Hopf Monoids of supercharacters

Nantel Bergeron

www.math.yorku.ca/bergeron
E-Mail: **bergeron@yorku.ca**

(with **M. Aguiar and N. Thiem**)

(C. Benedetti, A. Lauve, F. Saliola and many more)

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UNIVERSITY

Outline

- Supercharacter theory of $U_n(q)$ $SC(q)$ a Hopf algebra
- Symmetric functions in noncommutative variables $NCSym$ a Hopf algebra
- Isomorphism (based on 28 authors AIM paper)
- Hopf monoid (on species)
- Hopf monoids related to $U_n(q)$
- What can we do with this? conceptual and computational (Higman's conjecture, primitive,antipode formulas)

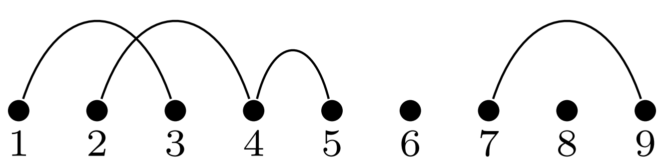
Supercharacter theory of $U_n(q)$

lumping conjugacy classes and characters together to get a more tame theory André, Diaconis-Isaac.

- Unipotent upper triangular matrices over finite Fields $\mathbf{F}_q$: $U_n(q)$.
- Superclasses in $U_n(q)$:

$$A \cong B \quad \leftrightarrow \quad (A - I) = DM(B - I)N$$

superclass representative has at most one 1 in each row and column (strictly above the diagonal).

$\lambda =$ 

Supercharacter theory of $U_n(q)$

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- Unipotent upper triangular matrices over finite Fields $\mathbf{F}_q$: $U_n(q)$.
- Superclasses in $U_n(q)$ λ
- Supercharacters χ^λ Hopf algebra structure see ArXive 28 author paper:

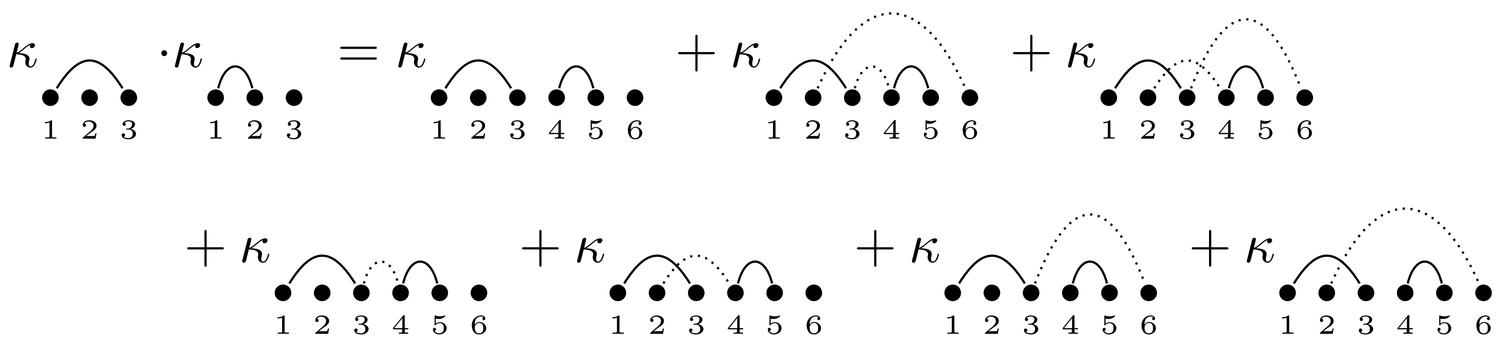
$$\Delta(\chi) = \sum_{A+B=[n]} \mathrm{Res}\,^{U_n(q)}_{U_{|A|}(q)\times U_{|B|}(q)}\,\chi$$

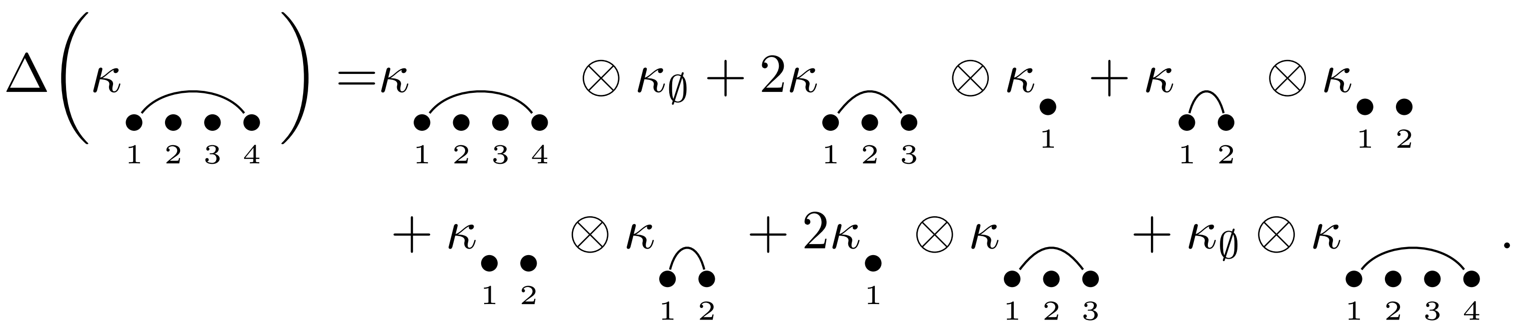
$$\chi \cdot \psi = \mathrm{Inf}\,^{U_{n+m}(q)}_{U_n(q)\times U_m(q)}\,\chi \otimes \psi = (\chi \otimes \psi) \circ \pi$$

where $\pi\colon U_{n+m}(q) \to U_n(q) \times U_m(q)$.

Supercharacter theory of $U_n(q)$

- Superclass functions κ_λ basis Hopf algebra structure is nice:

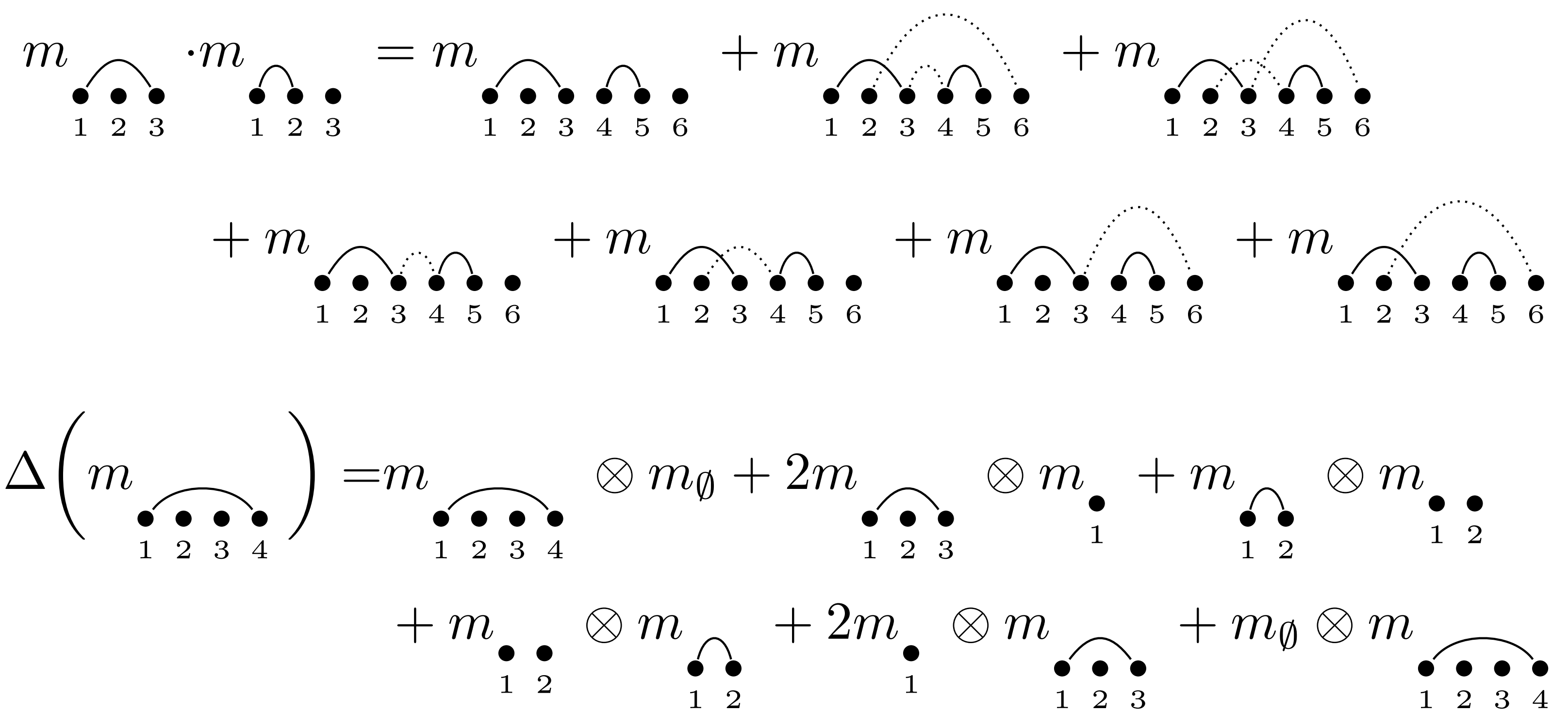




Symmetric functions in noncommutative variables

Wolf, Rosas-Sagan, Bergeron-Zabrocki, ... Reutenauer ...

- monomial symmetric functions m_λ basis (sum of orbit of a word)
- indexed by set partitions Hopf algebra structure is nice:



Isomorphism

• the Hopf algebra of symmetric functions in noncommutative variables is isomorphic to the Hopf algebra of supperclass functions.

• Now we want to better understand these structures.

• Where is q? [see nice paper by Bergeron-Thiem to appear in the volume dedicated to CR]

Hopf Monoid

A nice Combinatorial Hopf Algebra is indexed by combinatorial object with a lot of structure. It should have a lift at the level of species (Hopf monoid). These are "Hopf algebras" graded by finite sets instead of integers.

This explains much of the structures of the Combinatorial Hopf algebras with more elegant and simplified formulas.

Also we have many Combinatorial Hopf algebras associated to one Hopf monoid.

Aguiar-Mahajan

Hopf Monoid

- A Species $\mathbf{P}$ encode data: $\forall$ finite set $K \mapsto \mathbf{P}[K]$ vector space.

Typically, $\mathbf{P}[K]$ is the formal linear span of combinatorial objects of type "$\mathbf{P}$" (think Graphs, Set partitions, ...)

A bijection $K \xrightarrow{\sigma} T$ should induce a bijection $\mathbf{P}[K] \xrightarrow{\mathbf{P}[\sigma]} \mathbf{P}[T]$

and other natural conditions:

$\mathbf{P}[Id_K] = Id_{\mathbf{P}[K]}$,

$\mathbf{P}[\sigma \circ \tau] = \mathbf{P}[\sigma] \circ \mathbf{P}[\tau]$,

...

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- Monoidal structure '$\bullet$' on species:

$$(\mathbf{p} \bullet \mathbf{q})[K] = \bigoplus_{K=I \sqcup J} \mathbf{p}[I] \otimes \mathbf{q}[J]\,,$$

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• A Hopf Monoid $\mathbf{H}$ is a species such that $\mathbf{H}[\emptyset] = \mathbf{C}$ and structure maps m, Δ, u, ϵ, S.

(1) $m_{I,J} \colon \mathbf{H}[I] \otimes \mathbf{H}[J] \to \mathbf{H}[K], \qquad \forall I \sqcup J = K$

Associative multiplication $m \colon \mathbf{H} \bullet \mathbf{H} \to \mathbf{H}$ with unity $u \colon \mathbf{1} \to \mathbf{H}$

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(2) $\Delta_{I,J} \colon \mathbf{H}[K] \to \mathbf{H}[I] \otimes \mathbf{H}[J], \qquad \forall I \sqcup J = K$

Associative comultiplication $\Delta \colon \mathbf{H} \to \mathbf{H} \bullet \mathbf{H}$ with counit $\epsilon \colon \mathbf{H} \to \mathbf{1}$

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(2) $\Delta_{I,J} \colon \mathbf{H}[K] \to \mathbf{H}[I] \otimes \mathbf{H}[J], \qquad \forall I \sqcup J = K$

(3) $\Delta_{I,J} \circ m_{I',J'} = (m_{A,C} \otimes m_{B,D})(\mathrm{id}_A \otimes \beta_{B,C} \otimes \mathrm{id}_D)(\Delta_{A,B} \otimes \Delta_{C,D})$

where for $K = I \sqcup J = I' \sqcup J'$ there is unique

$A = I' \cap I$, $B = I' \cap J$, $C = J' \cap I$ and $D = J' \cap J$.

Hopf Monoid

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The antipode $S \colon \mathbf{H} \to \mathbf{H}$ is constructed (for free) recursively.

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These structure are more natural than they *look*:

(1) given two combinatorial objects, give a rule to build larger one

(2) Given a decomposition of the ground set, give ways to decompose combinatorial object

(3) (typically) Find bijection between two constructions.

Matrices indexed by finite set

Given a finite set I and a linear order ℓ on I.

$\mathrm{M}(I)$: $A = (a_{ij})_{i,j \in I}$, $a_{ij} \in \mathbb{F}_q$ for all $i, j \in I$.

$\mathrm{GL}(I)$: the invertible matrices in $\mathrm{M}(I)$.

$\mathrm{U}(I, \ell)$: the *upper ℓ-unitriangular* matrices

$$U = (u_{ij})_{i,j \in I},\ u_{ii} = 1 \text{ for all } i \in I,\ u_{ij} = 0 \text{ whenever } i >_\ell j.$$

Matrices indexed by finite set

$\mathrm{U}(I,\ell)$: the *upper ℓ-unitriangular* matrices

DIRECT SUM: Given $K = I \sqcup J$ and ℓ on K,

$$\begin{aligned} \sigma_{I,J}\colon \mathrm{U}(I,\ell|_I)\times \mathrm{U}(J,\ell|_J) &\to \mathrm{U}(K,\ell) \\ (A,B) &\mapsto A\oplus B \end{aligned}$$

MINORS: Given $K = I \sqcup J$ and ℓ_1, ℓ_2 on I, J (resp.),

$$\begin{aligned} \pi_{I,J}\colon \mathrm{U}(K,\ell_1\cdot\ell_2) &\to \mathrm{U}(I,\ell_1)\times \mathrm{U}(J,\ell_2) \\ U &\mapsto \left(U\big|_I, U\big|_J\right) \end{aligned}$$

Group Morphisms.

Hopf monoids using U

Direct sums and Minors allow us to define Hopf monoids:

$\mathbf{f}(\mathrm{U})$: The species of functions on U.

$$\mathbf{f}(\mathrm{U})[I] = \bigoplus_{\ell \in L[I]} \mathbf{f}\big(\mathrm{U}(I,\ell)\big),$$

where $L[I]$ is the set of linear orders on I and $\mathbf{f}\big(\mathrm{U}(I,\ell)\big)$ is the space of functions $\mathrm{U}(I,\ell) \to \mathbb{C}$

with basis: $\kappa_{(U,\ell)}$ the characteristic function of $U \in \mathrm{U}(I,\ell)$

$$\kappa_{(U,\ell)}(V,\tau) = \begin{cases} 1 & \text{if } \ell = \tau \text{ and } U = V, \\ 0 & \text{if not.} \end{cases}$$

Hopf monoids using U

Direct sums and Minors allow us to define Hopf monoids:

$\mathbf{f}(\mathrm{U})$: $\mathbf{f}(\mathrm{U})[I] = \bigoplus_{\ell \in L[I]} \mathbf{f}\big(\mathrm{U}(I,\ell)\big)$,

with basis: $\kappa_{(U,\ell)}$ the characteristic function of $U \in \mathrm{U}(I,\ell)$

Multiplication: $U \in \mathrm{U}(I,\ell)$, $V \in \mathrm{U}(J,\tau)$ and $I \sqcup J = K$,

$$\kappa_{(U,\ell)}\kappa_{(V,\tau)} = \sum_{\pi_{I,J}(W)=(U,V)} \kappa_{(W,\ell\cdot\tau)}$$

Comultiplication: for $W \in \mathrm{U}(K,\ell)$ and $I \sqcup J = K$,

$$\Delta_{I,J}\big(\kappa_{(W,\ell)}\big) = \sum_{\sigma_{I,J}(U,V)=W} \kappa_{(U,\ell|_I)} \otimes \kappa_{(V,\ell|_J)}$$

Associativity and compatibility follow from properties of $\pi_{I,J}$ and $\sigma_{I,J}$.

Hopf monoids using U

Direct sums and Minors allow us to define Hopf monoids:

$\mathbf{f}(\mathrm{U})$: $\mathbf{f}(\mathrm{U})[I] = \bigoplus_{\ell \in L[I]} \mathbf{f}\big(\mathrm{U}(I,\ell)\big)$; basis: $\big\{\kappa_{(U,\ell)}\big\}$.

$\mathbf{cf}(\mathrm{U})$: $\mathbf{cf}(\mathrm{U})[I] = \bigoplus_{\ell \in L[I]} \mathbf{cf}\big(\mathrm{U}(I,\ell)\big)$; basis: $\big\{\kappa_{(C,\ell)}\big\}$.

$\mathbf{cf}\big(\mathrm{U}(I,\ell)\big)$: the subspace of class functions on $\mathrm{U}(I,\ell)$

$\kappa_{(C,\ell)} = \sum_{U\in C} \kappa_{(U,\ell)}$ for a conjugacy class C.

$\mathbf{cf}(\mathrm{U}) \hookrightarrow \mathbf{f}(\mathrm{U})$ is Hopf submonoid.

This follow from the fact that $\pi_{I,J}$ and $\sigma_{I,J}$ are group morphisms.

Hopf monoids using U

Direct sums and Minors allow us to define Hopf monoids:

$\mathbf{f}(\mathrm{U})$: $\mathbf{f}(\mathrm{U})[I] = \bigoplus_{\ell \in L[I]} \mathbf{f}\big(\mathrm{U}(I,\ell)\big)$; basis: $\{\kappa_{(U,\ell)}\}$.

$\mathbf{cf}(\mathrm{U})$: $\mathbf{cf}(\mathrm{U})[I] = \bigoplus_{\ell \in L[I]} \mathbf{cf}\big(\mathrm{U}(I,\ell)\big)$; basis: $\{\kappa_{(C,\ell)}\}$.

$\mathbf{scf}(\mathrm{U})$: $\mathbf{scf}(\mathrm{U})[I] = \bigoplus_{\ell \in L[I]} \mathbf{scf}\big(\mathrm{U}(I,\ell)\big)$; basis: $\{\kappa_{(\lambda,\ell)}\}$.

$\mathbf{scf}\big(\mathrm{U}(I,\ell)\big)$: the subspace of superclass functions on $\mathrm{U}(I,\ell)$

$\kappa_{(\lambda,\ell)} = \sum_{U \in \lambda} \kappa_{(U,\ell)}$ for a superclass λ.

$$\mathbf{scf}(\mathrm{U}) \hookrightarrow \mathbf{cf}(\mathrm{U}) \hookrightarrow \mathbf{f}(\mathrm{U}) \text{ Hopf submonoids.}$$

This follow from the fact that $\pi_{I,J}$ and $\sigma_{I,J}$ are behave well with superclasses.

Hopf monoids using U

Direct sums and Minors allow us to define Hopf monoids:

f(U): $\mathbf{f}(\mathrm{U})[I] = \bigoplus_{\ell \in L[I]} \mathbf{f}\big(\mathrm{U}(I,\ell)\big)$; basis: $\big\{\kappa_{(U,\ell)}\big\}$.

cf(U): $\mathbf{cf}(\mathrm{U})[I] = \bigoplus_{\ell \in L[I]} \mathbf{cf}\big(\mathrm{U}(I,\ell)\big)$; basis: $\big\{\kappa_{(C,\ell)}\big\}$.

scf(U): $\mathbf{scf}(\mathrm{U})[I] = \bigoplus_{\ell \in L[I]} \mathbf{scf}\big(\mathrm{U}(I,\ell)\big)$; basis: $\big\{\kappa_{(\lambda,\ell)}\big\}$.

L: $\mathbf{scf}(\mathrm{U})[I] = \bigoplus_{\ell \in L[I]} \mathbb{C}1_\ell$; basis: $\big\{1_\ell\big\}$.

$$\mathbf{L} \hookrightarrow \mathbf{scf}(\mathrm{U}) \hookrightarrow \mathbf{cf}(\mathrm{U}) \hookrightarrow \mathbf{f}(\mathrm{U})$$ Hopf submonoids.

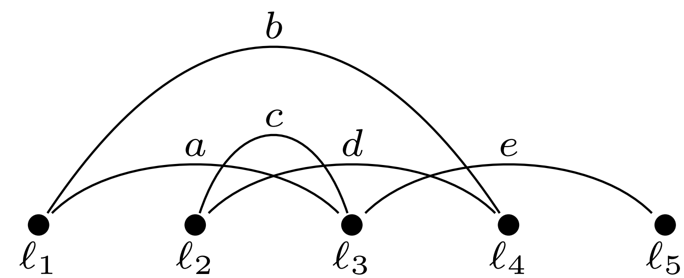
Combinatorial model for Hopf monoids

f(U); basis: $\{\kappa_{(U,\ell)}\}$; labeled graph on a linear order $\mathbf{L} \times \mathbf{G}_q$

cf(U); basis: $\{\kappa_{(C,\ell)}\}$;

scf(U); basis: $\{\kappa_{(\lambda,\ell)}\}$;

$$U = \begin{pmatrix} 1 & 0 & a & b & 0 \\ & 1 & c & d & 0 \\ & & 1 & 0 & e \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \Rightarrow$$



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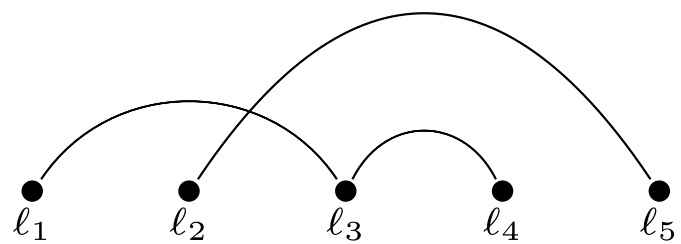
scf(U); basis: $\{\kappa_{(\lambda,\ell)}\}$; set partitions on a linear order $\mathbf{L} \times \mathbf{\Pi}$

$$U = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 1 \\ & & 1 & 1 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \quad \Rightarrow$$



Lots of representation properties are read on the picture

Combinatorial model for Hopf monoids

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cf(U); basis: $\{\kappa_{(C,\ell)}\}$; W I L D !!!

scf(U); basis: $\{\kappa_{(\lambda,\ell)}\}$; set partitions on a linear order $\mathbf{L} \times \mathbf{\Pi}$

???

Combinatorial model for Hopf monoids

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THEOREM

f(U), **cf**(U) and **scf**(U) are free monoid.

CONJECTURE[Higman]

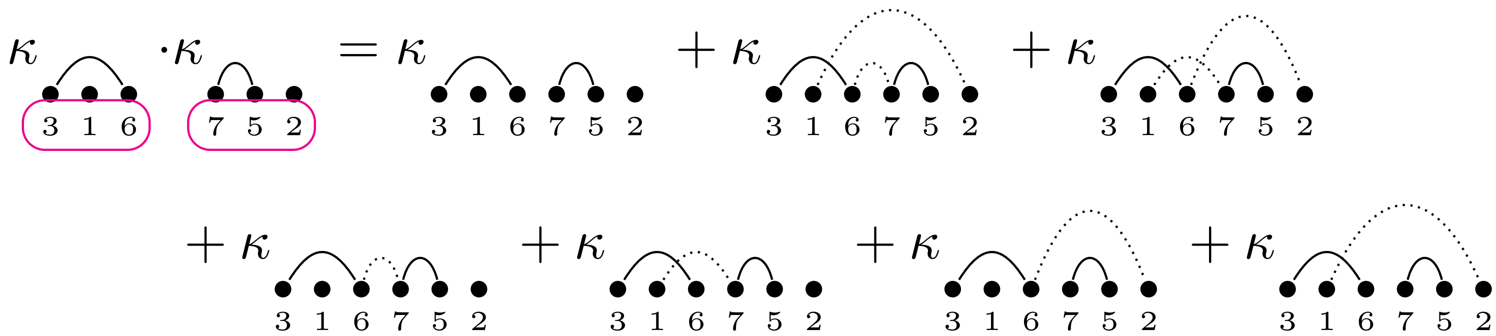
$\dim \mathbf{cf}\big(\mathrm{U}(I,\ell)\big)$ is a positive polynomial in $q-1$.

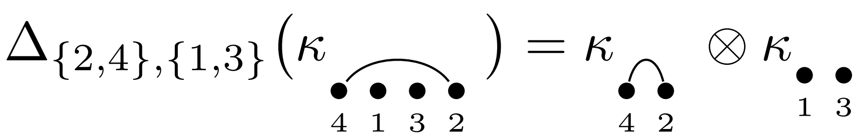
CONJECTURE(refinement)

The number of free generators of $\mathbf{cf}\big(\mathrm{U}(I,\ell)\big)$
is a positive polynomial in $q-1$.

we can also derive upper and lower bound for these number using $\mathbf{scf}(\mathrm{U}) \hookrightarrow \mathbf{cf}(\mathrm{U}) \hookrightarrow \mathbf{f}(\mathrm{U})$

Hopf Monoid scf(U)



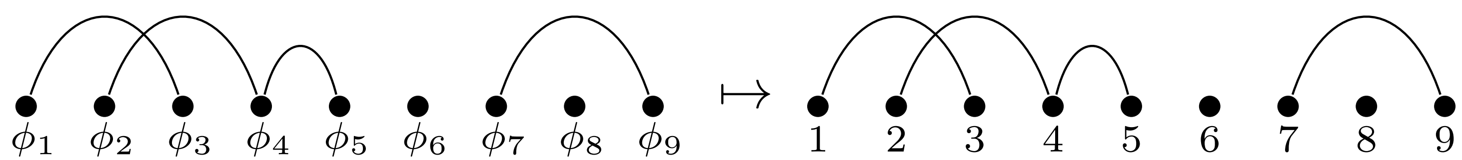


What can we do with this?

- Functor that send Hopf monoid to Hopf algebras

$$\mathbf{scf}(\mathrm{U}) \mapsto NCSym$$

$$(\phi, \lambda) \mapsto \phi^{-1} \circ \lambda$$



The structure stay the same but the linear order is forgotten.

What can we do with this?

• Gives new basis (that depend on q)

• The simplified structure of **scf**(U) gives multiplicity free and cancelation free formula for the antipode of $\kappa_{(\phi,\lambda)}$ and other basis.

Functor that send **scf**(U) to $NCSym$

• Gives new basis (that depend on q)

• Gives cancelation free formula for the antipode of $\kappa_{(\phi,\lambda)}$ and other basis.

• Gives us some understanding of antipode for supercharacter χ^λ. In particular we show that given $\lambda = \lambda^{(1)}|\cdots|\lambda^{(k)}$ with its unique factorization into atomics.

$$S(\chi^\lambda) = (-1)^k S(\chi^{\lambda^{(k)}|\cdots|\lambda^{(1)}}) + \text{lower term in some order.}$$