Positivity in Combinatorial Hopf Algebras

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(with **S. Assaf and F. Sottile**)
(and with **C. Benedetti**)

Outline

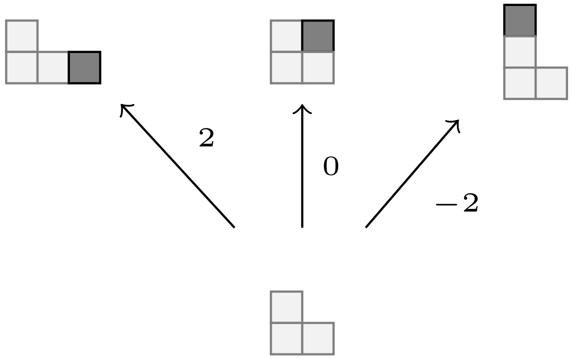
- Encoding Pieri rule in SYM, Schubert and dual k-Schur.
- Pieri Operators and quasisymmetric functions $K_{[u,v]}$.
- For SYM, Schubert and dual k-Schur:

  $K_{[u,v]}$ is symmetric and we want Schur Positivity.
- Dual Knuth Equivalence graph for $K_{[u,v]}$.

Pieri Rule in SYM

$$S_\lambda S_{(1)} = \sum_{\mu/\lambda \text{ a box}} S_\mu = \sum_{c\in\mathbb{Z}} S_{\mathbf{u}_c(\lambda)}\,.$$

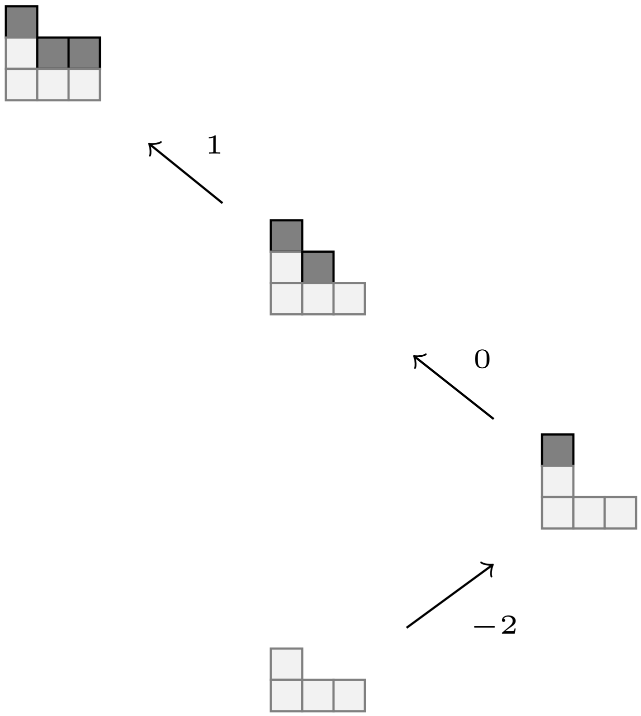
In Young Lattice:



$$\begin{aligned} \mathbf{u}_c : \ \mathbb{Z}\mathcal{Y} \ &\longrightarrow \quad \mathbb{Z}\mathcal{Y}, \\ \lambda \ &\longmapsto \ \begin{cases} \lambda & \text{if } \lambda/\mu \text{ a box in } (i,j) : i-j=c \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Pieri Rule in SYM

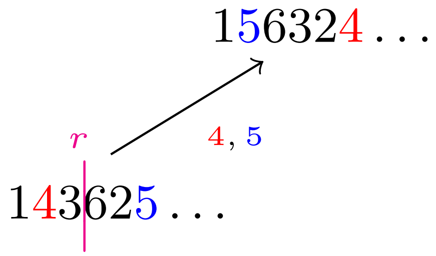
$$S_\lambda S_{(m)} = \sum_{\mu/\lambda \ m\text{-row strip}} S_\mu = \sum_{i_1<i_2<\cdots<i_m} S_{\mathbf{u}_{i_m}\cdots\mathbf{u}_{i_2}\mathbf{u}_{i_1}(\lambda)} \, .$$



Pieri Rule in Schubert

$$\mathfrak{S}_u h_{(m)}(x_1, \ldots, x_r) = \sum_{\substack{b_1 < b_2 < \ldots < b_m \\ a_i < b_i}} \mathfrak{S}_{\mathbf{u}^r_{a_m b_m} \cdots \mathbf{u}^r_{a_2 b_2} \mathbf{u}^r_{a_1 b_1}(u)}$$

In Bruhat order:



$$\begin{aligned} \mathbf{u}^r_{ab} \colon \ \mathbb{Z}\mathcal{S}_\infty &\longrightarrow \mathbb{Z}\mathcal{S}_\infty, \\ u &\longmapsto \begin{cases} (a\,b)u & \text{if } \begin{array}{l} u \lessdot (a\,b)u \\ u^{-1}(a) \leq r < u^{-1}(b) \end{array} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Pieri Rule in dual k-Schur

$$\mathfrak{S}_u^{(k)} h_{(m)} = \sum_{\substack{b_1 < b_2 < \ldots < b_m \\ b_i - a_i \leq k+1}} \mathfrak{S}^{(k)}_{u\mathbf{t}_{a_1 b_1} \cdots \mathbf{t}_{a_m b_m}}$$

In k-affine Bruhat order:

$$|8\,\bar{6}\,3\,\bar{1}\,4\,13|$$

$$\cdots \bar{8}\,1|\overline{12}\,2\,\bar{3}\,\bar{7}\,\bar{2}\,7\,\underbrace{|\bar{6}\,8\,3\,\bar{1}\,4\,13|}_{\text{main}}\,\overset{7,\ 8}{\bar{0}\,14}\cdots$$

$$\begin{aligned} \mathbf{t}_{ab} \colon \mathbb{Z}W^0 &\longrightarrow \mathbb{Z}W^0, \\ u &\longmapsto \begin{cases} ut_{a,b} & \text{if } u \lessdot ut_{a,b} \text{ and } u(a) \leq 0 < u(b) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Pieri operator of POSETs

In Young lattice:

$$H_m = \sum_{i_1<i_2<\cdots<i_m} \mathbf{u}_{i_m} \cdots \mathbf{u}_{i_2}\mathbf{u}_{i_1} \,.$$

In Bruhat order:

$$H_m^{(r)} = \sum_{\substack{b_1<b_2<\ldots<b_m \\ a_i<b_i}} \mathbf{u}^r_{a_m b_m} \cdots \mathbf{u}^r_{a_2 b_2}\mathbf{u}^r_{a_1 b_1} \,.$$

In k-affine Bruhat order of $W = \widetilde{A}_k$:

$$H_m = \sum_{\substack{b_1<b_2<\ldots<b_m \\ b_i-a_i\leq k+1}} \mathbf{t}_{a_1 b_1} \cdots \mathbf{t}_{a_m b_m} \,.$$

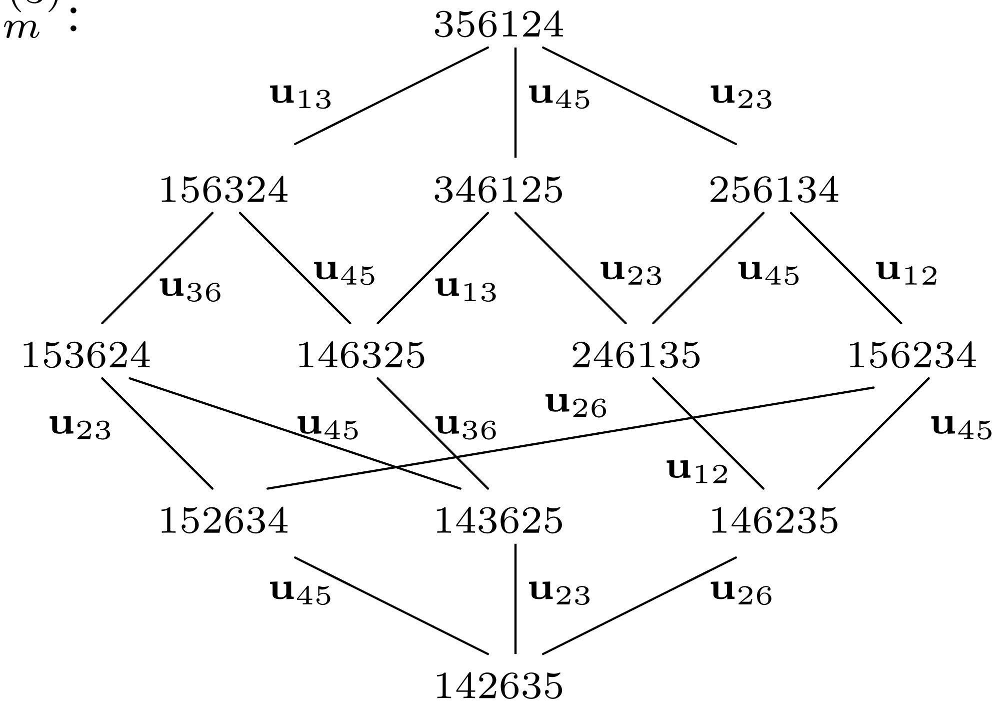
Quasisymmetric function $K_{[u,v]}$

Given Pieri Operator H_m we construct a quasisymmetric function

$$K_{[u,w]} = \sum_{\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_\ell)} \langle H_{\alpha_\ell} \cdots H_{\alpha_2} H_{\alpha_1}(u), w \rangle M_\alpha .$$

Quasisymmetric function $K_{[u,v]}$

Bruhat order with $H_m^{(3)}$:



$$\begin{aligned} K_{[u,v]_3} &= 2M_{22} + M_{31} + M_{13} + 4M_{211} + 4M_{121} + 4M_{112} + 8M_{1111} \\ &= F_{13} + 2F_{121} + 2F_{22} + F_{112} + F_{31} + F_{211} \\ &= S_{31} + S_{22} + S_{211}. \end{aligned}$$

It is Schur positive !

Quasisymmetric function $K_{[u,v]}$

Given Pieri Operator H_m we construct a quasisymmetric function

$$K_{[u,w]} = \sum_{\alpha=(\alpha_1,\alpha_2,\dots,\alpha_\ell)} \langle H_{\alpha_\ell} \cdots H_{\alpha_2} H_{\alpha_1}(u), w\rangle M_\alpha .$$

THEOREM A: [B-Mikytiuk-Sottile-vanWilligenburg]

$$\boxed{\text{If } H_m H_n = H_n H_m, \text{ then } K_{[u,v]} \text{ is symmetric.}}$$

$K_{[\lambda,\mu]}$ for SYM

Using the Pieri Operators H_m on Young lattice:

THEOREM B1: [Classic result]

(a) $H_m H_n = H_n H_m$, hence $K_{[\lambda,\mu]}$ is symmetric.

(b) $K_{[\lambda,\mu]} = \sum_{\alpha} d^{\alpha}_{\lambda,\mu} F_{\alpha}$, where

$d^{\alpha}_{\lambda,\mu}$ counts the paths in $[\lambda, \mu]$ with descent given by α.

(c) $K_{[\lambda,\mu]} = \sum_{\nu} c^{\mu}_{\lambda,\nu} S_{\nu}$, where $c^{\mu}_{\lambda,\nu}$ are the structure constants in:

$$\boxed{S_\lambda S_\nu = \sum_{\mu} c^{\mu}_{\lambda,\nu} S_\mu}$$

In fact $K_{[\lambda,\mu]} = S_{\mu/\lambda}$ is the skew-Schur function

$K_{[u,w]_r}$ for Schubert

Using the Pieri Operators $H_m^{(r)}$ on Bruhat order:

THEOREM B2: [B-sottile]

(a) $H_m^{(r)} H_n^{(r)} = H_n^{(r)} H_m^{(r)}$, hence $K_{[u,w]_r}$ is symmetric.

(b) $K_{[u,w]_r} = \sum_\alpha d^\alpha_{u,w,r} F_\alpha$, where

$d^\alpha_{u,w,r}$ counts the paths in $[u,w]_r$ with descent given by α.

(c) $K_{[u,w]_r} = \sum_\lambda c^w_{u,\lambda,r} S_\lambda$,

where $c^w_{u,\lambda,r}$ are the structure constants in:

$$\boxed{\mathfrak{S}_u \cdot S_\lambda(x_1, x_2, \ldots, x_r) = \sum_w c^w_{u,\lambda,r} \mathfrak{S}_w}$$

$K_{[u,w]}$ for k-Schur

Using the Pieri Operators H_m on k-affine Bruhat order $W = \widetilde{A}_k$

THEOREM B3: [(a) Lam, (b)(c) BMSvW, B-Benedetti]

(a) $H_m H_n = H_n H_m$, hence $K_{[u,w]}$ is symmetric.

(b) $K_{[u,w]} = \sum_{\alpha} d^{\alpha}_{u,w} F_{\alpha}$, where

$d^{\alpha}_{u,w}$ counts the paths in $[u, w]$ with descent given by α.

(c) For $u, w \in W^0$, Grassmannian permutations,

$K_{[u,w]} = \sum_{\lambda} c^{w}_{u,\lambda} S_{\lambda}$, where $c^{w}_{u,\lambda}$ are the structure constants in:

$$\boxed{\mathfrak{S}^{(k)}_{u} S_{\lambda} = \sum_{w} c^{w}_{u,\lambda} \mathfrak{S}^{(k)}_{w}}$$

Schur Positivity of $K_{[u,w]}$

For SYM, Schubert and k-Schur, the Schur Positivity of $K_{[u,w]}$ is mostly guaranteed by geometry. But we are interested to show positivity with a combinatorial construction:

THEOREM C1: [classic]

The $c^{\mu}_{\lambda,\nu}$ are the Littlewood-Richardson numbers.
Many combinatorial constructions exist.

THEOREM C2: [Assaf-B-Sottile]

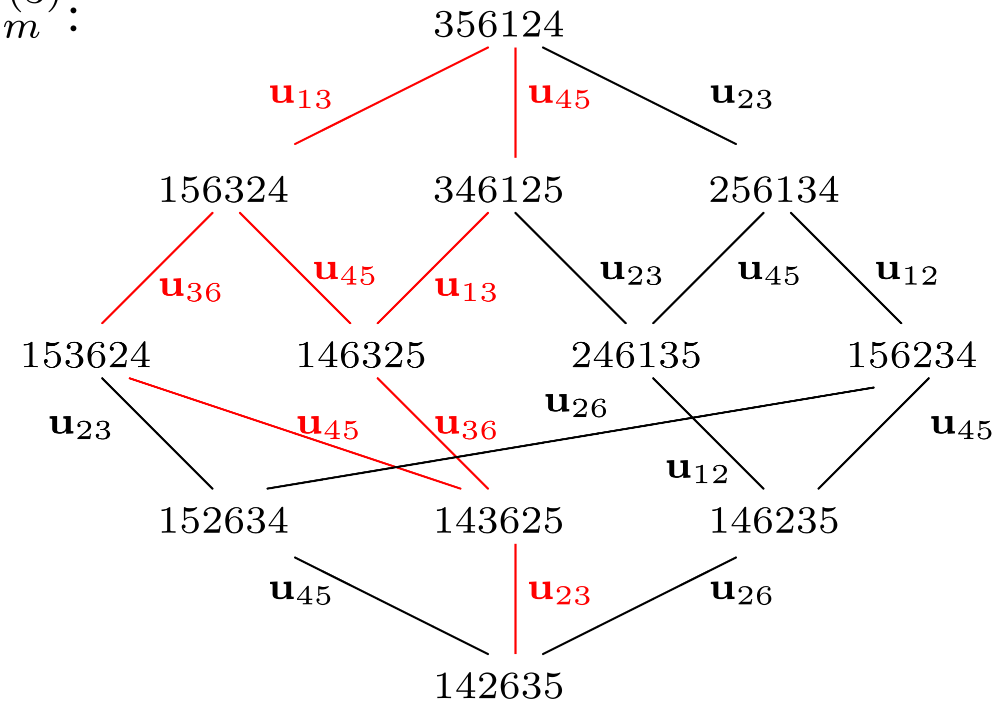
The $c^{w}_{u,\lambda,r}$ are combinatorially positive.

THEOREM C3 (in PROGRESS): [B-Benedetti]

The $c^{w}_{u,\lambda}$ are combinatorially positive.

Dual Knuth Equivalence graph for $K_{[u,w]}$

Bruhat order with $H_m^{(3)}$:

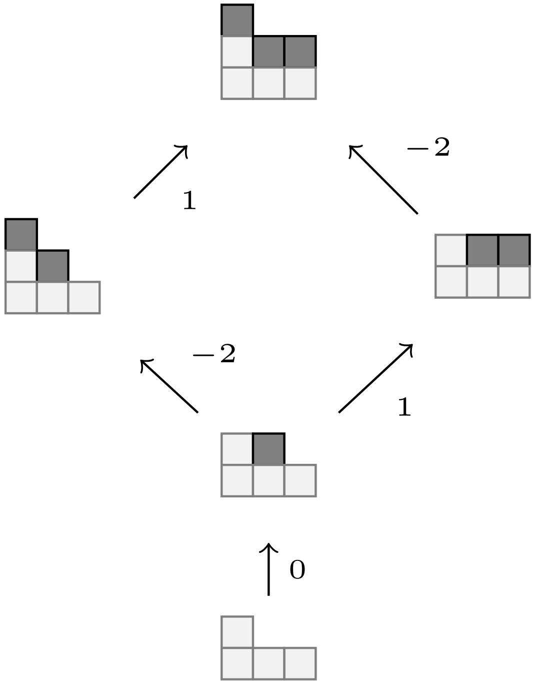


$$K_{[u,v]_3} = F_{13} + 2F_{121} + 2F_{22} + F_{112} + F_{31} + F_{211}$$

$$S_{31} = F_{31} + F_{22} + F_{13}, \qquad S_{22} = F_{22} + F_{121}$$

$$S_{211} = F_{211} + F_{121} + F_{112}$$

Classical Dual Knuth Equivalence for $K_{[u,w]}$



$$\begin{aligned} \mathbf{u}_c\mathbf{u}_r\mathbf{u}_t &\leftrightarrow \mathbf{u}_c\mathbf{u}_t\mathbf{u}_r \quad &&\text{if } r < c < t, \\ \mathbf{u}_r\mathbf{u}_t\mathbf{u}_c &\leftrightarrow \mathbf{u}_t\mathbf{u}_r\mathbf{u}_c \quad &&\text{if } r < c < t. \end{aligned}$$

Breaks interval nicely into Schur functions.

Dual Knuth Equivalence for Schubert $K_{[u,w]_r}$

with $a < b < c < d$

(A) $\mathbf{u}_{\gamma c}\mathbf{u}_{\alpha a}\mathbf{u}_{\beta b} \leftrightarrow \mathbf{u}_{\alpha a}\mathbf{u}_{\gamma c}\mathbf{u}_{\beta b},$

$\mathbf{u}_{\beta b}\mathbf{u}_{\alpha a}\mathbf{u}_{\gamma c} \leftrightarrow \mathbf{u}_{\beta b}\mathbf{u}_{\gamma c}\mathbf{u}_{\alpha a},$ if $\{a, \alpha\} \cap \{c, \gamma\} = \emptyset,$

(B) $\mathbf{u}_{bc}\mathbf{u}_{ab}\mathbf{u}_{bd} \leftrightarrow \mathbf{u}_{ac}\mathbf{u}_{cd}\mathbf{u}_{bc},$

$\mathbf{u}_{bd}\mathbf{u}_{ab}\mathbf{u}_{bc} \leftrightarrow \mathbf{u}_{bc}\mathbf{u}_{cd}\mathbf{u}_{ac},$

(C) $\mathbf{u}_{\beta b}\mathbf{u}_{\alpha a}\mathbf{u}_{ac} \leftrightarrow \mathbf{u}_{\alpha a}\mathbf{u}_{ac}\mathbf{u}_{\beta b},$

$\mathbf{u}_{ac}\mathbf{u}_{\alpha a}\mathbf{u}_{\beta b} \leftrightarrow \mathbf{u}_{\beta b}\mathbf{u}_{ac}\mathbf{u}_{\alpha a},$ if $\{\alpha, a, c\} \cap \{b, \beta\} = \emptyset.$

Does not break interval nicely into Schur functions!!!

But this satisfies Assaf's weak dual equivalence (checking more than 20 000 cases by computer). A combinatorial induction shows positivity.

Dual Knuth Equivalence for k-Schur $K_{[u,w]}$ –I–

$\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv \mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd},$ if $a < b = c < e < f < d$ and $\bar{a} = \bar{d}$

$\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ec} \equiv \mathbf{t}_{ec}\mathbf{t}_{a,b-|c-e|}\mathbf{t}_{ed},$ if $a < b < e < f < c < d$ and $f \neq \bar{a} = \bar{d} < \bar{b} = \bar{c}$

$\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv \mathbf{t}_{d-n,c}\mathbf{t}_{b-n,a}\mathbf{t}_{ef},$ if $a < b < e < c < d$ and $\bar{a} = \bar{d} < \bar{e} < \bar{b} = \bar{c}$

$\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ec} \equiv \mathbf{t}_{\bar{d}c}\mathbf{t}_{\bar{b}a}\mathbf{t}_{ec},$ if $a < b < e < c < d$ and $\bar{a} = \bar{d} = \bar{e}$, $\bar{b} = \bar{c}$

$\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv \mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd},$ if $a < b < c < e < f < d$ and $\bar{a} = \bar{d}$, $\bar{b} = \bar{c}$

$u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv u\mathbf{t}_{d,c+r}\mathbf{t}_{b-r,a}\mathbf{t}_{ef},$ if $a < b < e < f \leq c < d$, $\bar{a} = \bar{d}$, $u(c) \leq 0$, $u(d) \leq 0$

$u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd},$ if $a < b < c < e < f < d$, $\bar{a} = \bar{d}$, $u(c) \leq 0$, $u(d) \leq 0$

$u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv u\mathbf{t}_{cd}\mathbf{t}_{b-r,b}\mathbf{t}_{ef},$ if $a < b < e < f \leq c < d$, $\bar{a} = \bar{d}$, $u(d) > 0$

$u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd},$ if $a < b < c < e < f < d$, $\bar{a} = \bar{d}$, $u(d) > 0$

$u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv u\mathbf{t}_{d-r,c}\mathbf{t}_{b,a+r}\mathbf{t}_{ef},$ if $a < b < e < f \leq c < d$,$\bar{b} = \bar{c}$, $\bar{e} = \bar{d}$ or $\bar{e} < \bar{a}, u(a$

$u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ec} \equiv u\mathbf{t}_{ec}\mathbf{t}_{a,b-|c-e|}\mathbf{t}_{ed},$ if $a < b < e < c < d$, $\bar{b} = \bar{c}$, $\bar{e} > \bar{a}, u(a+r) > u(b)$

$u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd},$ if $a < b < e < f < c < d$, $\bar{b} = \bar{c}, u(a+r) > u(b) > 0$

$u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv u\mathbf{t}_{d-r,d}\mathbf{t}_{ab}\mathbf{t}_{ef},$ if $a < b < e < f \leq c < d$, $\bar{b} = \bar{c}$, $\bar{e} \geq \bar{d}$, $u(a+r) \leq$

Dual Knuth Equivalence for k-Schur $K_{[u,w]}$ –II–

$$u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd}, \qquad \text{if } a<b<c<e<f<d,\ \bar{b}=\bar{c},\ \ \bar{e}\geq\bar{d},\ u(a+r)$$

$$u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv u\mathbf{t}_{cd}\mathbf{t}_{a,b-|d-c|}\mathbf{t}_{ef}, \qquad \text{if } a<b<e<f<d, c>b,\ \bar{a}<\bar{c}<\bar{b}=\bar{d},\ \bar{e}\neq\bar{a},$$

$$u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv u\mathbf{t}_{c,d-|b-a|}\mathbf{t}_{ab}\mathbf{t}_{ef}, \qquad \text{if } a<b<e<f\leq c<d,\ \bar{c}<\bar{a}<\bar{b}=\bar{d},\ \ u(a)<$$

$$u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd}, \qquad \text{if } a<b<c<e<f<d,\ \bar{c}<\bar{a}<\bar{b}=\bar{d},\ \ u(a)<$$

$$\mathbf{t}_{ab}\mathbf{t}_{bc}\mathbf{t}_{db} \equiv \mathbf{t}_{db}\mathbf{t}_{ad}\mathbf{t}_{dc}, \qquad \text{if } a<d<b<c \text{ and } \bar{a}=\bar{c}$$

$$\mathbf{t}_{ab}\mathbf{t}_{bc}\mathbf{t}_{ab} \equiv \mathbf{t}_{ab}\mathbf{t}_{\bar{b}c}\mathbf{t}_{ab}, \qquad \text{if } a<b<c \text{ and } \bar{a}=\bar{c}$$

$$\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{eb} \equiv \mathbf{t}_{\bar{d}c}\mathbf{t}_{\bar{b}a}\mathbf{t}_{eb}, \qquad \text{if } a\geq e<b<c<d \text{ and } \bar{a}=\bar{d}, \bar{b}=\bar{d}$$

$$u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{eb} \equiv u\mathbf{t}_{eb}\mathbf{t}_{ae}\mathbf{t}_{c-|b-e|,d}, \qquad \text{if } a<e<b<c<d,\ \bar{b}=\bar{c}, \bar{e}>\bar{a},\ u(a+r)>u($$

$$u\mathbf{t}_{eb}\mathbf{t}_{ae}\mathbf{t}_{c-|b-e|,d} \equiv u\mathbf{t}_{eb}\mathbf{t}_{d-r,d}\mathbf{t}_{ab}, \qquad \text{if } a<e<b<c<d,\ \bar{b}=\bar{c}, \bar{e}>\bar{a},\ u(a+r)\leq 0$$

$$u\mathbf{t}_{ea}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{eb}\mathbf{t}_{c,d-|a-e|}\mathbf{t}_{ea}, \qquad \text{if } c<d<e<a<b,\ \bar{c}<\bar{e},\ \bar{a}=\bar{d},\ u(b-r)\leq 0$$

$$u\mathbf{t}_{ea}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ea}\mathbf{t}_{d,c+r}\mathbf{t}_{b-r,a}, \qquad \text{if } c<d<e<a<b,\ \bar{c}>\bar{e},\ \bar{a}=\bar{d},\ u(b-r)\leq 0$$

$$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef}, \qquad \text{if } c<d<a<e<f<b,\ \ \bar{a}=\bar{d},\ u(b-r)\leq 0$$

$$u\mathbf{t}_{ea}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{eb}\mathbf{t}_{c,d-|a-e|}\mathbf{t}_{ea}, \qquad \text{if } c<d<e<a<b,\ \bar{c}<\bar{e},\ \bar{a}=\bar{d},\ u(b-r)\leq 0$$

Dual Knuth Equivalence for k-Schur $K_{[u,w]}$ –III–

$$u\mathbf{t}_{ea}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ea}\mathbf{t}_{d,c+r}\mathbf{t}_{b-r,a}, \quad \text{if } c<d<e<a<b,\ \bar{c}>\bar{e},\ \bar{a}=\bar{d},\ u(b-r)\leq 0$$

$$u\mathbf{t}_{ea}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{eb}\mathbf{t}_{c,d-|a-e|}\mathbf{t}_{ea}, \quad \text{if } c<d<e<a<b,\ \bar{c}<\bar{e},\ \bar{a}=\bar{d},\ u(b-r)\leq 0$$

$$u\mathbf{t}_{ea}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ea}\mathbf{t}_{d,c+r}\mathbf{t}_{b-r,a}, \quad \text{if } c<d<e<a<b,\ \bar{c}>\bar{e},\ \bar{a}=\bar{d},\ u(b-r)\leq 0$$

$$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef}, \quad \text{if } c<d<a<e<f<b,\ \bar{a}=\bar{d},\ u(b-r)\leq 0$$

$$u\mathbf{t}_{ea}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ea}\mathbf{t}_{c,c+r}\mathbf{t}_{ab}, \quad \text{if } c<d<e<a<b,\ \bar{c}\leq e,\ \bar{a}=\bar{d},\ u(b-r)>0$$

$$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef}, \quad \text{if } c<d<a<e<f<b,\ \bar{a}=\bar{d},\ u(b-r)>0$$

$$u\mathbf{t}_{ea}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ea}\mathbf{t}_{d-r,c}\mathbf{t}_{b,a+r}, \quad \text{if } c<d<e<a<b,\ \bar{c}\neq\bar{e}\leq\bar{d},\ \bar{b}=\bar{c},\ u(c)>0$$

$$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef}, \quad \text{if } c<d<a<e<f<b,\ \bar{b}=\bar{c},\ u(c)>0$$

$$u\mathbf{t}_{ea}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ea}\mathbf{t}_{cd}\mathbf{t}_{a,a+r}, \quad \text{if } c<d<e<a<b,\ \bar{b}=\bar{c},\ u(c)\leq 0$$

$$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef}, \quad \text{if } c<d<a<e<f<b,\ \bar{b}=\bar{c},\ u(c)\leq 0$$

$$u\mathbf{t}_{ea}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ef}\mathbf{t}_{c+|b-a|}\mathbf{t}_{ab}, \quad \text{if } c<d<e<a<b,\ \bar{a}=\bar{c}<\bar{b}<\bar{d},\ u(b)>0$$

$$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ef}\mathbf{t}_{d,c-r}\mathbf{t}_{b-r,a}, \quad \text{if } c<d<e<f<a<b,\ \bar{a}=\bar{d}, e\neq\bar{b},\ u(b-r)\leq$$

$$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ef}\mathbf{t}_{cd}\mathbf{t}_{ab}, \quad \text{if } e<c<d<f<a<b,\ \bar{a}=\bar{d},\ u(b-r)\leq 0$$

Dual Knuth Equivalence for k-Schur $K_{[u,w]}$ –IV–

$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ef}\mathbf{t}_{c,c+r}\mathbf{t}_{ab},$ if $c < d < e < f < a < b,\ \bar{a} = \bar{d}, f \neq \bar{c},\ u(b-r) >$

$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ef}\mathbf{t}_{cd}\mathbf{t}_{ab},$ if $e < c < d < f < a < b,\ \bar{a} = \bar{d},\ u(b-r) > 0$

$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ef}\mathbf{t}_{d-r,c}\mathbf{t}_{b,a+r},$ if $c \leq e \leq d < f < a < b,\ \bar{b} = \bar{c},\ u(c) > 0$

$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ef}\mathbf{t}_{cd}\mathbf{t}_{ab},$ if $e < c < d < f < a < b,\ \bar{b} = \bar{c},\ u(b-r) \leq 0$

$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ef}\mathbf{t}_{cd}\mathbf{t}_{a.a+r},$ if $c < d \leq e < f < a < b,\ \bar{b} = \bar{c}, f \neq \bar{c},\ u(c) \leq 0$

$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ef}\mathbf{t}_{cd}\mathbf{t}_{ab},$ if $e < c < d < f < a < b,\ \bar{b} = \bar{c},\ u(c) \leq 0$

$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ef}\mathbf{t}_{cd}\mathbf{t}_{\bar{d}b},$ if $c < d \leq e < f < a < b,\ \bar{a} = \bar{c} < \bar{d} < \bar{b} \neq e,\ u(c+$

$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ef}\mathbf{t}_{cd}\mathbf{t}_{ab},$ if $e < c < d < f < a < b,\ \bar{a} = \bar{c},\ u(c+r) \leq 0$

$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ef}\mathbf{t}_{c+|b-a|,d}\mathbf{t}_{ab},$ if $c < d \leq e < f < a < b,\ \bar{a} = \bar{c} < \bar{b} < \bar{d},\ u(b) > 0$

$u\mathbf{t}_{ef}\mathbf{t}_{ab}\mathbf{t}_{cd} \equiv u\mathbf{t}_{ef}\mathbf{t}_{cd}\mathbf{t}_{ab},$ if $e < c < d < f < a < b,\ \bar{a} = \bar{c} < \bar{b} < \bar{d},\ u(b) > 0$

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Dual Knuth Equivalence for k-Schur $K_{[u,w]}$ –V–

$$\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv \mathbf{t}_{cd}\mathbf{t}_{ab}\mathbf{t}_{ef} \quad \text{if } a < b < e < f < c < d,\ \bar{a} < \bar{c} < \bar{b} < \bar{d}$$

$$\mathbf{t}_{ab}\mathbf{t}_{cd}\mathbf{t}_{ef} \equiv \mathbf{t}_{cd}\mathbf{t}_{ab}\mathbf{t}_{ef} \quad \text{if } a < b < e < f < c < d,\ \bar{c} < \bar{a} < \bar{d} < \bar{b}$$

AND A FEW MORE THAT WE OMITED AT THIS TIME...

ADIOS

MUCHAS GRACIAS