \\ | \\ 3 \\ | \\ 4 \\ | \\ 1 \end{smallmatrix}, \begin{smallmatrix} 3 \\ | \\ 2 \\ | \\ 1 \\ | \\ 4 \end{smallmatrix}, \begin{smallmatrix} 2 \\ | \\ 3 \\ | \\ 1 \\ | \\ 4 \end{smallmatrix}, \begin{smallmatrix} 2 \\ | \\ 1 \\ | \\ 3 \\ | \\ 4 \end{smallmatrix} \right\}$$

$$F_{(2)} \cdot F_{(1,1)} = F_{(3,1)} + F_{(2,2)} + F_{(2,1,1)} + F_{(1,3)} + F_{(1,2,1)} + F_{(1,1,2)}$$

Quasisymmetric functions (Graded Algebra)

Graded vector space: $QSym = \bigoplus_{n\geq 0} QSym_n$

Graded associative multiplication: $m\colon QSym \otimes QSym \to QSym$

$$\begin{array}{ccc} QSym \otimes QSym \otimes QSym & \xrightarrow{\;m \otimes \mathbf{1}\;} & QSym \otimes QSym \\ {\scriptstyle \mathbf{1} \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ QSym \otimes QSym & \xrightarrow{\;m\;} & QSym \end{array}$$

Graded unity: $u\colon \mathbb{Q} \to QSym$

$$u(1) = 1$$

$QSym$ is a graded algebra

Quasisymmetric functions (Graded Co-Algebra)

Graded vector space: $QSym = \bigoplus_{n \geq 0} QSym_n$

Graded associative comultiplication: $\Delta : QSym \rightarrow QSym \otimes QSym$

$$\varphi(x_1, x_2, \ldots) \in QSym$$

Let $X = x_1, x_2, \ldots$

Quasisymmetric functions (Graded Co-Algebra)

Graded vector space: $QSym = \bigoplus_{n \geq 0} QSym_n$

Graded associative comultiplication: $\Delta : QSym \to QSym \otimes QSym$

$$\varphi(X) \in QSym$$

substitute $Y + Z = y_1, y_2, \ldots, z_1, z_2, \ldots$ in φ and rewrite:

$$\varphi(Y + Z) = \sum \varphi^{(1)}(Y)\varphi^{(2)}(Z)$$

This defines for us $\Delta = \sum \varphi^{(1)} \otimes \varphi^{(2)}$

Quasisymmetric functions (Graded Co-Algebra)

Graded vector space: $QSym = \bigoplus_{n \geq 0} QSym_n$

Graded associative comultiplication: $\Delta : QSym \to QSym \otimes QSym$

EXAMPLE: $F_{(2,1)} = \overline{F}_{P,\gamma}$ where $(P,\gamma) = \begin{matrix} 2 \\ \cdot & < \\ 3 \\ \cdot & \leq \\ 1 \end{matrix}$

$$\overline{F}_{\substack{2\\3\\1}}(Y+Z) = \overline{F}_{\substack{2\\3\\1}}(Y) + \overline{F}_{\substack{3\\1}}(Y)\overline{F}_{2}(Z) + \overline{F}_{1}(Y)\overline{F}_{\substack{2\\3}}(Z) + \overline{F}_{\substack{2\\3\\1}}(Z)$$

$$\Delta(F_{(2,1)}) = F_{(2,1)} \otimes 1 + F_{(2)} \otimes F_{(1)} + F_{(1)} \otimes F_{(1,1)} + 1 \otimes F_{(2,1)}$$

Quasisymmetric functions (Graded Co-Algebra)

Graded vector space: $QSym = \bigoplus_{n\geq 0} QSym_n$

Graded associative comultiplication: $\Delta : QSym \to QSym \otimes QSym$

$\varphi((X+Y)+Z) = \varphi(X+(Y+Z))$, hence Δ is coassociative:

$$\begin{array}{ccc} QSym & \xrightarrow{\quad \Delta \quad} & QSym \otimes QSym \\ {\scriptstyle \Delta}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{1}\otimes\Delta} \\ QSym \otimes QSym & \xrightarrow{\quad \Delta\otimes\mathbf{1} \quad} & QSym \otimes QSym \otimes QSym \end{array}$$

Quasisymmetric functions (Graded Co-Algebra)

Graded vector space: $QSym = \bigoplus_{n \geq 0} QSym_n$

Graded associative comultiplication: $\Delta : QSym \to QSym \otimes QSym$

$$\begin{array}{ccc} QSym & \xrightarrow{\Delta} & QSym \otimes QSym \\ \Delta \downarrow & & \downarrow \mathbf{1} \otimes \Delta \\ QSym \otimes QSym & \xrightarrow{\Delta \otimes \mathbf{1}} & QSym \otimes QSym \otimes QSym \end{array}$$

Graded counity: $\epsilon : QSym \to \mathbb{Q}$

$$\epsilon(\varphi) = \varphi(0, 0, \ldots)$$

$QSym$ is a graded coalgebra

Quasisymmetric functions (Graded Hopf Algebra)

Graded vector space: $QSym = \bigoplus_{n\geq 0} QSym_n$

Associative multiplication: $m\colon QSym \otimes QSym \to QSym$

Unity: $u\colon \mathbb{Q} \to QSym$

Associative comultiplication: $\Delta\colon QSym \to QSym \otimes QSym$

Counity: $\epsilon\colon QSym \to \mathbb{Q}$

Compatibility: Δ and ϵ are algebra morphism

$$(f\cdot g)(Y+Z) = f(Y+Z)\cdot g(Y+Z)$$

$$(f\cdot g)(0,0,\dots) = f(0,0,\dots)\cdot g(0,0,\dots)$$

Quasisymmetric functions (Graded Hopf Algebra)

Graded vector space: $QSym = \bigoplus_{n\geq 0} QSym_n$

Associative multiplication: $m : QSym \otimes QSym \to QSym$

Unity: $u : \mathbb{Q} \to QSym$

Associative comultiplication: $\Delta : QSym \to QSym \otimes QSym$

Counity: $\epsilon : QSym \to \mathbb{Q}$

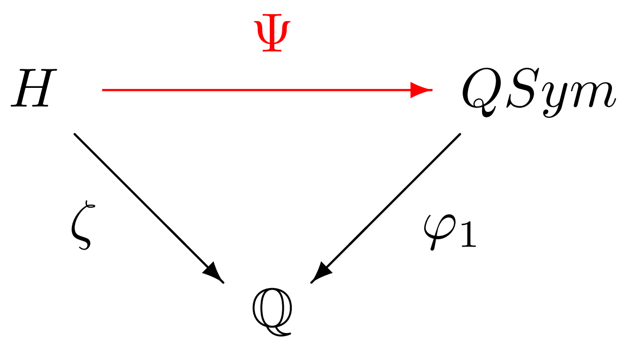
Compatibility: Δ and ϵ are algebra morphism

ANTIPODE: $S : QSym \to QSym$

Combinatorial Hopf Algebra

[Aguiar-Bergeron-Sottile]

$H = \bigoplus_{n \geq 1} H_n$ graded Hopf algebra with character $\zeta : H \to \mathbb{Q}$



where $\varphi_1(f) = f(1, 0, 0, \cdots)$.

MORE to come... part II