Why a supercharacter theoretic Hopf Monoid of set partitions?

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Nantel Bergeron

www.math.yorku.ca/bergeron
E-Mail: **bergeron@yorku.ca**

with **M. Aguiar and N. Thiem**

Outline

- Hopf monoid of supercharacters $\mathbf{SC}(q)$ (recall)

What can we do with this? conceptual and computational (primitive and antipode formulas).

- The functor $\overline{K}\colon \mathbf{Sp} \to \mathbf{gr\text{-}Vec}$.
- Antipode formulas (in Power-sum basis).
- Antipode results (in supercharacter basis).
- Construction of primitives.

Hopf Monoid

A nice Combinatorial Hopf Algebra is indexed by combinatorial objects with a lot of structure. It should have a lift at the level of species (Hopf monoid).

This explains much of the structures of Combinatorial Hopf algebras with more elegant and simplified formulas.

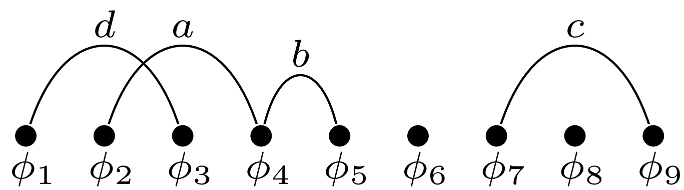
Also we have many Combinatorial Hopf algebras associated to one Hopf monoid.

Aguiar-Mahajan

Hopf Monoid of Supercharacters

- **SC** is the species of Set of arcs with Total order: (ϕ, λ)

$$(\phi, \lambda) =$$



There are several bases:

$$\mathbf{SC}[K] = \mathbb{C}[\chi_{(\phi,\lambda)}] = \mathbb{C}[\kappa_{(\phi,\lambda)}] = \mathbb{C}[P_{(\phi,\lambda)}]$$

- Supercharacter basis: $\chi_{(\phi,\lambda)}$
- Superclass function basis: $\kappa_{(\phi,\lambda)}$
- Power sum basis: $P_{(\phi,\lambda)} = \sum_{\mu \subset \lambda} \kappa_{(\phi,\mu)}$

Hopf Monoid of Supercharacters

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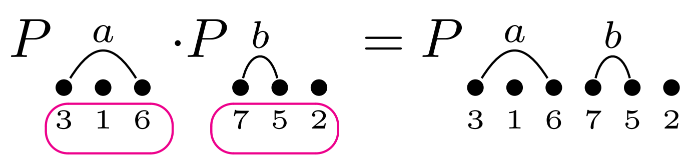
Given $I \sqcup J = K$

(1) $P_{(\phi,\lambda)} \cdot P_{(\tau,\nu)} = \mathrm{Inf}_{U_I^{\phi}(q) \times U_J^{\tau}(q)}^{U_K^{\phi\tau}(q)} \left(P_{(\phi,\lambda)} \otimes P_{(\tau,\nu)}\right) = P_{(\phi\tau,\lambda\cup\nu)}$

This is the same as for supercharacter theory but with a twist:



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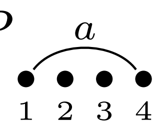
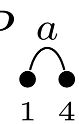
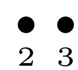
(2) $\Delta_{I,J}(P_{(\phi,\lambda)}) = \mathrm{Res}_{U_I^{\phi|_I}(q) \times U_J^{\phi|_J}(q)}^{U_K^{\phi}(q)} (P_{(\phi,\mu)})$

$$= \begin{cases} P_{(\phi|_I,\mu_I)} \otimes P_{(\phi|_J,\mu_J)} & \text{if } \mu = \mu_I \cup \mu_J, \\ 0 & \text{otherwise.} \end{cases}$$

This is much simpler than a usual comultiplication:

$\Delta_{\{1,4\},\{2,3\}}\big(P$ 

$\big) = P$ 

$\otimes P$ 

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(3) "Check" that all axioms are fulfilled.

The functor $\overline{K}$: **Sp** $\to$ **gr-Vec** .

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$$\overline{K}[\mathbf{G}] = \bigoplus_{n\geq 0} \mathbf{G}[n] \Big/ \langle \phi - \sigma[\phi] : \sigma \in S_n, \phi \in \mathbf{G}[n] \rangle$$

This functor send Hopf Monoids to graded Hopf algebras and preserve all structures (including antipode).

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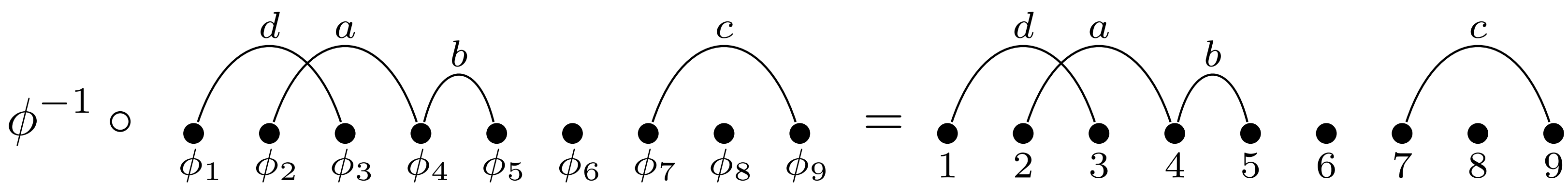
$$\mathbf{SC} \mapsto SC$$

$$P_{(\phi,\lambda)} \mapsto P_{\phi^{-1}\circ\lambda}$$

Here



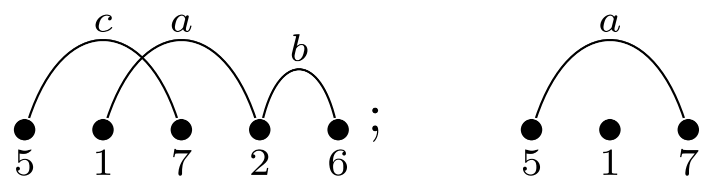
The antipode formula .

Some definitions: Atomic

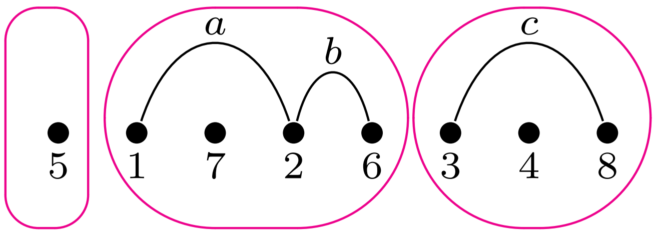
We say that (ϕ, λ) is atomic if

$$(\phi, \lambda) = (\pi\tau, \mu \cup \nu) \implies \pi = \emptyset \text{ or } \tau = \emptyset$$

Atomics:



NON Atomic:



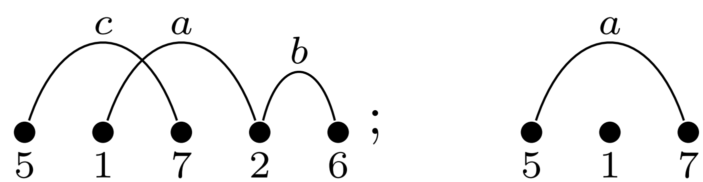
The antipode formula .

Some definitions: Atomic; Increasing with respect to ϕ

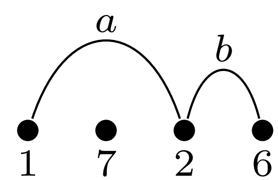
We say that (τ, λ) is increasing with respect to ϕ if

$$i \overset{a}{\frown} j \in \lambda \text{ where } i <_\tau j \qquad \Longrightarrow \qquad i <_\phi j$$

Increasing with respect to $\phi = (1, 2, 6, 5, 7)$:



NON increasing with respect to $\phi = (1, 6, 2, 7)$:



The antipode formula .

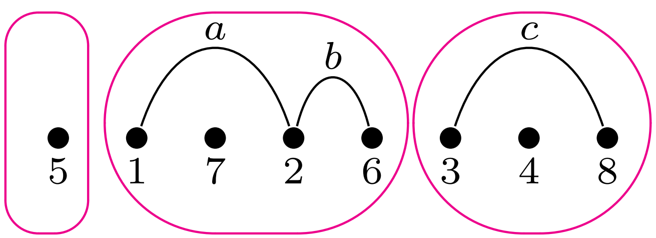
Some definitions: Atomic; Increasing with respect to ϕ; Increasing Atomic with respect to ϕ

We say that (τ, λ) is increasing atomic with respect to ϕ if (τ, λ) is increasing and for the unique factorization into atomics

$$(\tau, \lambda) = (\tau_1 \tau_2 \cdots \tau_k, \lambda^{(1)} \cup \lambda^{(2)} \cup \cdots \cup \lambda^{(k)})$$

we have that $\tau = \tau_1 \tau_2 \cdots \tau_k$ is also the unique factorization into maximal rising subsequences with respect to ϕ

Increasing atomic with respect to $\phi = (1, 5, 7, 3, 4, 2, 6, 8)$



But NOT increasing atomic with respect to $\phi = (1, 5, 3, 4, 2, 6, 7, 8)$

The antipode formula .

Some definitions: Atomic; Increasing with respect to ϕ; Increasing Atomic with respect to ϕ

THEOREM

$$S(P_{(\phi,\lambda)}) = \sum_{(\tau,\lambda)\ \text{increasing atomic}} (-1)^k P_{(\tau,\lambda)}$$

where $\tau = \tau_1\tau_2\cdots\tau_k$ is the unique factorization into maximal increasing subsequences with respect to ϕ.

REMARK This formula is multiplicity and cancellation free.

REMARK Under the functor $\overline{K}$ we get a cancellation free formula:

$$S(P_\lambda) = \sum_{\mu} (-1)^k c_{\lambda,\mu} P_\mu$$

where

$$c_{\lambda,\mu} = \{\tau \in S_n : (\tau,\lambda) \text{ increasing atomic wrt Id}, \mu = \tau^{-1} \circ \lambda\}$$

The antipode for supercharacter .

Given $(\phi,\lambda)=(\phi_1\phi_2\cdots\phi_k,\lambda^{(1)}\cup\lambda^{(2)}\cup\cdots\cup\lambda^{(k)})$ maximal atomic decomposition

THEOREM

$$S(\chi^{(\phi,\lambda)})=(-1)^k S(\chi^{(\phi_k\cdots\phi_2\phi_1,\lambda)})+\text{lower term in some order}$$

This answers a question from our AIM workshop

THEOREM For $\tau=(1,2,...,n)$

$$S(\chi^{(\tau,1\overset{a}{\frown}n)})=\sum_{\phi\in L[n]}\ \sum_{\mu\in\mathcal{S}_q^{\phi}[n]\cap\mathcal{S}_q^{\tau}[n]} z^{\mu}_{\phi,\tau}(q-1)\chi^{(\phi,\mu)},$$

The antipode for supercharacter .

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We have an exact expression for this... but it is a nice mess

Primitive (if time left)

Given Any Hopf Monoid $\mathbf{H}$ and for $k \in K$, we define

$$\Psi_{(k,K)} = \sum_{\substack{J=(J_1|\ldots|J_\ell)\models K \\ k\in J_1}} (-1)^{\ell-1} m_J \circ \Delta_J$$

THEOREM We have a projection

$$\Psi_{(k,K)} : \mathbf{H}[K] \to \mathcal{P}(\mathbf{H})[K]$$

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THEOREM The $\Psi_{(k,K)}\big(P_{(\phi,\lambda)}\big)$ for atomic (ϕ,λ) are algebraically independent generators of $\mathcal{P}(\mathbf{SC})$.

COROLLARY [Lauve-Mastnak] atomic $\overline{\mathbf{K}}\Big[\Psi_{(k,K)}\big(P_{(\phi,\lambda)}\big)\Big]$ are algebraically independent generators of the primitives of SC.