Hopf (pipe) dreams

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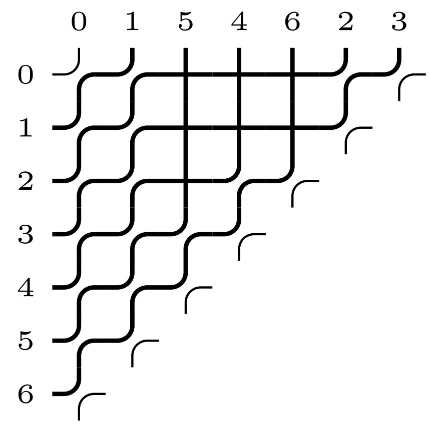
Outline

- Pipe Dreams
- Hopf Algebra of pipe dreams
- some Hopf subalgebra and enumarations

Pipe Dreams



A pipe dream $P \in \mathbf{\Pi}_6$ where $\omega_P = [1,5,4,6,2,3]$.

$$\omega\colon \bigoplus_{n\geq 0} \mathbf{\Pi}_n \longrightarrow \bigoplus_{n\geq 0} S_n$$

comultiplication of Pipe Dreams

$$\Delta_n\colon \quad \mathbf{\Pi}_n \quad \to \quad \bigoplus_{\gamma=0}^{n} \mathbf{\Pi}_\gamma \otimes \mathbf{\Pi}_{n-\gamma}$$

$$P \quad \mapsto \quad \sum_{\gamma \in GD(\omega_P)\cup\{0,n\}} \Delta_{\gamma,n-\gamma}(P)$$

comultiplication of Pipe Dreams

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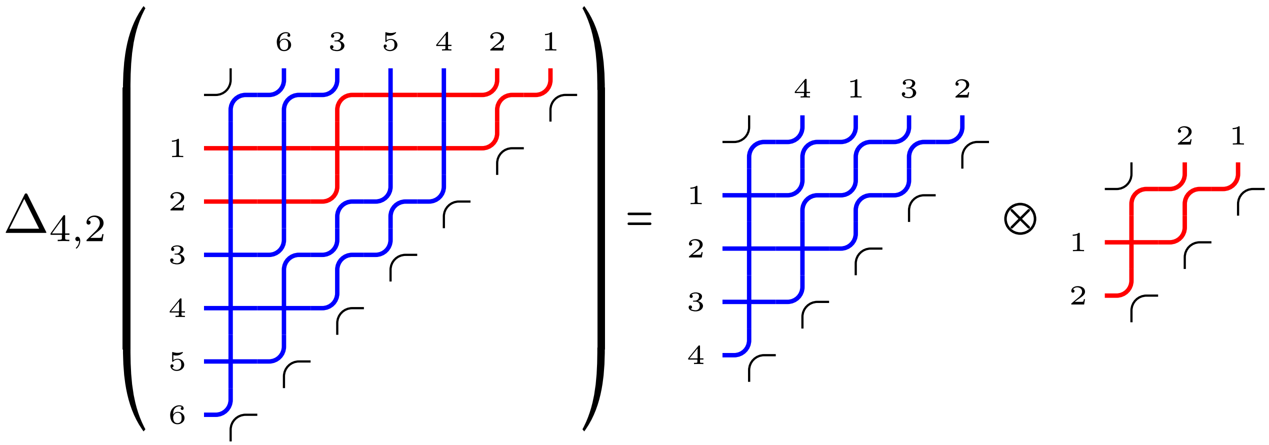
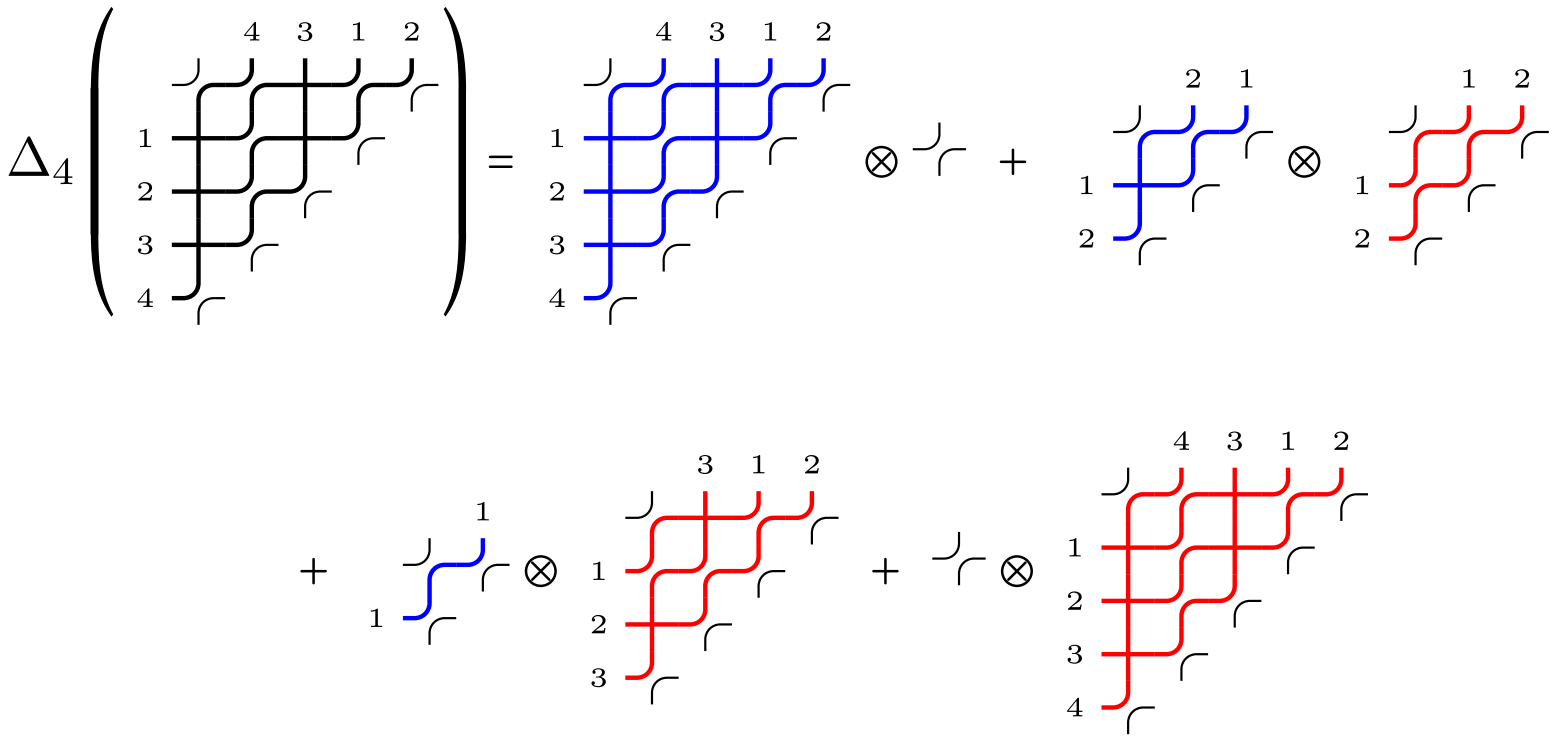


comultiplication of Pipe Dreams



comultiplication on permutation

$$\Delta_{\gamma,n-\gamma}\colon \quad \mathbf{\Pi}_n \quad \to \quad \mathbf{\Pi}_\gamma \otimes \mathbf{\Pi}_{n-\gamma}$$

comultiplication on permutation

$$\begin{array}{rccc} \Delta_{\gamma,n-\gamma}\colon & \mathbf{\Pi}_n & \rightarrow & \mathbf{\Pi}_\gamma \otimes \mathbf{\Pi}_{n-\gamma} \\ & \downarrow^{\omega} & & \downarrow^{\omega\otimes\omega} \\ & S_n & \rightarrow & S_\gamma \otimes S_{n-\gamma} \end{array}$$

comultiplication on permutation

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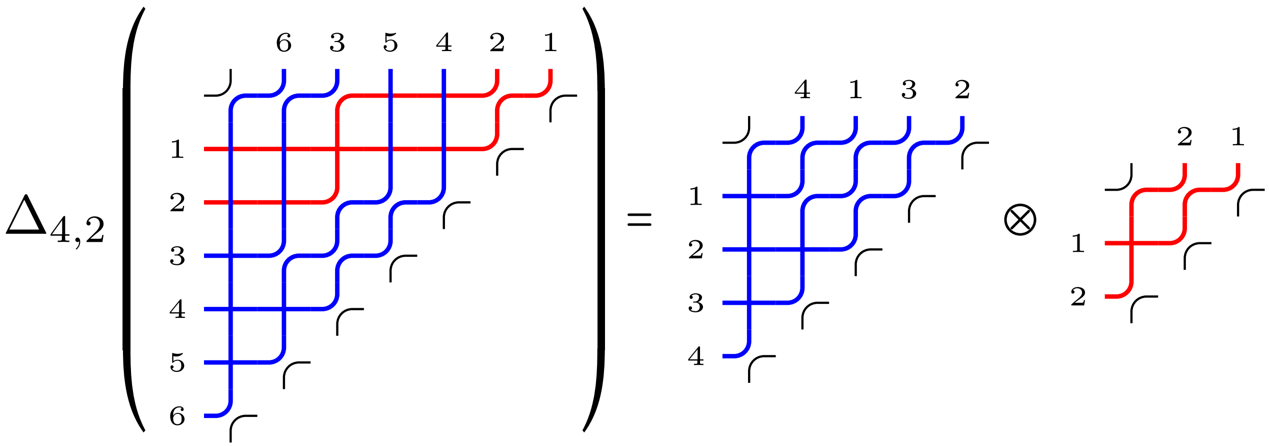
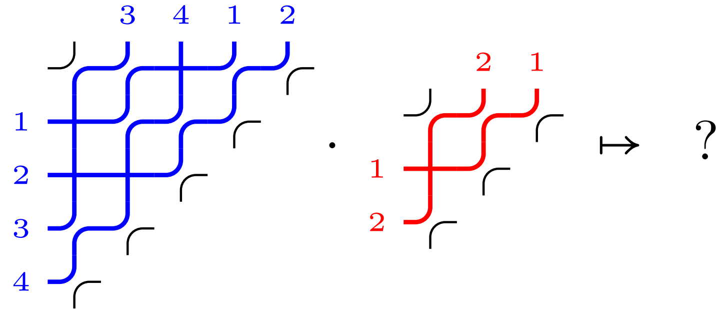


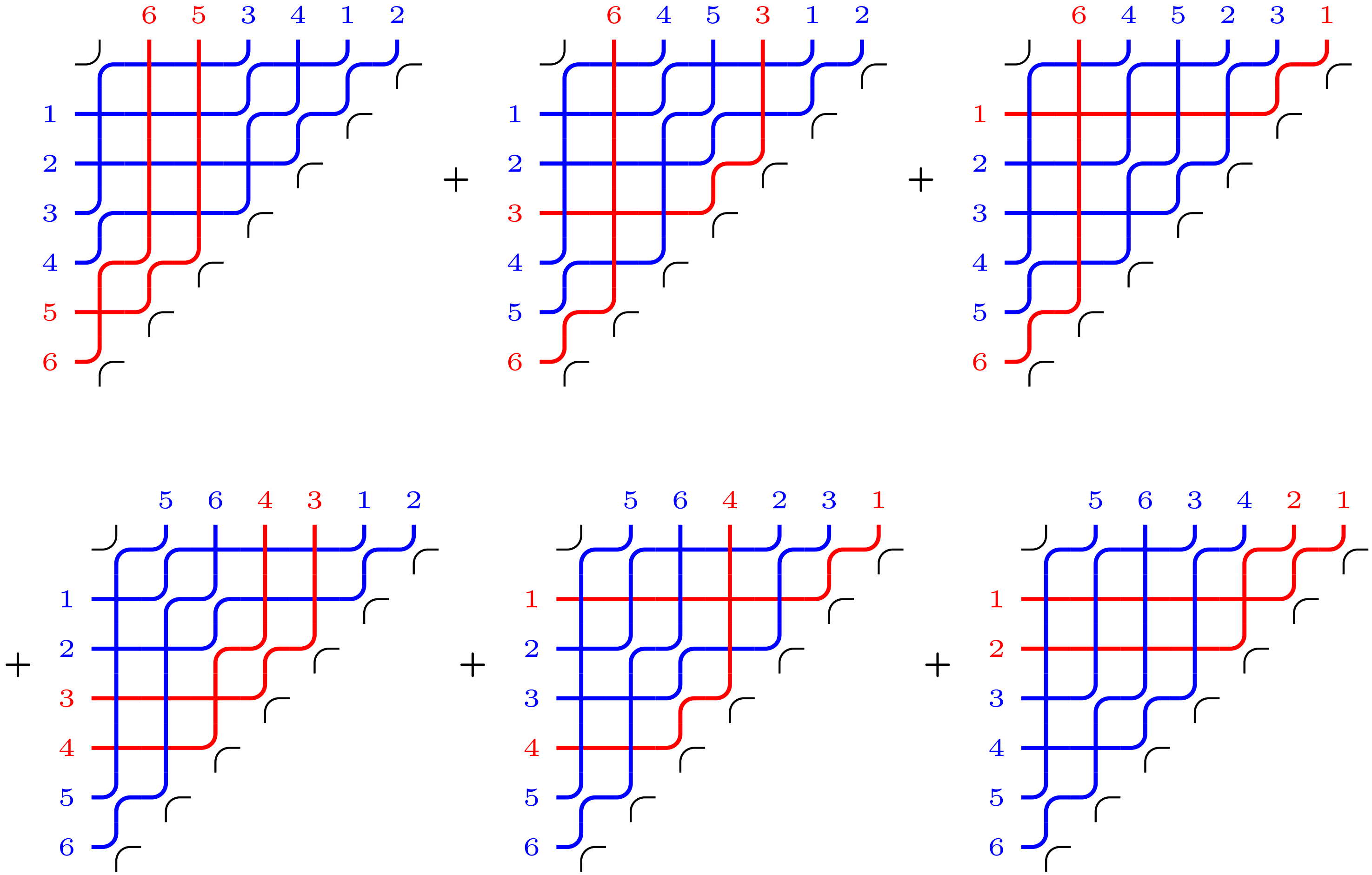
$$635421 \quad \mapsto \quad 4132 \otimes 21$$

multiplication of Pipe Dreams

$$\mu_{r,s}\colon \quad \mathbf{\Pi}_r \otimes \mathbf{\Pi}_s \quad \to \quad \mathbf{\Pi}_{r+s}$$
$$P \cdot Q \quad \mapsto \quad P \curvearrowleft \Delta^{GD(\omega_P)+1}(Q)$$



multiplication of Pipe Dreams



multiplication of permutations

$$\begin{array}{cccc} \mu_{r,s}\colon & \mathbf{\Pi}_r \otimes \mathbf{\Pi}_s & \to & \mathbf{\Pi}_{r+s} \\ & \downarrow^{\omega\otimes\omega} & & \downarrow^{\omega} \\ & S_r \otimes S_s & \to & S_{r+s} \\ & u \cdot v & \mapsto & u \curvearrowleft \Delta^{GD(u)+1}(v) \end{array}$$

$3412 \cdot 21 \mapsto \ ?$

multiplication of permutations

$$\begin{array}{llcc} \mu_{r,s}\colon & \mathbf{\Pi}_r \otimes \mathbf{\Pi}_s & \to & \mathbf{\Pi}_{r+s} \\ & \downarrow^{\omega\otimes\omega} & & \downarrow^{\omega} \\ & S_r \otimes S_s & \to & S_{r+s} \\ & u \cdot v & \mapsto & u \curvearrowleft \Delta^{GD(u)+1}(v) \end{array}$$

$3412 \cdot 21 \mapsto\ ?$

$3412 = 12 \bullet 12 \qquad\qquad 21 = 1 \bullet 1$

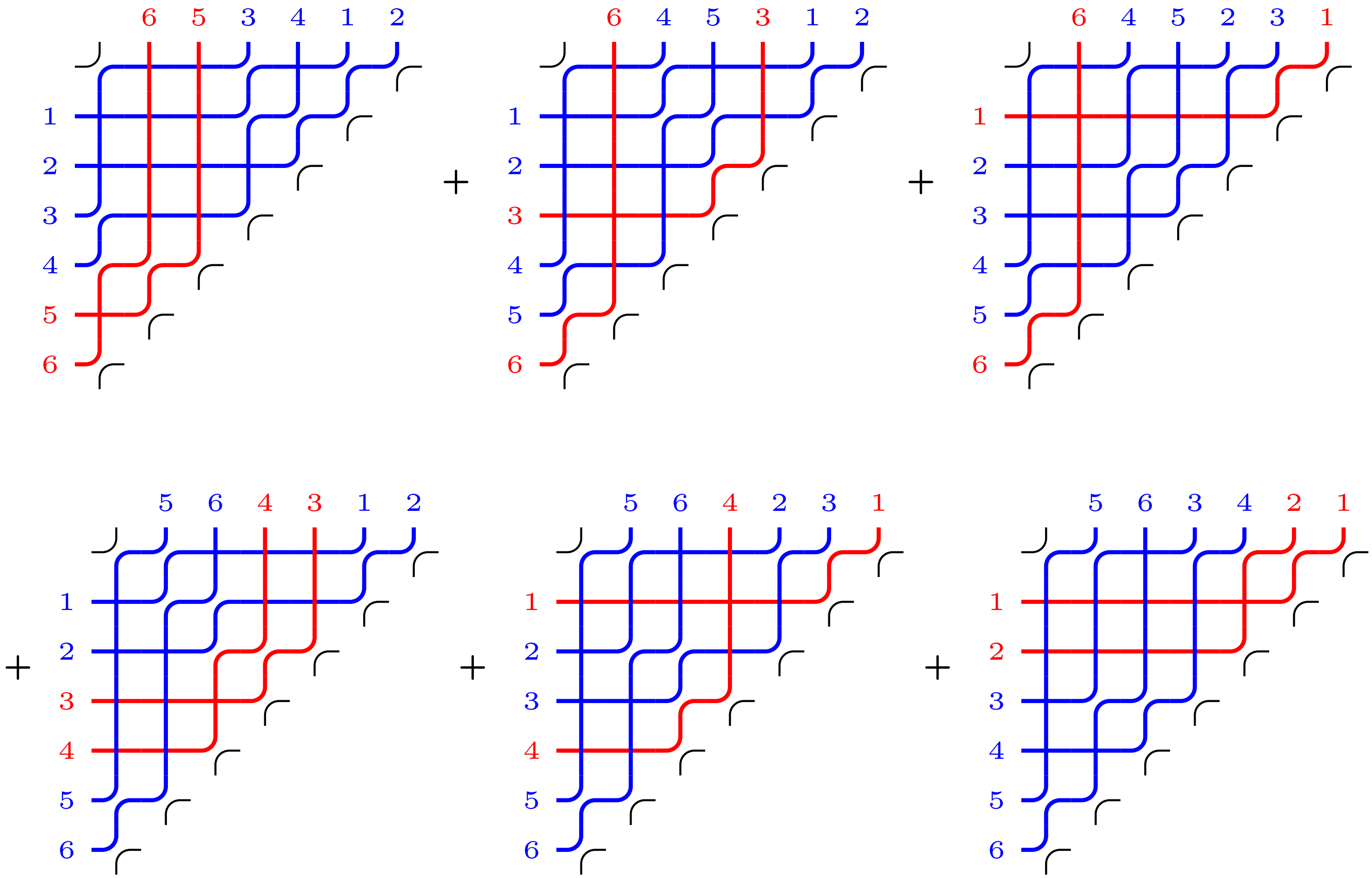
multiplication of permutations

$$\begin{array}{cccc} \mu_{r,s}\colon & \mathbf{\Pi}_r \otimes \mathbf{\Pi}_s & \to & \mathbf{\Pi}_{r+s} \\ & \big\downarrow^{\omega\otimes\omega} & & \big\downarrow^{\omega} \\ & S_r \otimes S_s & \to & S_{r+s} \end{array}$$

$$\begin{aligned} 3412 \cdot 21 = (12 \bullet 12) \sqcup\!\sqcup_{\bullet} (1 \bullet 1) \\ = \ 12 \bullet 12 \bullet 1 \bullet 1 \ + \ 12 \bullet 1 \bullet 12 \bullet 1 \ + \ 1 \bullet 12 \bullet 12 \bullet 1 \\ + \ 12 \bullet 1 \bullet 1 \bullet 12 \ + \ 1 \bullet 12 \bullet 1 \bullet 12 \ + \ 1 \bullet 1 \bullet 12 \bullet 12 \\ = \ 563421 \ + \ 564231 \ + \ 645231 \\ + \ 564312 \ + \ 645312 \ + \ 653412 \end{aligned}$$

multiplication of permutations

$$3412 \cdot 21 = 563421 + 564231 + 645231 + 564312 + 645312 + 653412$$



Two Hopf Algebras

$\mathbf{k\Pi} = \bigoplus_{n\geq 0} \mathbf{k\Pi}_n$

$\mathbf{kS} = \bigoplus_{n\geq 0} \mathbf{k}S_n$

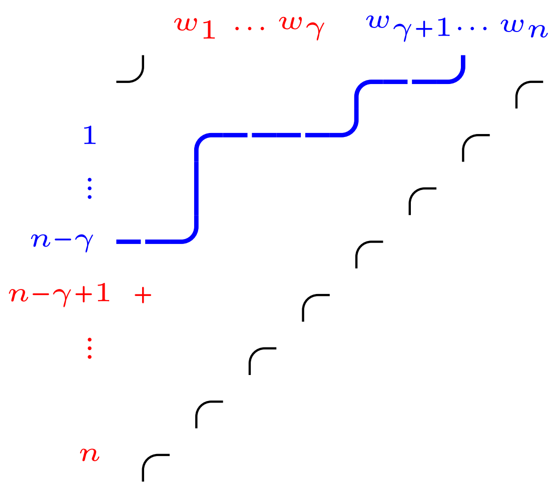
$$\omega\colon \mathbf{k\Pi} \to \mathbf{kS}$$

$\mathbf{k\Pi}$: new

$\mathbf{kS}$: new? What is relation with Malvenuto-Reutenauer

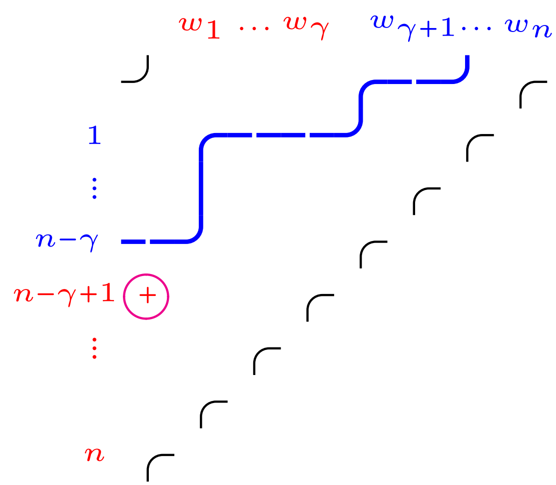
kΠ is free and cofree

Generators:



kΠ is free and cofree

Generators:



$\mathbf{k\Pi}_S$ for a (fixed) set of atomics S

Given a set of atomics S [atomic=no global descent]

$$\mathbf{\Pi}_S = \{P \in \mathbf{\Pi} : Atomics(\omega_P) \subseteq S\}$$

Example S={1}: $\mathbf{k\Pi}_{\{1\}} \cong$ Loday-Ronco Hopf Algebra

— dim deg n = C_{n+1} (Catalan)

— number of generators deg n = C_n

$\mathbf{k\Pi}_S$ for a (fixed) set of atomics S

Given a set of atomics S [atomic=no global descent]

$$\mathbf{\Pi}_S = \{P \in \mathbf{\Pi} : Atomics(\omega_P) \subseteq S\}$$

Example S={12}: $\mathbf{k\Pi}_{\{12\}}$

— number of generators deg n = C_{2n+1}

$\mathbf{k\Pi}_S$ for a (fixed) set of atomics S

Given a set of atomics S [atomic=no global descent]

$$\mathbf{\Pi}_S = \{P \in \mathbf{\Pi} : Atomics(\omega_P) \subseteq S\}$$

Example S={213}: $\mathbf{k\Pi}_{\{213\}}$

— number of generators deg n = C_{3n+2}

$\mathbf{k\Pi}_S$ for a (fixed) set of atomics S

Given a set of atomics S [atomic=no global descent]

$$\mathbf{\Pi}_S = \{P \in \mathbf{\Pi} : Atomics(\omega_P) \subseteq S\}$$

Example S={3214}: $\mathbf{k\Pi}_{\{3214\}}$

— number of generators deg n = C_{4n+3}

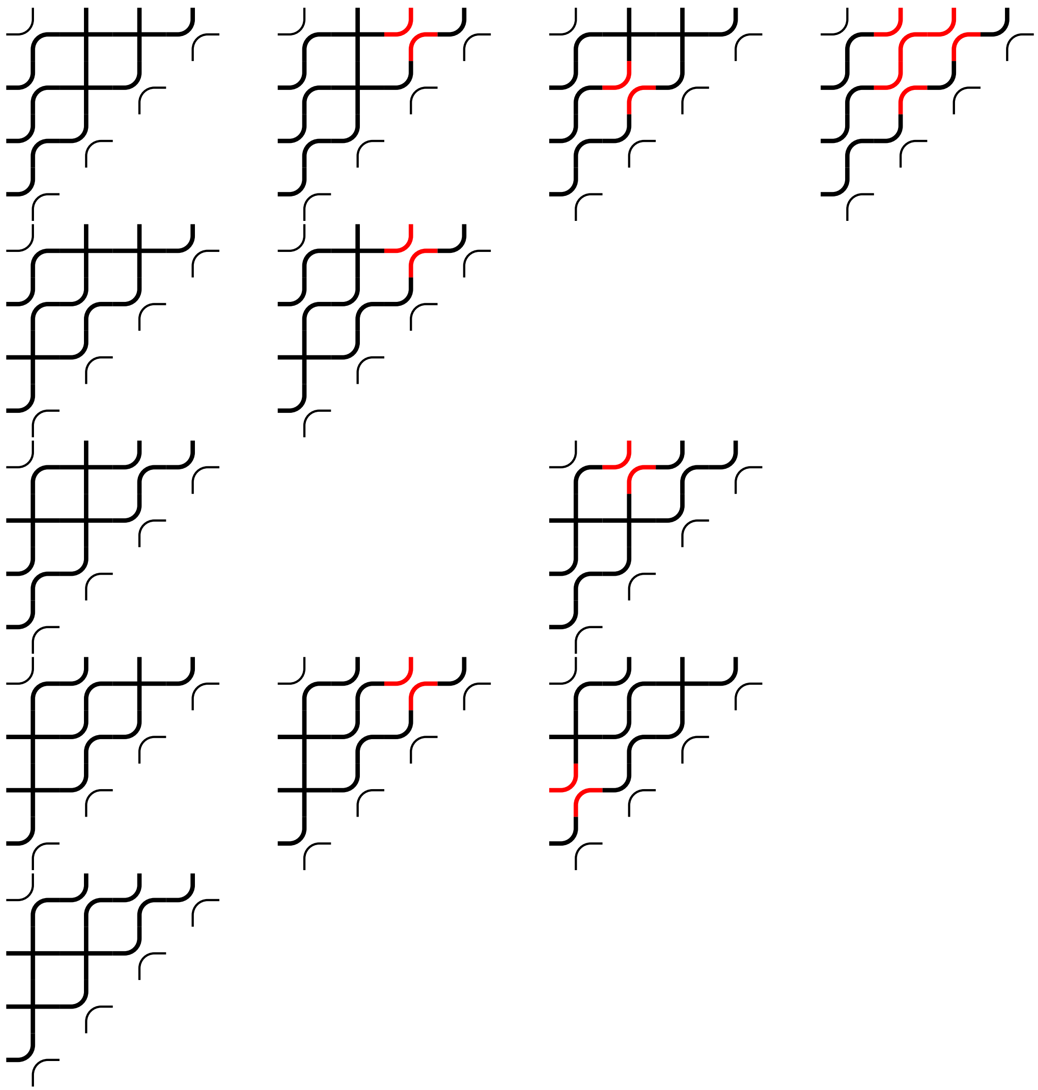
$\mathbf{k\Pi}_S$ for $S = \{1, 12, 123, \ldots\}$

Example S={1,12,123,...}: $\mathbf{k\Pi}_S$

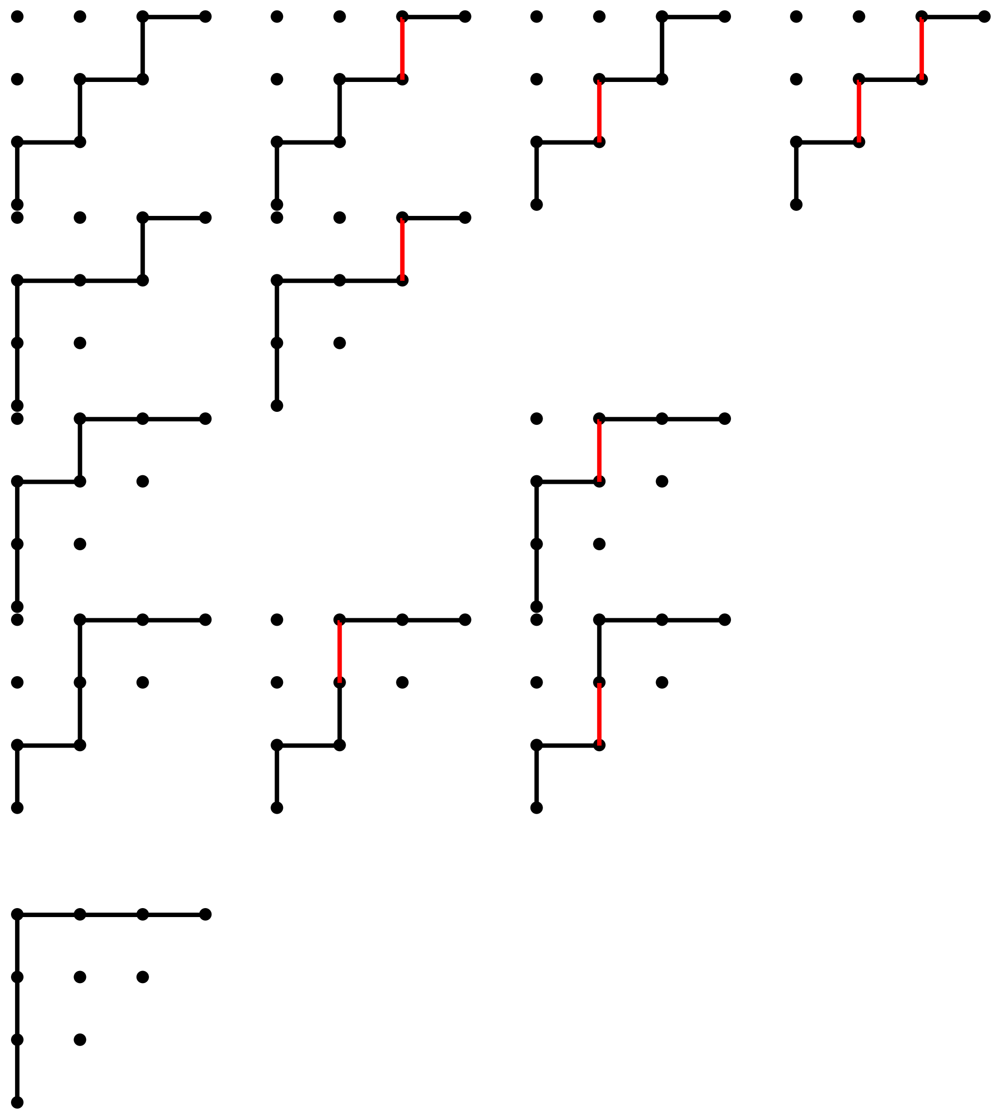
— dim deg n = number of paths in positive quadran with steps

[Bousquet-Mélou, Mishna]

$\mathbf{k\Pi}_S$ for $S = \{1, 12, 123, \ldots\}$



$\mathbf{k\Pi}_S$ for $S = \{1, 12, 123, \ldots\}$



감사해요 K A M S A H A E Y O

M E R C I

T H A N K S

G R A C I A S