LR Rule for Schuberts vs Grassmannians

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(with **S. Assaf, F. Sottile**)

Outline

- Schubert polynomials and Schubert varieties $c^w_{u,v}$
- Schubert polynomials as Grassmanian module $c^w_{u,v(\lambda,k)}$
- Monk's rule and k-Bruhat order $[u,w]_k$
- Action of B-S algebra on permutation $h_r = \sum \mathbf{u}_{\alpha,\beta}$
- B-S Quasisymmetric function of $[u,w]_k$ $F_{[u,w]_k}$

$$F_{[u,w]_k} \overset{THM}{=} \sum c^w_{u,v(\lambda,k)} S_\lambda$$

- - - Sami Assaf

- We show combinatorially $c^w_{u,v(\lambda,k)} \geq 0$
- When do we have a rule for constucting $c^w_{u,v(\lambda,k)}$

Schubert polynomials and Schubert varieties

L-S defined polynomial representatives for Schubert classes in the cohomology of flag manifolds:

For w in $\mathcal{S}_\infty$, the Schubert polynomial $\mathfrak{S}_w \in \mathbb{Z}[x_1, x_2, \ldots]$.

Schubert polynomials form an additive basis for this ring. Thus the identity

$$\mathfrak{S}_u \cdot \mathfrak{S}_v \quad = \quad \sum_w c_{u\,v}^{w} \mathfrak{S}_w$$

A (classical) theorem shows: $c_{u\,v}^{w}$ enumerate flags in a suitable triple intersection of Schubert varieties. (this is related to intersection product in the cohomology ring)

Schubert polynomials as Grassmanian module

Given λ and $k > 0$ there is a (grassmannian) permutation $v(\lambda, k)$

$$S_{\lambda}(x_1, x_2, \ldots, x_k) = \mathfrak{S}_{v(\lambda,k)}$$

A particular case, Monk's rule: for $\lambda = \square$, $v(\lambda, k) = s_k$

$$c^{w}_{u,\, s_k} = \begin{cases} 1 & \text{if } \ell(w) = \ell(u) + 1,\ w = u t_{a,b},\ a \le k < b \\ 0 & \text{otherwise} \end{cases}$$

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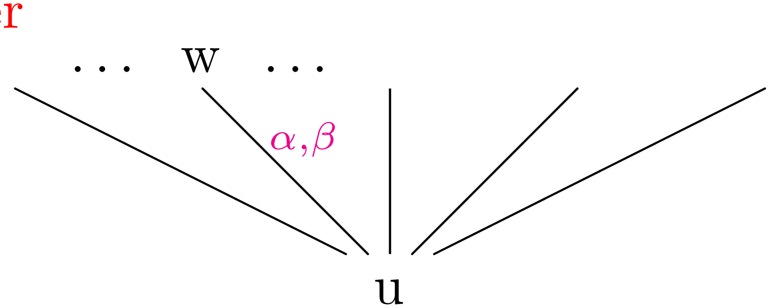
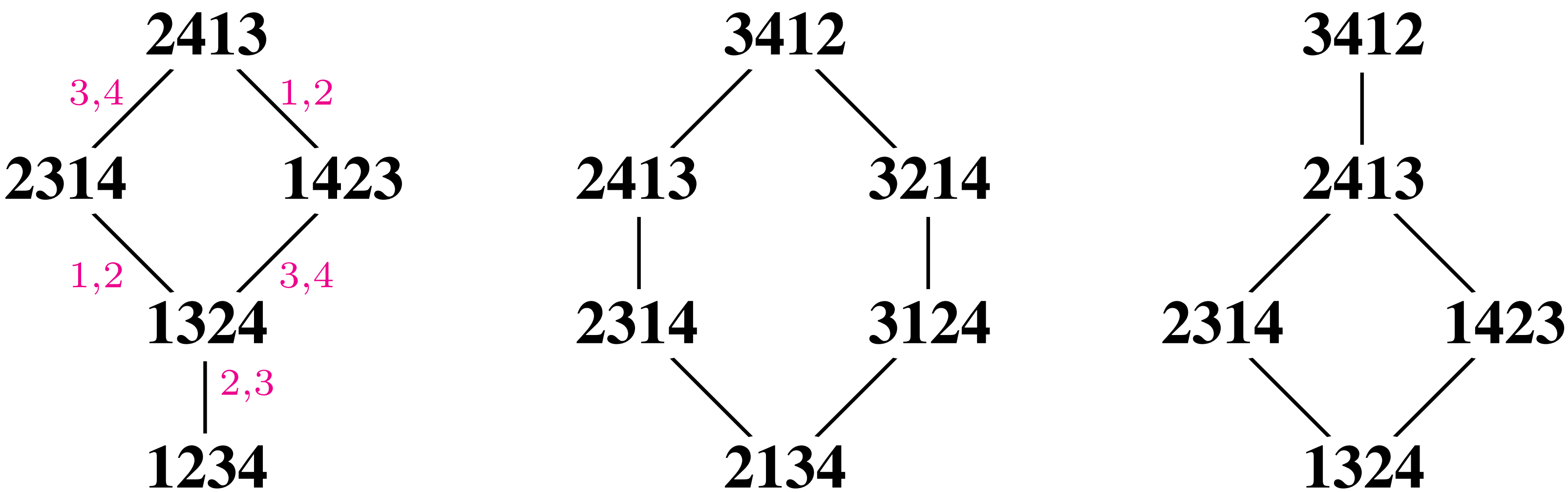
k-Bruhat order



$\alpha = u(a)$

$\beta = u(b)$

Schubert polynomials as Grassmanian module



Number of chains $C_{[u,w]_k}$ in $[u,w]_k$ of rank d

$$\mathfrak{S}_u\Big(h_1(x_1,\ldots,x_k)\Big)^d = \sum_w C_{[u,w]_k}\mathfrak{S}_w$$

$$= \mathfrak{S}_u\Big(\sum_\lambda f^\lambda S_\lambda\Big) = \sum_w \Big(\sum_\lambda f^\lambda c^w_{u,v(\lambda.k)}\Big)\mathfrak{S}_w$$

Action of B-S algebra on permutation

The combinatorics of chains in $[u, w]_k$ should be interesting and full of nice structure... we had some nice results but never could solve the problem of giving a rule for computing the $c^w_{u,v(\lambda.k)}$

Action of B-S algebra on permutation

B-S algebra $\mathcal{A}$: generators $\mathbf{u}_{\alpha,\beta}$ for each $0 < \alpha < \beta$, and relations

$$\begin{aligned}
\mathbf{u}_{\beta\gamma}\mathbf{u}_{\gamma\delta}\mathbf{u}_{\alpha\gamma} &\equiv \mathbf{u}_{\beta\delta}\mathbf{u}_{\alpha\beta}\mathbf{u}_{\beta\gamma}, && \text{if } \alpha < \beta < \gamma < \delta,\\
\mathbf{u}_{\alpha\gamma}\mathbf{u}_{\gamma\delta}\mathbf{u}_{\beta\gamma} &\equiv \mathbf{u}_{\beta\gamma}\mathbf{u}_{\alpha\beta}\mathbf{u}_{\beta\delta}, && \text{if } \alpha < \beta < \gamma < \delta,\\
\mathbf{u}_{\alpha\beta}\mathbf{u}_{\gamma\delta} &\equiv \mathbf{u}_{\gamma\delta}\mathbf{u}_{\alpha\beta}, && \text{if } \beta < \gamma \text{ or } \alpha < \gamma < \delta < \beta,\\
\mathbf{u}_{\alpha\gamma}\mathbf{u}_{\beta\delta} &\equiv \mathbf{u}_{\beta\delta}\mathbf{u}_{\alpha\gamma} \equiv 0, && \text{if } \alpha \leq \beta < \gamma \leq \delta,\\
\mathbf{u}_{\beta\gamma}\mathbf{u}_{\alpha\beta}\mathbf{u}_{\beta\gamma} &\equiv \mathbf{u}_{\alpha\beta}\mathbf{u}_{\beta\gamma}\mathbf{u}_{\alpha\beta} \equiv 0, && \text{if } \alpha < \beta < \gamma.
\end{aligned}$$

An analogue for this problem to the nil-coxeter algebra.

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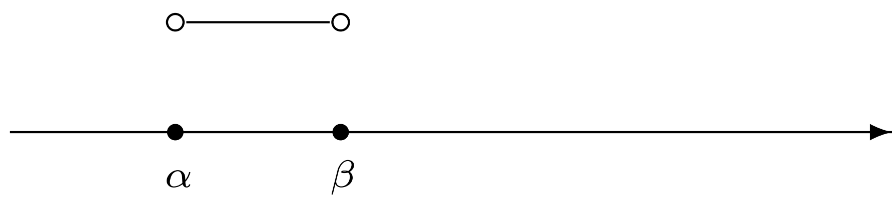
$\mathbf{u}_{\alpha\beta}$



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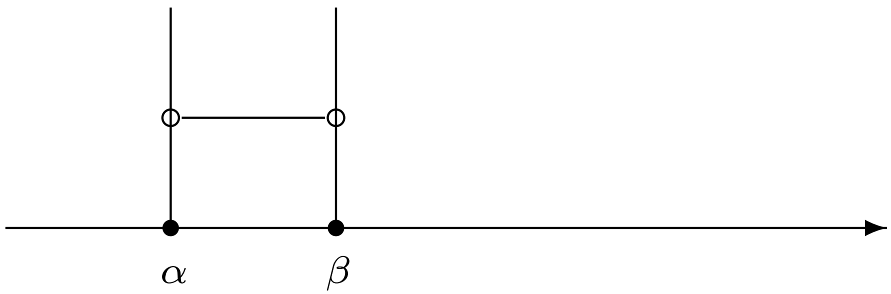

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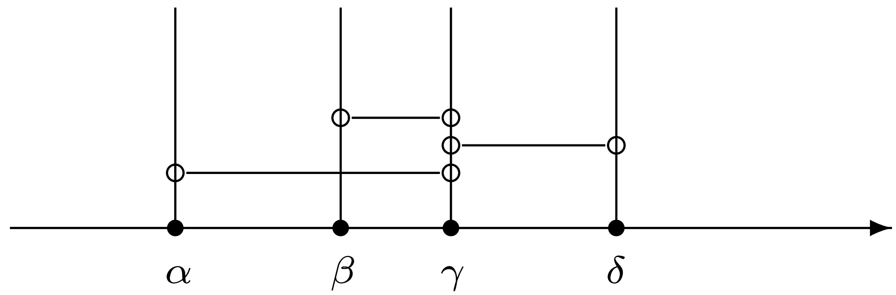
$\mathbf{u}_{\beta\gamma}\mathbf{u}_{\gamma\delta}\mathbf{u}_{\alpha\gamma}$



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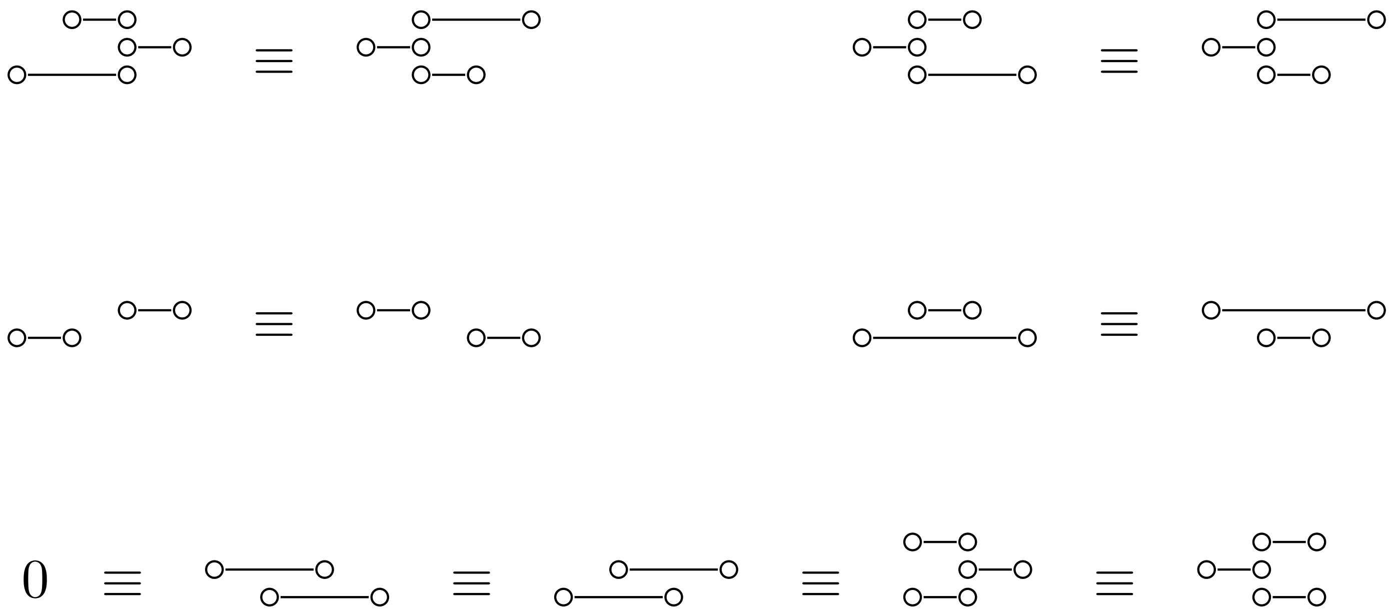

Action of B-S algebra on permutation

B-S algebra $\mathcal{A}$: generators $\mathbf{u}_{\alpha,\beta}$ for each $0 < \alpha < \beta$, and relations

The algebra $\mathcal{A}$ acts on permutations $(\mathcal{S}_\infty, k)$ as follow:

$$\mathbf{u}_{\alpha,\beta}(u) = \begin{cases} (\alpha\ \beta)u & \text{if } u <_k (\alpha\ \beta)u \text{ is a cover} \\ 0 & \text{otherwise} \end{cases}$$

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Let $\overline{\mathcal{A}}$ be the homogeneous series over $\mathcal{A}$. the action is locally finite so $\overline{\mathcal{A}}$ also acts on $(\mathcal{S}_\infty, k)$. Consider (in $\overline{\mathcal{A}}$)

$$\mathbf{h}_d = \sum_{\substack{\alpha_1 < \alpha_2 < \cdots < \alpha_d \\ \alpha_i < \beta_i}} \mathbf{u}_{\alpha_d,\beta_d} \cdots \mathbf{u}_{\alpha_2,\beta_2} \mathbf{u}_{\alpha_1,\beta_1}$$

Action of B-S algebra on permutation

B-S algebra $\mathcal{A}$: generators $\mathbf{u}_{\alpha,\beta}$ for each $0 < \alpha < \beta$, and relations

THEOREM[S,B-S,A-B-S] (mostly old stuff)

(a) The subalgebra $\langle \mathbf{h}_d : d \geq 1 \rangle$ in $\overline{\mathcal{A}}$ is commutative.

(b) Given k, the action $\mathbf{h}_d(u) = \sum c^w_{u,v((d),k)}\ w$.

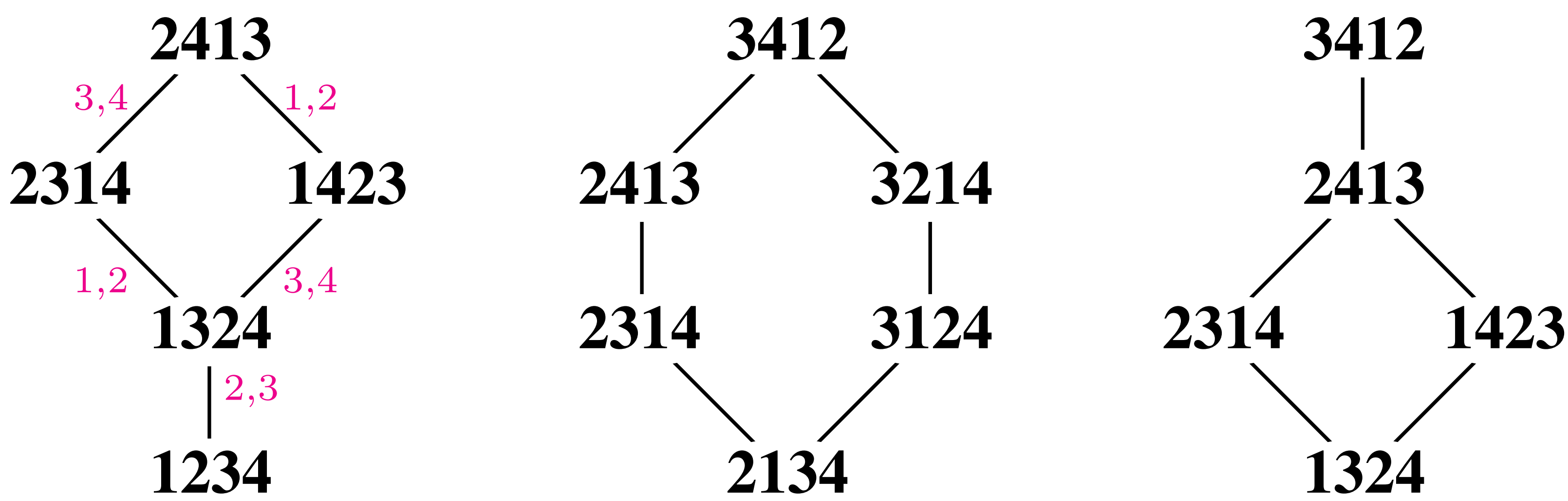
(c) Given k, the action $\mathbf{S}_\lambda(u) = \sum c^w_{u,v(\lambda,k)}\ w$.

(d) The quasisymmetric function

$$F_{[u,w]_k} \stackrel{def}{=} \sum_{C \in Chain([u,w]_k)} F_{D(C)}$$

$$\stackrel{thm}{=} \sum_{\lambda} c^w_{u,v(\lambda,k)} S_\lambda$$

Quasysymmetric function $F_{[u,w]_k}$



Chains: words $u_{3,4}u_{1,2}u_{2,3}$ and $u_{1,2}u_{3,4}u_{2,3}$ (NOT in $\mathcal{A}$)

Descent: $D(u_{\alpha_d,\beta_d}\cdots u_{\alpha_2,\beta_2}u_{\alpha_1,\beta_1}) = \{i : \alpha_i > \alpha_{i+1}\}$

$$D(u_{3,4}u_{1,2}u_{2,3}) = \{1\} \qquad D(u_{1,2}u_{3,4}u_{2,3}) = \{2\}$$

Quasysymmetric function $F_{[u,w]_k}$

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Fundamental quasisymmetric functions indexed by $J \subset \{1, 2, \ldots, n-1\}$

$$F_J = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n \\ j \in J \implies i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

$$F_{[u,w]_k} = F_{\{1\}} + F_{\{2\}} = S_{(2,1)}$$

Assaf's strong/weak dual-equivalence graphs

Given a combinatorial set $\mathcal{C}$ equipped with a map $D\colon \mathcal{C} \to 2^{n-1}$

$$F_{\mathcal{C}} \;=\; \sum_{C\in\mathcal{C}} F_{D(C)}$$

(a) When is $F_{\mathcal{C}} = \sum c^{\lambda}_{\mathcal{C}} S_\lambda$ symmetric function?

(weak DEG $\implies$ combinatorial proof)

(b) Given (a) can we get a positive rule for $c^{\lambda}_{\mathcal{C}}$?

(strong DEG $\implies$ combinatorial construction)

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Given a combinatorial set $\mathcal{C}$ equipped with a map $D\colon \mathcal{C} \to 2^{n-1}$

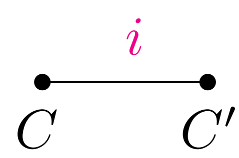
$$F_{\mathcal{C}} = \sum_{C \in \mathcal{C}} F_{D(C)}$$

We (strongly) need involution $\theta_i\colon \mathcal{C} \to \mathcal{C}$ for $2 \leq i \leq n-1$ such that

(a) $\theta_i(C) = C' \neq C$ exactly when $\{i-1, i\} \cap D(C) = \{j\}$ and

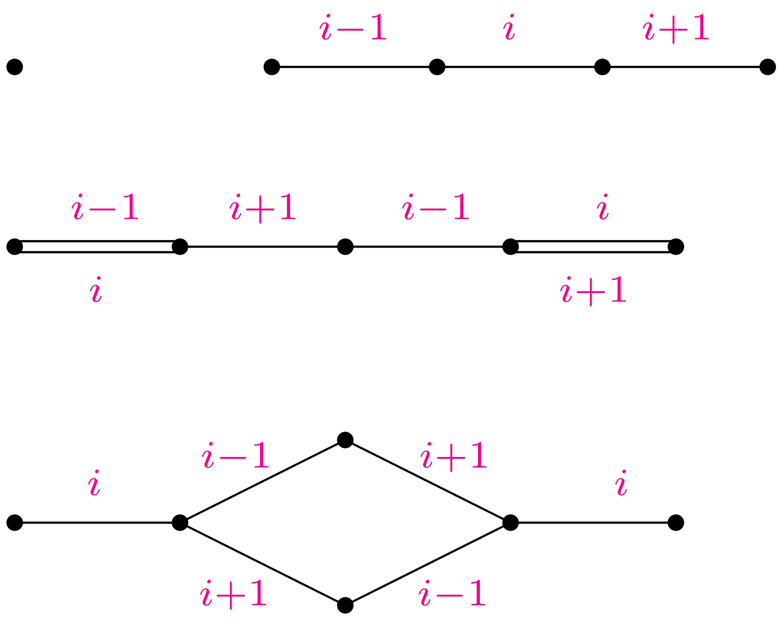
- $\{i-1, i\} \cap D(C') = ((\{i-1, i\} \cap D(C)) - \{j\}) \cup \{j\}^c$
- $D(C') - [i-2, i+1] = D(C) - [i-2, i+1]$
- $j = i$ and $i-2 \notin D(C) \implies i-2 \notin D(C')$
- $j = i$ and $i+1 \in D(C) \implies i+1 \in D(C')$

We construct a colored multi Graph on $\mathcal{C}$: For $\theta_i(C) = C' \neq C$



Assaf's strong/weak dual-equivalence graphs

(b) The connected components induced by θ_{1-1}; θ_i; θ_{i+1} is one of the following



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Given a combinatorial set $\mathcal{C}$ equipped with a map $D\colon \mathcal{C} \to 2^{n-1}$

$$F_{\mathcal{C}} = \sum F_{D(C)}$$

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(a) $\theta_i(C) = C' \neq C$ with some restriction on $D(C)$ and $D(C')$

(b) The connected components $\theta_{1-1}; \theta_i; \theta_{i+1}$ are special

(c) if $|i-j| > 2$, then $\theta_i \circ \theta_j = \theta_j \circ \theta_i$

(d) for all C, C' in a component $\theta_2; \theta_3; \cdots ; \theta_i$ we can find a path with at most one edge i.

Assaf's ~~strong~~ WEAK dual-equivalence graphs

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(a) $\theta_i(C) = C' \neq C$ with some restriction on $D(C)$ and $D(C')$

non trivial $\left|Comp(\theta_i; \theta_{i+1})\right| = 2$ or 4 and $\left|Comp(\theta_{i-1}; \theta_{i+1})\right| \geq 4$

~~(b) The connected components $\theta_{1-1}; \theta_i; \theta_{i+1}$ are special~~

(c) if $|i-j| > 2$, then $\theta_i \circ \theta_j = \theta_j \circ \theta_i$

~~(d) for all C, C' in a component $\theta_2; \theta_3; \cdots; \theta_i$ we cant find a path with at most one edge i.~~

Assaf's weak dual-equivalence graphs

Given a combinatorial set $\mathcal{C}$ equipped with a map $D\colon \mathcal{C} \to 2^{n-1}$

$$F_{\mathcal{C}} \;=\; \sum F_{D(C)}$$

We (weakly) need involution $\theta_i\colon \mathcal{C} \to \mathcal{C}$ for $2 \leq i \leq n-1$ such that

(a) $\theta_i(C) = C' \neq C$ with some restriction on $D(C)$ and $D(C')$

(b) non trivial component have cardinality

$$\left|Comp(\theta_i;\, \theta_{i+1})\right| = 2 \text{ or } 4 \text{ and } \left|Comp(\theta_{i-1};\, \theta_{i+1})\right| \geq 4$$

(c) if $|i-j| > 2$, then $\theta_i \circ \theta_j = \theta_j \circ \theta_i$

Example: Schur function if time

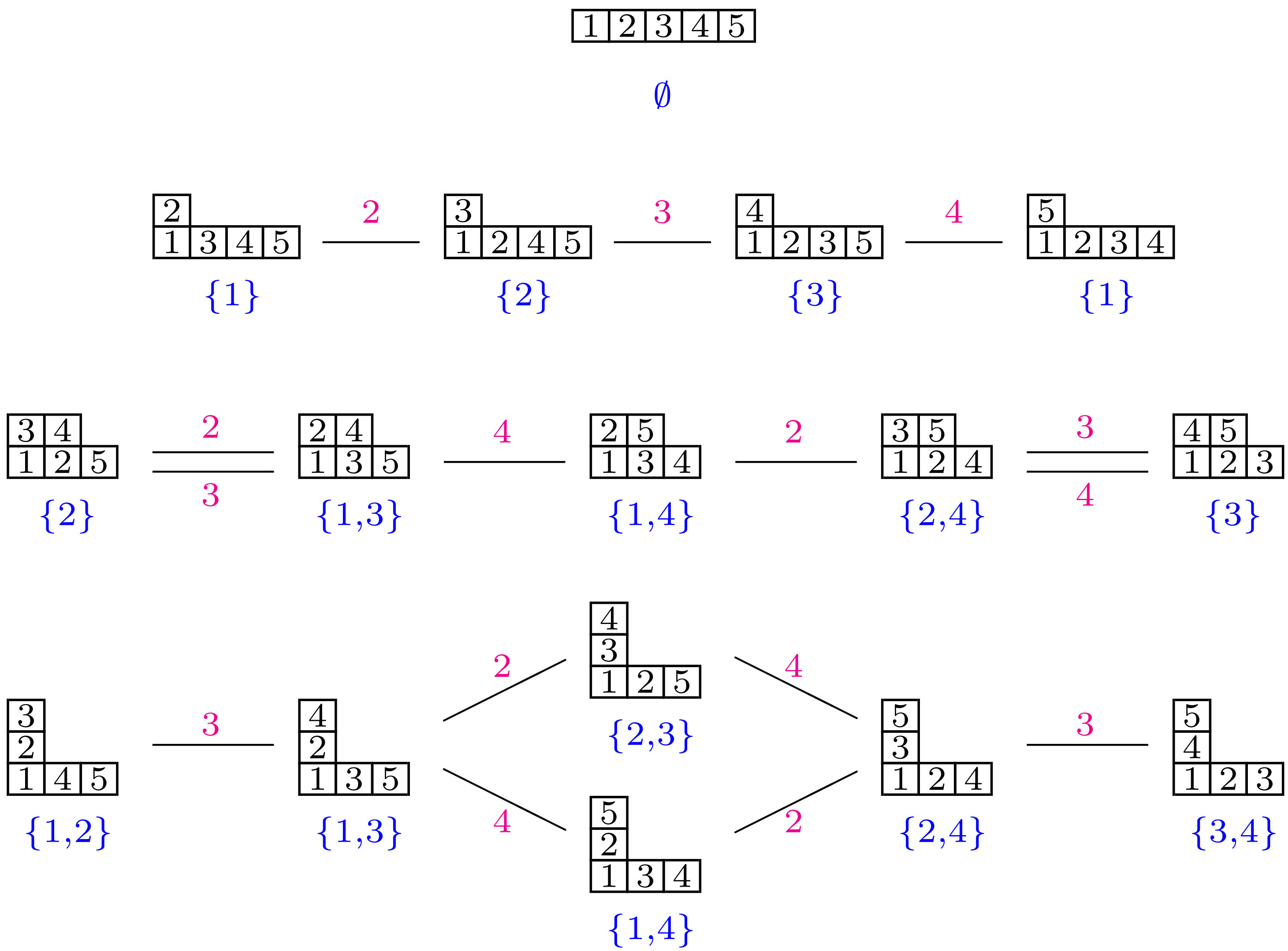
Let $\mathcal{C} = SYT_\lambda$ the set of standard Young tableau of shape λ.

For $T \in SYT_\lambda$, let $D(T) = \{i : i+1 \text{ is NW of } i \in T\}$. We have

$$S_\lambda = F_{\mathcal{S}YT_\lambda} \;=\; \sum F_{D(C)}$$

We let $\theta_i(T) = T'$ where we do dual-Knuth equivalence:

Example: Schur function if time



Back to Schubert and $F_{[u,w]_k}$

$$F_{[u,w]_k} = \sum_{C \in [u,w]_k} F_{D(C)}$$

Do we have θ_i?

Francois ask a question

You know... euh... if... you... mrrr mrrr...euh...is there other functions you can look at and do this?

YES! In the paper "non-commutative Pierri operator" [BMWS], we give many quasisymmetric functions like this that encode problems that are still open... we plan to re-visite this soon.

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Do we have θ_i? Yes, in position $i-1, i, i+1$ do

$$\begin{aligned}
&\mathrm{u}_{c\gamma}\mathrm{u}_{a\alpha}\mathrm{u}_{b\beta} \mapsto \mathrm{u}_{a\alpha}\mathrm{u}_{c\gamma}\mathrm{u}_{b\beta}, && \text{if } \{a,\alpha\} \cap \{c,\gamma\} = \emptyset \text{ and } a < b < c,\\
&\mathrm{u}_{\beta\gamma}\mathrm{u}_{\alpha\beta}\mathrm{u}_{\beta\delta} \mapsto \mathrm{u}_{\alpha\gamma}\mathrm{u}_{\gamma\delta}\mathrm{u}_{\beta\gamma}, && \text{if } \alpha < \beta < \gamma < \delta,\\
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Strong axioms: (a) $\surd$

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Strong axioms: (a) $\surd$ (c) $\surd$ (b) ??? Exhaustive computer search (20,000 cases)...

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&\mathrm{u}_{\beta\delta}\mathrm{u}_{\alpha\beta}\mathrm{u}_{\beta\gamma} \mapsto \mathrm{u}_{\beta\gamma}\mathrm{u}_{\gamma\delta}\mathrm{u}_{\alpha\gamma}, && \text{if } \alpha < \beta < \gamma < \delta,\\
&\mathrm{u}_{c\gamma}\mathrm{u}_{ac}\mathrm{u}_{b\beta} \mapsto \mathrm{u}_{b\beta}\mathrm{u}_{c\gamma}\mathrm{u}_{ac}, && \text{if } \{a,c,\gamma\} \cap \{b,\beta\} = \emptyset \text{ and } a < b < c,\\
&\mathrm{u}_{b\beta}\mathrm{u}_{ac}\mathrm{u}_{c\gamma} \mapsto \mathrm{u}_{ac}\mathrm{u}_{c\gamma}\mathrm{u}_{b\beta}, && \text{if } \{a,c,\gamma\} \cap \{b,\beta\} = \emptyset \text{ and } a < b < c,\\
&\mathrm{u}_{b\beta}\mathrm{u}_{a\alpha}\mathrm{u}_{c\gamma} \mapsto \mathrm{u}_{b\beta}\mathrm{u}_{c\gamma}\mathrm{u}_{a\alpha}, && \text{if } \{a,\alpha\} \cap \{c,\gamma\} = \emptyset \text{ and } a < b < c,
\end{aligned}$$

Strong axioms: (a) $\surd$ (c) $\surd$ (b) ??? Exhaustive computer search (20,000 cases)... NO :(

138 cases does not satisfy strong (b)... but they satisfy weak (b)

Conclusion about $F_{[u,w]_k}$

$$F_{[u,w]_k} = \sum_{C \in [u,w]_k} F_{D(C)} = \sum_{\lambda} c^{w}_{u,v(\lambda,k)} S_{\lambda}$$

If wu^{-1} does not contain one of the pattern

6325471; 534261; 6543271; 7543216; 642315; 7325416,

then the conditions are STRONG and we can give a rule for the $c^{w}_{u,v(\lambda,k)}$. Otherwise the condition are weak and we have to work harder (maybe possible but not at all trivial or ellegant)

The known rule are included above (Kogan, Belligan, Lennart). Belliban was on the right track by excluding nesting