Primitive Orthogonal Idempotents For R-Trivial Monoids

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(with **C. Berg, S. Bhargava and F. Saliola**)

Outline

- Origin of problem

  (Norton; Krob-Thibon; Schocker; Denton; BBBS)

- Weakly Ordered Monoids (WOM) (Schocker)

- WOM $\iff$ R-trivial (Steinberg)

- Constructing Orthogonal Idempotents (BBBS)

Origin of problem

Norton 79: Constructs analogue of Young Idempotents η_α for $H_n(0)$ (type A).

$H_n(0)\eta_\alpha$ gives all projective indecomposable (not simple)

YEAH! .... but the η_α are not idempotents nor orthogonal ... Oh!

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BBBS 10: Constructs Orthogonal Idempotents for WOM, generalizing Norton's elements ...

Weakly Ordered Monoids (WOM) (Schocker)

Brown (and Saliola) were very successful in constructing orthogonal idempotents for left regular bands

Left Regular Band W: $x^2 = x$ and $xyx = xy$

Support map $Supp\colon W \to L$:
$L = W/\sim$ where $x \sim y$ iff $x = xy$ and $y = yx$, is a lattice.

Radical of kW: $\sqrt{kW} = \ker(Supp)$, in particular $kW \Big/ \sqrt{kW} \cong kL$.

Orthogonal idempotents of kW: for $J \in L$, take x such that $Supp(x) = j$.

$$e_J = x\left(1 - \sum_{K>J} e_K\right).$$

Weakly Ordered Monoids (WOM) (Schocker)

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Manfred notice the similarity between LRB and the way I was understanding the radical of $H_n(0)$. He proposed to study the following class of monoid that contains both LRB and Hecke algebra at $q = 0$ of all finite types.

Weakly Ordered Monoids (WOM) (Schocker)

- Preorder: $u \leq v \iff uw = v$ for some $w \in W$

When is this an order?

Weakly Ordered Monoids (WOM) (Schocker)

- Preorder: $u \leq v \iff uw = v$ for some $w \in W$
- W is WOM: if we have a lattice L and two maps $C, D\colon W \to L$

where

(1) C surjective morphism of monoids.

(2) $uv \leq u$ and $u \leq uv \implies C(v) \leq D(u)$.

(3) $C(v) \leq D(u) \implies uv = u$.

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PROP. A [Schocker]: For W WOM, $\leq$ is an order and

$$\boxed{\sqrt{kW} = \ker(C)}$$

IDEMPOTENT ???

Weakly Ordered Monoids (WOM) (Schocker)

- LRB: $C = D = Supp$
- Hecke Algebra $H_n(0)$:
  - Generated by T_i; $T_i^2 = T_i$ and braid relations.
  - $D(T_w)$ Descent map $= \{i : T_w T_i = T_w\}$.
  - $C(T_w)$ Content map $= \{i : T_i$ occur in $T_w\}$.
  - The order $\leq$ is the weak order on W.

WOM $\iff$ R-trivial (Steinberg)

W is R-trivial if

$$xW = yW \implies x = y.$$

WOM $\iff$ **R-trivial** (Steinberg)

W is R-trivial if $\quad \left[\, xW = yW \implies x = y \,\right].$

PROP. B

$$\boxed{\leq \text{ is an order } \iff W \text{ is R-trivial}}$$

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PROP. B

$\leq$ is an order $\iff$ W is R-trivial

PROP. C

W is WOM $\iff$ W is R-trivial

WOM $\iff$ R-trivial (Steinberg)

W is R-trivial if $\left[\, xW = yW \implies x = y \, \right]$.

PROP. C

$$W \text{ is WOM} \iff W \text{ is R-trivial}$$

"$\Longleftarrow$"

- $L = \{We : e \text{ idempotent }\}$ with inclusion
- $C(x) = Wx^{\omega}$ where

$$x \le x^2 \le ... \le x^N = x^{N+1} = x^{\omega}$$

- $D(x) = C(e)$ where

$$e = \max_{\le}\{s \in W : xs = x\}.$$

Constructing Idempotents (BBBS and S(?!))

Let W be WOM generated by $G = \{g_1, g_2, \ldots\}$.

- From Prop. A: $kW \big/ \sqrt{kW} \cong L$ semisimple commutative.

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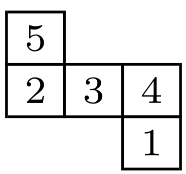
- For each $J \in L$ we first construct an analogue of Norton

$$\eta_J = A_J T_J$$

In the case of Hecke algebra of type A at q=0



$$T_J = T_5 \cdot T_2 T_3 T_2 T_4 T_3 T_2 \cdot T_1$$

$$A_J = \overline{T}_2 \overline{T}_5 \cdot \overline{T}_3 \cdot \overline{T}_1 \overline{T}_4 \text{ where } \overline{T}_i = 1 - T_i$$

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$$\eta_J = A_J T_J$$

$$T_J = \Big(\prod_{\substack{g \in G \\ C(g) \leq J}} g^{\omega} \Big)^{\omega}$$

$$T_J x = T_J \text{ for all } x \text{ such that } C(x) \leq J$$

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$$\eta_J = A_J T_J$$

$$A_J = \Big(\prod_{\substack{g \in G \\ C(g) \not\leq J}} (1 - g^\omega) \Big)^\omega$$

$$xA_J = 0 \text{ for all } x \text{ such that } C(x) \not\leq J$$

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Can we do tHAT !?

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$$\eta_J = A_J T_J$$

$$A_J = \Big(\prod_{\substack{g \in G \\ C(g) \not\leq J}} (1 - g^\omega) \Big)^{\omega}$$

$$x A_J = 0 \text{ for all } x \text{ such that } C(x) \not\leq J$$

PROP. D [BBBS] Yes we can!

Constructing Idempotents (BBBS and S(?!))

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- From Prop. A: $kW \Big/ \sqrt{kW} \cong L$ semisimple commutative.
- For each $J \in L$ we first construct $\eta_J = A_J T_J$
- Almost orthogonal: $J \not\leq K \implies \eta_J \eta_K = 0$.

A_J and T_J are both idempotents but... η_J IS NOT.

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- Build an. idempotent: $P_J = \left(\sum_{n \geq 0} \eta_J (1 - \eta_J)^n \right)^2$

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Is is Finite?

PROP. E [BBBS] $\boxed{\eta_J^2 (1 - \eta_J)^N = 0 \text{ for some } N > 0}$

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PROP. F [BBBS] $\boxed{P_J^2 = P_J}$

$$\sum_{n=0}^{N} x(1 - x)^n = 1 - (1 - x)^{N+1}.$$

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THEOREM [BBBS] $\boxed{e_J = P_J \Big(1 - \sum_{K > J} e_K \Big)}$

This is the Left Regular Band trick of Saliola and Brown.

Someone must be smilling