Base Immaculée
et les coefficients de LR

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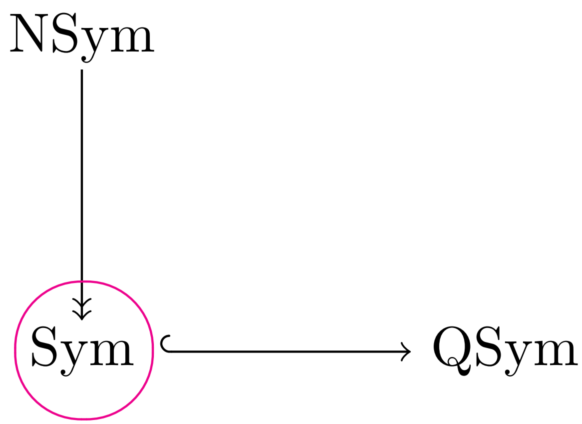
Aperçu

- NSym, Sym, QSym Algèbres de Hopf Combinatoire.
- Base Immaculée I
- Retour vers Sym
- Base Immaculée II: créée à partir du vide
- Les coefficient de Littlewood-Richardson.
- Polytope de ruches.

Sym, NSym, QSym

$$\begin{array}{ccc} \mathrm{NSym} & & \\ \big\downarrow\!\!\downarrow & & \\ \mathrm{Sym} & \hookrightarrow & \mathrm{QSym} \end{array}$$

Sym, NSym, QSym



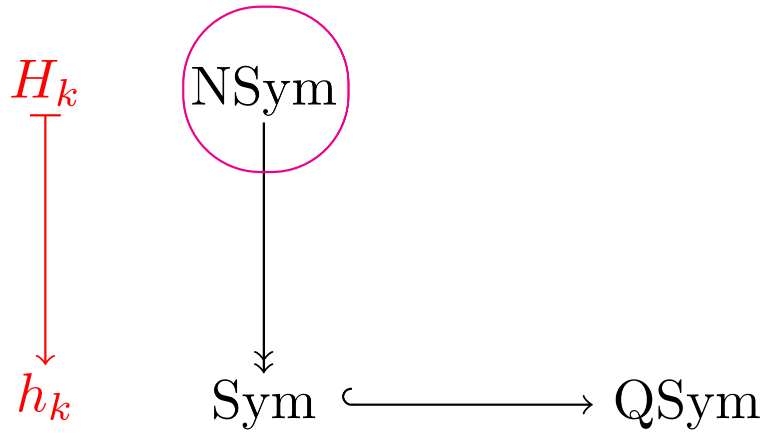
Sym: Symmetric functions

Basis: e_λ (Elementary); m_λ (monomials); p_λ (Power sums);

$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$ (Homogeneous) $\sum_{k \geq 0} h_k t^k = \prod_{i \geq 1} \frac{1}{1 - x_i t}$

$s_\lambda = \det(h_{\lambda_i + j - i})$ (Schur)

Sym, NSym, QSym



Sym: Symmetric functions $= \mathbb{Z}[h_1, h_2, \ldots]$

NSym: noncommutative symmetric functions $= \mathbb{Z}\langle H_1, H_2, \ldots \rangle$

Sym, NSym, QSym

$$\mathfrak{S}_\alpha \longmapsto s_\alpha = \det\left(h_{\alpha_i+j-i}\right)$$

$$\mathrm{NSym} \twoheadrightarrow \mathrm{Sym} \hookrightarrow \mathrm{QSym}$$

Sym: Symmetric functions

NSym: noncommutative symmetric functions = $\mathbb{Z}\langle H_1, H_2, \ldots \rangle$

a quest for Schur function in NSym

Immaculate noncommutative symmetric functions [B-B-S-S-Z]

$$\boxed{\mathfrak{S}_\alpha := \sum_{\sigma \in S_m} (-1)^\sigma H_{\alpha_1+\sigma_1-1} H_{\alpha_2+\sigma_2-2} \cdots H_{\alpha_m+\sigma_m-m}}$$

Sym, NSym, QSym

$$\begin{array}{ccc} \mathrm{NSym} & & \\ \downarrow\!\!\downarrow & & \\ \mathrm{Sym} & \hookrightarrow & \mathrm{QSym} \quad {\color{red}\mathfrak{S}^*_\alpha} \end{array}$$

Sym: Symmetric functions

NSym: noncommutative symmetric functions = $\mathbb{Z}\langle H_1, H_2, \ldots \rangle$

QSym: Quasisymmetric functions = $NSym^*$

Littlewood-Richardson in Sym

THEOREM Littlewood-Richardson ($c^{\nu}_{\mu,\lambda} \geq 0$)

$$s_\mu s_\lambda = \sum_\beta c^{\nu}_{\mu,\lambda} s_\nu$$

$c^{\nu}_{\mu,\lambda}$ is the number of column strict tableaux of shape ν/μ, content λ and Yamanouchi.

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$c^{\nu}_{\mu,\lambda}$ is the number of column strict tableaux of shape ν/μ, content λ and Yamanouchi.

$c^{\nu}_{\mu,\lambda}$ is the number of integral points in a certain polytope in $\binom{m}{2}$-dimension where $m = \max\{\ell(\alpha), \ell(\lambda), \ell(\beta)\}$.

$$x_{00}$$

$$x_{01} \quad x_{11}$$

$$x_{02} \quad x_{12} \quad x_{22}$$

Littlewood-Richardson in Sym

$c^{\nu}_{\mu,\lambda}$ is the number of integral points in a certain polytope with $\binom{m}{2}$ variables x_{ij}:

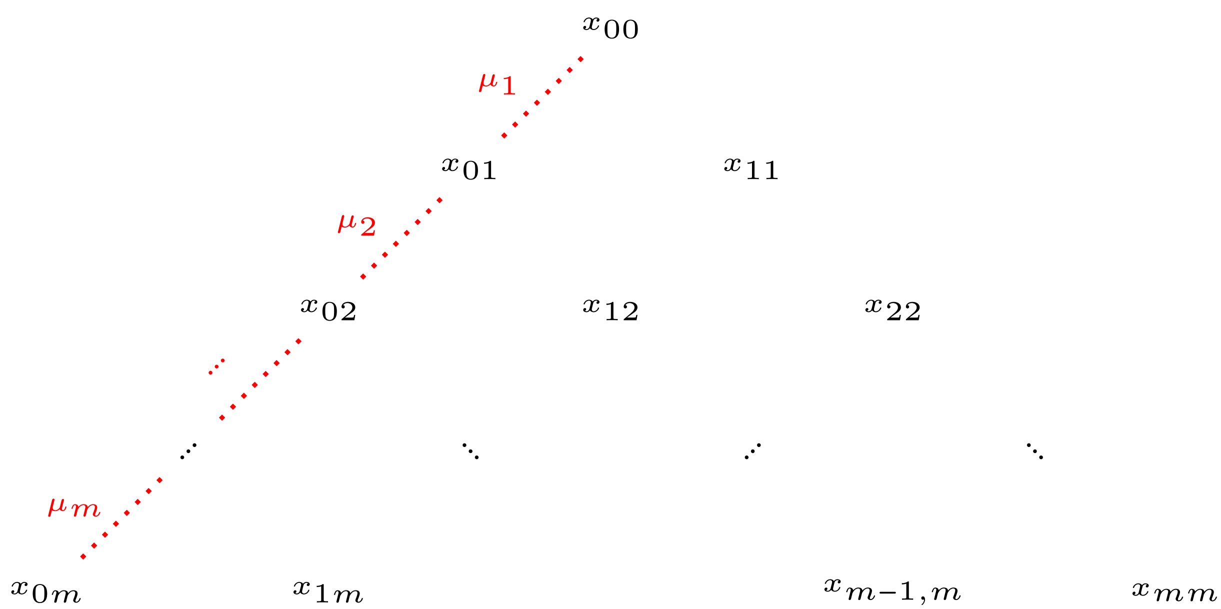
$$
\begin{array}{ccccccccc}
 & & & & x_{00} & & & & \\
 & & & x_{01} & & x_{11} & & & \\
 & & x_{02} & & x_{12} & & x_{22} & & \\
 & \iddots & & \ddots & & \iddots & & \ddots & \\
x_{0m} & & x_{1m} & & & & x_{m-1,m} & & x_{mm}
\end{array}
$$

Littlewood-Richardson in Sym

$x_{0j} - x_{0\,j-1} = \mu_j$ $\quad x_{jj} - x_{j-1\,j-1} = \nu_j$ $\quad x_{im} - x_{i-1\,m} = \lambda_i$

$x_{ij} - x_{i\,j-1} \geq x_{i-1\,j} - x_{i-1\,j-1}$ $\quad x_{ij} - x_{i-1\,j} \geq x_{i+1\,j+1} - x_{i\,j+1}$

$x_{i-1\,j} - x_{i-1\,j-1} \geq x_{i\,j+1} - x_{ij}$



Littlewood-Richardson in Sym

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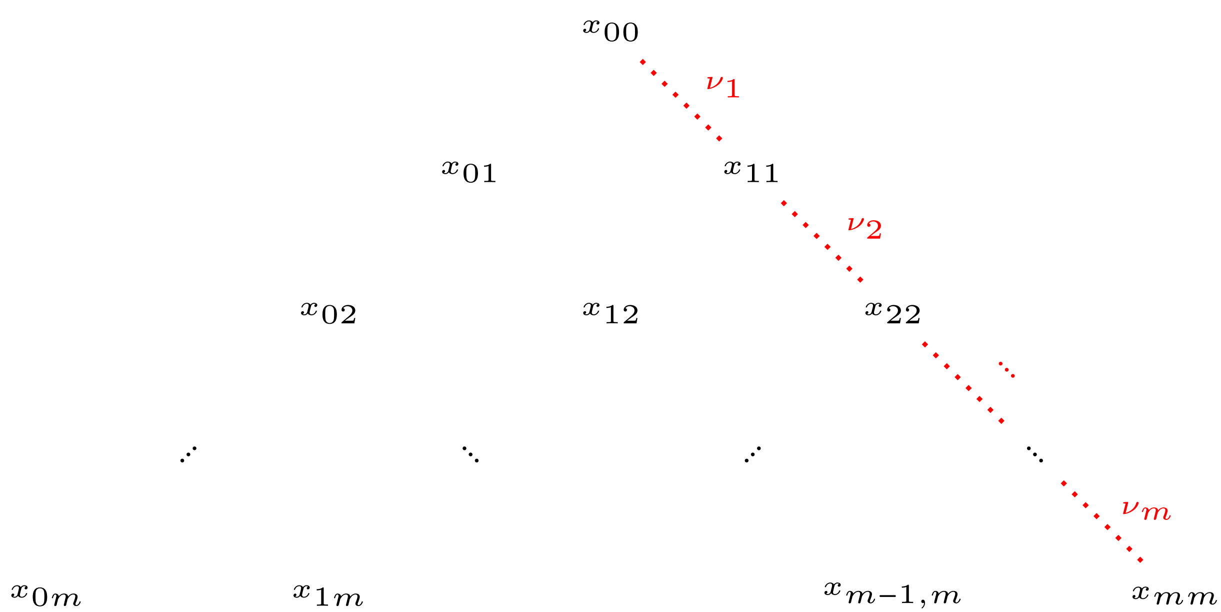

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$$x_{01} \qquad x_{11}$$

$$x_{02} \qquad x_{12} \qquad x_{22}$$

$$\iddots \qquad \ddots \qquad \iddots \qquad \ddots$$

$$\lambda_1 \qquad \cdots \qquad \lambda_m$$

$$x_{0m} \cdots\cdots x_{1m} \cdots\cdots x_{m-1,m} \cdots\cdots x_{mm}$$

Littlewood-Richardson in Sym

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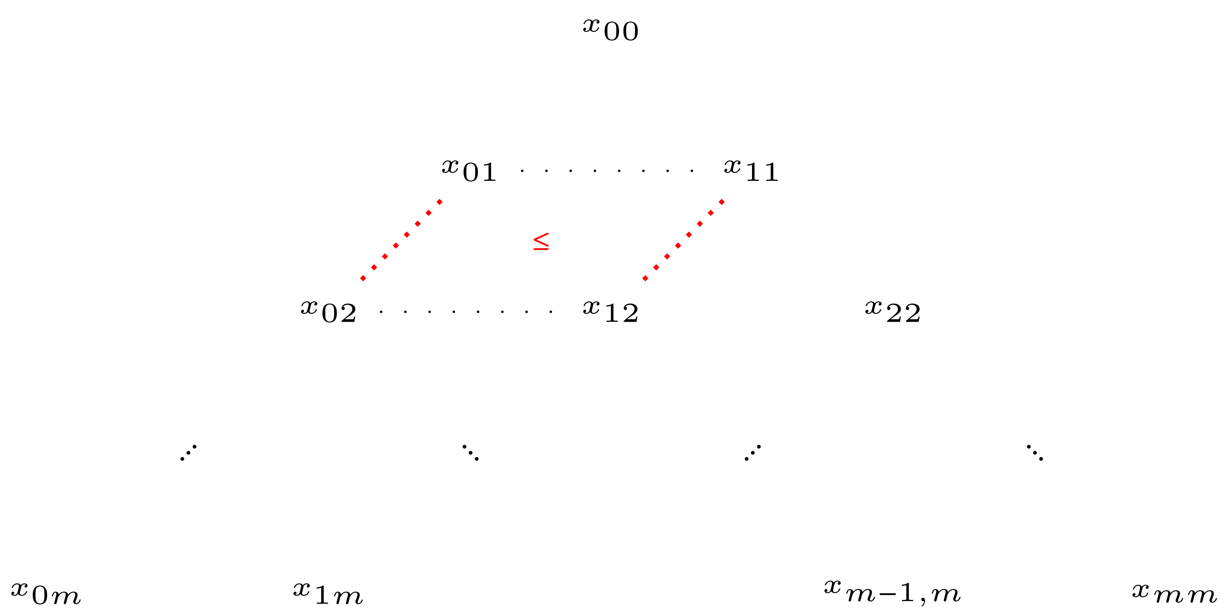

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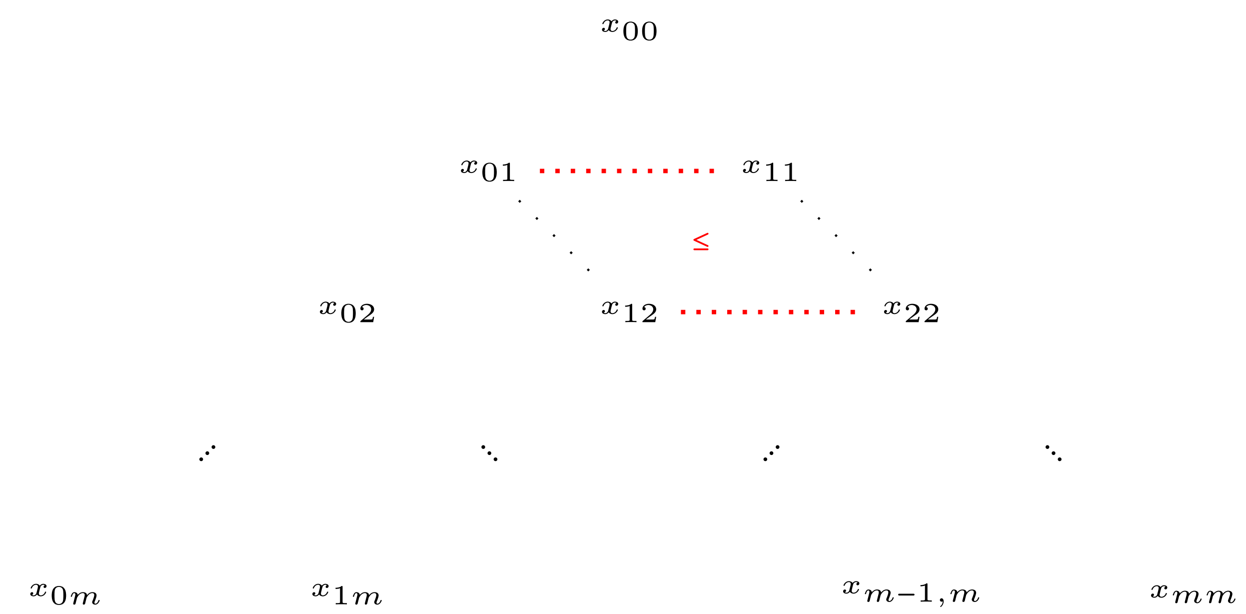

Littlewood-Richardson in Sym

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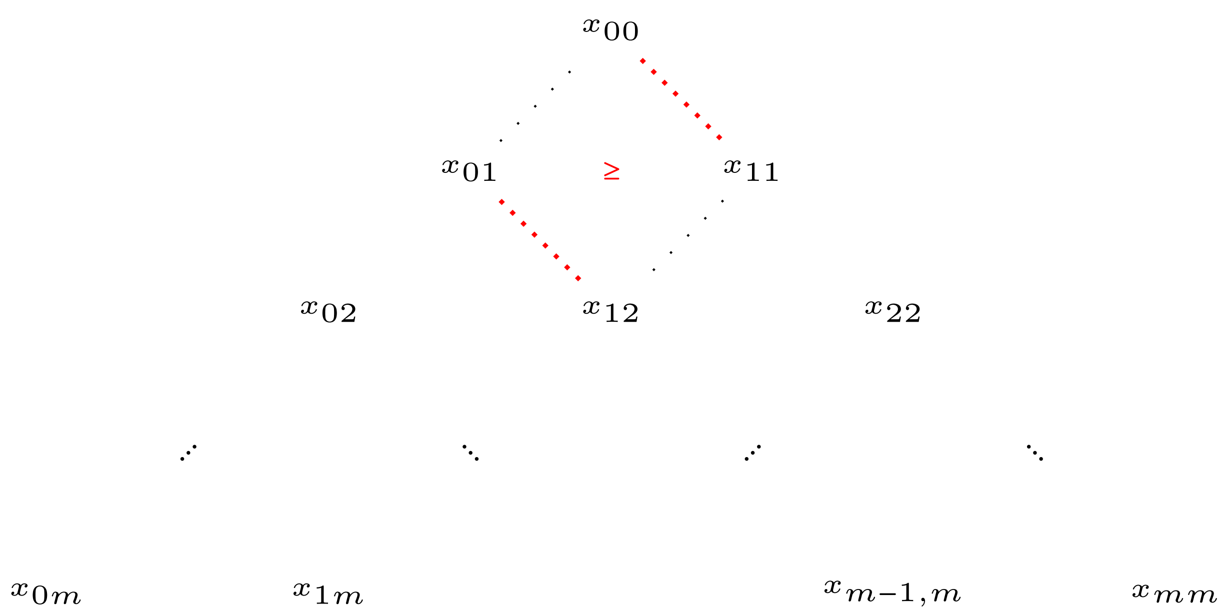

Immaculate noncommutative symmetric functions

$$\mathfrak{S}_\alpha := \sum_{\sigma \in S_m} (-1)^\sigma H_{\alpha_1+\sigma_1-1} H_{\alpha_2+\sigma_2-2} \cdots H_{\alpha_m+\sigma_m-m}$$

where we use the convention that $H_0 = 1$ and $H_{-m} = 0$ for $m > 0$.

Immaculate noncommutative symmetric functions

$$\mathfrak{S}_\alpha := \sum_{\sigma \in S_m} (-1)^\sigma H_{\alpha_1+\sigma_1-1} H_{\alpha_2+\sigma_2-2} \cdots H_{\alpha_m+\sigma_m-m}$$

We have $\mathfrak{S}_\alpha = H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_\ell}$ + higher lex term

Hence

$\{\mathfrak{S}_\alpha\}$ is a basis of NSym

Immaculate noncommutative symmetric functions

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$\{\mathfrak{S}_\alpha\}$ is a basis of NSym

Creation operator $\mathbb{B}_m : \mathrm{NSym}_n \to \mathrm{NSym}_{m+n}$

$$\mathbb{B}_m := \sum_{i \geq 0} (-1)^i H_{m+i} F_{1^i}^{\perp},$$

$F_\alpha = R_\alpha^* \in \mathrm{QSym} = \mathrm{NSym}^*$ and $\langle F_{1^i}^{\perp} \varphi, F_\alpha \rangle = \langle \varphi, F_{1^i} F_\alpha \rangle$

$$\mathfrak{S}_\alpha = \mathbb{B}_{\alpha_1} \mathbb{B}_{\alpha_2} \cdots \mathbb{B}_{\alpha_\ell} 1$$

immaculately concieved?

Immaculate noncommutative symmetric functions

$$\mathfrak{S}_\alpha := \sum_{\sigma \in S_m} (-1)^\sigma H_{\alpha_1+\sigma_1-1} H_{\alpha_2+\sigma_2-2} \cdots H_{\alpha_m+\sigma_m-m}$$

$\{\mathfrak{S}_\alpha\}$ is a basis of NSym

$$\mathfrak{S}_\alpha = \mathbb{B}_{\alpha_1} \mathbb{B}_{\alpha_2} \cdots \mathbb{B}_{\alpha_\ell} 1$$

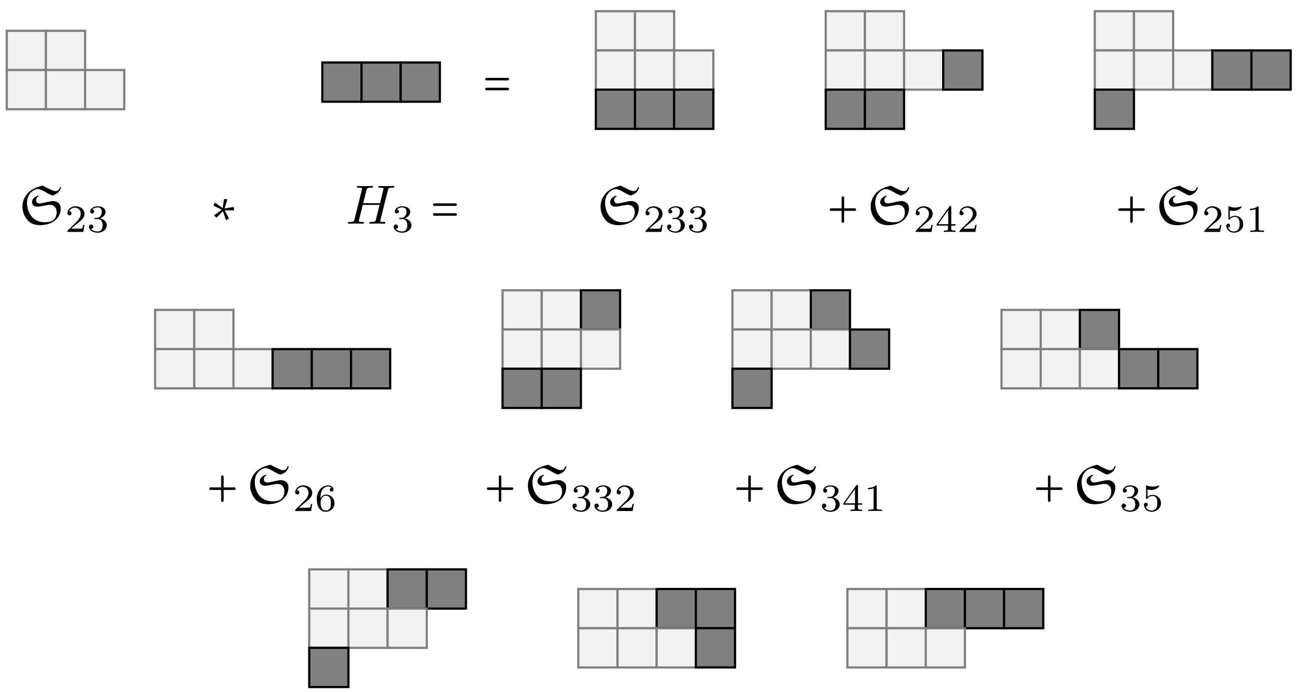
THEOREM Pieri Rule

$$\mathfrak{S}_\alpha H_s = \sum_{\alpha \subset_s \beta} \mathfrak{S}_\beta$$

$\alpha \subset_s \beta$: $\quad |\beta| = |\alpha| + s, \quad \alpha_j \le \beta_j, \quad \ell(\beta) \le \ell(\alpha) + 1.$

Example of Pieri Rule

$$\mathfrak{S}_\alpha H_s = \sum_{\alpha \subset_s \beta} \mathfrak{S}_\beta$$



$$\mathfrak{S}_{23} * H_3 = \mathfrak{S}_{233} + \mathfrak{S}_{242} + \mathfrak{S}_{251} + \mathfrak{S}_{26} + \mathfrak{S}_{332} + \mathfrak{S}_{341} + \mathfrak{S}_{35} + \mathfrak{S}_{431} + \mathfrak{S}_{44} + \mathfrak{S}_{53}$$

Noncommutative Littlewood-Richardson

THEOREM Pieri Rule

$$\mathfrak{S}_\alpha H_s = \sum_{\alpha \subset_s \beta} \mathfrak{S}_\beta$$

THEOREM Noncommutative Littlewood-Richardson ($c^{\beta}_{\alpha,\lambda} \geq 0$)

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_{\beta} c^{\beta}_{\alpha,\lambda} \mathfrak{S}_\beta$$

Noncommutative Littlewood-Richardson

THEOREM Pieri Rule

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THEOREM Noncommutative Littlewood-Richardson ($c^\beta_{\alpha,\lambda} \geq 0$)

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_\beta c^\beta_{\alpha,\lambda} \mathfrak{S}_\beta$$

In general $c^\beta_{\alpha,\delta} \ngeq 0$

left Pieri Rule $c^\beta_{(s),\delta}$ is ± 1 or 0. [B-Sánchez-Zabrocki]

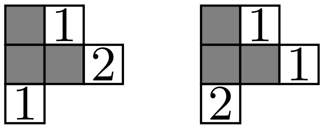
Noncommutative Littlewood-Richardson $c^{\beta}_{\alpha,\lambda}$

THEOREM Noncommutative Littlewood-Richardson ($c^{\beta}_{\alpha,\lambda} \geq 0$)

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_\beta c^{\beta}_{\alpha,\lambda} \mathfrak{S}_\beta$$

$c^{\beta}_{\alpha,\lambda}$ is the number of immaculate tableaux (row strict, only the first column strict), of shape α/β and content λ and Yamanouchi.

For example $C^{(2,3,1)}_{(1,2),(2,1)} = 2$:



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THEOREM Noncommutative Littlewood-Richardson ($c^{\beta}_{\alpha,\lambda} \geq 0$)

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_\beta c^{\beta}_{\alpha,\lambda} \mathfrak{S}_\beta$$

$c^{\beta}_{\alpha,\lambda}$ is the number of immaculate tableaux (row strict, only the first column strict), of shape α/β and content λ and Yamanouchi.

THEOREM for any ν such that $\ell(\nu) \leq \ell(\alpha)$

$$c^{\beta}_{\alpha,\lambda} = c^{\beta+\nu}_{\alpha+\nu,\lambda}$$

Noncommutative Littlewood-Richardson $c^{\beta}_{\alpha,\lambda}$

THEOREM Noncommutative Littlewood-Richardson ($c^{\beta}_{\alpha,\lambda} \geq 0$)

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_\beta c^{\beta}_{\alpha,\lambda} \mathfrak{S}_\beta$$

$c^{\beta}_{\alpha,\lambda}$ is the number of integral points in a certain polytope in $\binom{m}{2}$-dimension where $m = \max\{\ell(\alpha), \ell(\lambda), \ell(\beta)\}$.

$$\begin{matrix} & & x_{00} & & \\ & x_{10} & & x_{11} & \\ x_{20} & & x_{21} & & x_{22} \end{matrix}$$

Noncommutative Littlewood-Richardson $c^{\beta}_{\alpha,\lambda}$

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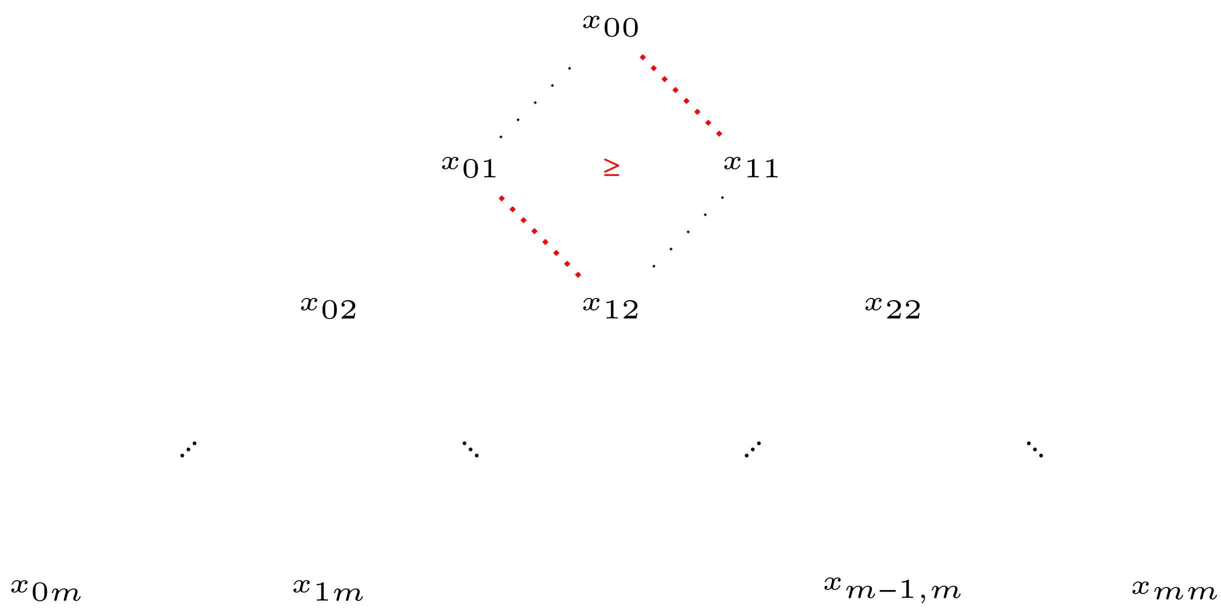
$x_{ij} - x_{i\,j-1} \geq x_{i-1\,j} - x_{i-1\,j-1} \qquad x_{ij} - x_{i-1\,j} \geq x_{i+1\,j+1} - x_{i\,j+1}$

$N(x_{i-1\,j} - x_{i-1\,j-1}) \geq x_{i\,j+1} - x_{ij}$ where $N = m^2$.



Noncommutative LR $c_{\alpha,\lambda}^{\beta}$ (idea of proof)

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_{\sigma \in S_m} (-1)^\sigma \mathfrak{S}_\alpha H_{[\lambda_1+\sigma_1-1, \lambda_2+\sigma_2-2, \ldots, \lambda_m+\sigma_m-m]},$$

$$= \sum_{\sigma \in S_m} \sum_{\substack{sh(T)=\gamma/\alpha \\ c(T)=\lambda+\sigma-Id}} (-1)^\sigma \mathfrak{S}_\gamma.$$

$$\mathfrak{S}_\alpha \mathfrak{S}_\lambda = \sum_{T \in \mathfrak{T}_\alpha^\lambda} (-1)^{\sigma(T)} \mathfrak{S}_{sh(T)}.$$

where

$$\mathfrak{T}_\alpha^\lambda = \left\{ T : \begin{array}{c} T \text{ is a skew immaculate tableau} \\ \text{of inner skew shape } \alpha \text{ for which} \\ c(T) - \lambda + Id \text{ is a permutation} \end{array} \right\}$$

Dual Immaculate and representation

THEOREM Noncommutative Kotska

$$H_\beta = \sum_{\alpha \geq_\ell \beta} K_{\alpha,\beta} \mathfrak{S}_\alpha$$

Dually:

$$\mathfrak{S}^*_\alpha = \sum_{\alpha \geq_\ell \beta} K_{\alpha,\beta} M_\beta$$

Naturally grouping terms leads to

$$\mathfrak{S}^*_\alpha = \sum_T F_{D(T)}$$

T runs over Standard Immaculate of shape α and $D(T)$ is the descent set of T.

Dual Immaculate and representation

$$\mathfrak{S}^*_\alpha = \sum_T F_{D(T)}$$

Action of $H_n(0)$