Antipode in a Hopf Algebra of Subword Complexes

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Plan of the talk

1. Preliminaries
2. Subword Complexes
3. A Hopf Algebra of Subword Complexes
4. Antipode

Preliminaries

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group of permutations of $\{1, \ldots, n+1\}$

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length of w: smallest r such that $w = s_{i_1} \ldots s_{i_r}$
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reduced expression for w: expression for w of minimal length

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In this talk: **finite Coxeter groups**
(like the symmetric group,
generated by reflections and satisfying relations $s_i s_j^{m_{ij}} = 1$)

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(W, S) finite Coxeter system

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Definition (Knutson–Miller, 2004)

The subword complex $\Delta(Q, \pi)$ is the simplicial complex whose

faces $\longleftrightarrow$ subwords P of Q such that $Q \setminus P$ contains a reduced expression of π

Knutson–Miller. Gröbner geometry of Schubert polynomials. Ann. Math., 161(3), '05
Knutson–Miller. Subword complexes in Coxeter groups. Adv. Math., 184(1), '04

Subword complexes - Example

In type A_2:

$W = \mathbb{S}_3$, $S = \{s_1, s_2\} = \{(1\ 2), (2\ 3)\}$

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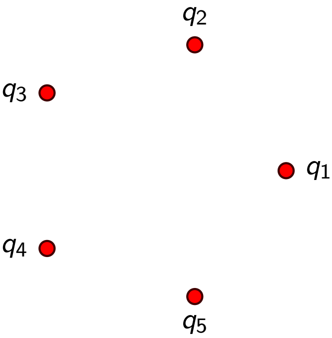

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$$Q = \begin{array}{c} (\quad , \quad , s_1, s_2, s_1) \\ q_1, q_2, \quad , \quad , \end{array}$$ and $\pi = [3\ 2\ 1] = s_1 s_2 s_1 = s_2 s_1 s_2$

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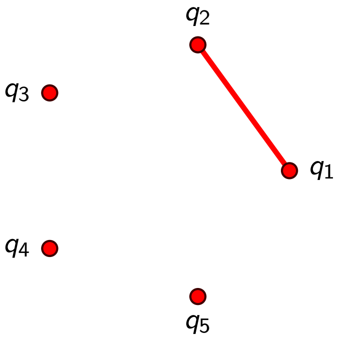

Subword complexes - Example

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$W = \mathbb{S}_3$, $S = \{s_1, s_2\} = \{(1\ 2), (2\ 3)\}$

$Q = \begin{matrix} (s_1, \quad , \quad , s_2, s_1) \\ , {\color{red}q_2}, {\color{red}q_3}, \quad , \end{matrix}$ and $\pi = [3\ 2\ 1] = s_1 s_2 s_1 = s_2 s_1 s_2$

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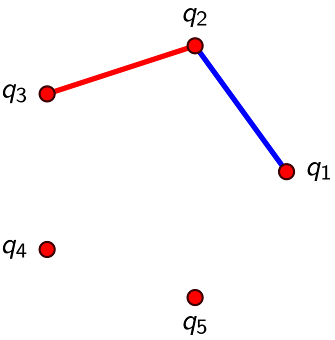

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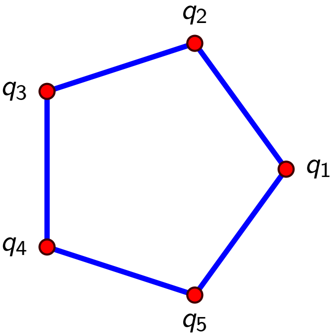

Subword complexes - Example 2

In type A_3:

$W = \mathbb{S}_4$, $S = \{s_1, s_2, s_3\} = \{(1\ 2), (2\ 3), (3\ 4)\}$

$$Q = \begin{pmatrix} s_1, s_2, s_3, s_1, s_2, s_3, s_1, s_2, s_1 \\ q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9 \end{pmatrix} \text{ and } \pi = [4\ 3\ 2\ 1].$$

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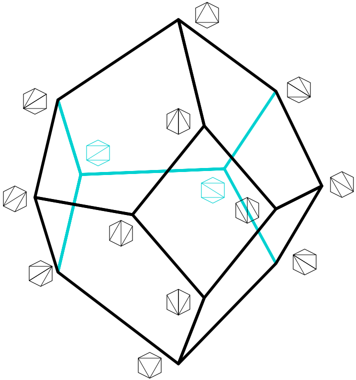
$\Delta(Q, \pi)$ is isomorphic to the dual of the associahedron



Hopf algebras

Theorem

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Hopf algebra: $(H, m, u, \Delta, \varepsilon, S)$ such that objects can be multiplied and comultiplied in a compatible way. Also there is an antipode.

$$S\colon H \to H$$

where

$$m(S \otimes Id)\Delta = m(Id \otimes S)\Delta = u\varepsilon\,.$$

Hopf algebra of subword complexes

Y_n: equivalent classes of tuples (W, Q, π, I) where W is a finite Coxeter group of rank n, and I is a facet of an irreducible subword complex $\Delta(Q, \pi)$.

Theorem (Bergeron–C., 2014$^+$)

The graded vector space

$$k[Y_\infty] := \bigoplus_{n \geq 0} k[Y_n]$$

may be equipped with a structure of graded Hopf algebra.

Comultiplication

Definition

A flat of a collection of vectors is a subset obtained by intersecting the collection with a subspace.

Main ingredient to define the comultiplication:

The flats of $\mathrm{r}(I, Q)$ encode the root function of subword complexes of smaller rank

$$(W_F, Q_F, \pi_F, I_F)$$

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Main ingredient to define the comultiplication:
The flats of $\mathrm{r}(I, Q)$ encode the root function of subword complexes of smaller rank

$$(W_F, Q_F, \pi_F, I_F)$$

Moreover, the resulting subword complex is isomorphic to the link of a face in the initial subword complex

Comultiplication

$$\Delta\left((W,Q,\pi,I)\right) := \sum_{\text{2-flat decomp. of } \mathrm{R}(I)} (W_{F_1},Q_{F_1},\pi_{F_1},I_{F_1}) \otimes (W_{F_2},Q_{F_2},\pi_{F_2},I_{F_2})$$

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where a 2-flat decomposition is a pair of flats $F_1, F_2 \subset R(I)$ such that

$$V = \text{sp}(R(I_{F_1})) \oplus \text{sp}(R(I_{F_2}))$$

$$(W, Q, \pi, I) = (A_3, Q, [4\ 3\ 2\ 1], \{1, 4, 5, 6, 7, 9, 11, 12, 13\})$$

$$Q = (\ s_1, s_2, s_3, s_1, s_2, s_3, s_1, s_2, s_3, s_1, s_2, s_3, s_1, s_2, s_3\ ),$$

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$r(i) = Q_{<i}(\alpha_{Q_i})$, for example $r(3) = s_2(\alpha_3) = \alpha_2 + \alpha_3 = \alpha_{23}$

$$r = (\ \alpha_1, \alpha_2, \alpha_{23}, \alpha_{12}, \alpha_3, \overline{\alpha}_{23}, \alpha_{12}, \alpha_3, \overline{\alpha}_2, \alpha_{123}, \alpha_{12}, \overline{\alpha}_2, \overline{\alpha}_{123}, \alpha_{12}, \alpha_1\ ),$$

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a 2-Flats

$F_1 = (\ \alpha_1, \alpha_1\ )$

$F_2 = (\ \alpha_{12}, \alpha_3, \alpha_{12}, \alpha_3, \alpha_{123}, \alpha_{12}, \overline{\alpha}_{123}, \alpha_{12},\ )$

$V = \mathrm{sp}(\alpha_1) \oplus \mathrm{sp}(\alpha_3, \alpha_{123}, \alpha_{12})$

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A term of $\Delta(W, Q, \pi, I)$:

$$(A_1, (s_1, s_1), [2\ 1], \{1\}) \otimes (A_2, (s_1, s_2, s_1, s_2, s_1, s_2, s_1, s_2), [3\ 2\ 1], \{1, 2, 3, 6, 7\})$$

Multiplication

$$(W, Q, \pi, I) \cdot (W', Q', \pi', I') = (\overline{W}, QQ', \pi\pi', II'_{\uparrow}),$$

where,
$\overline{W}$: Coxeter group generated by $S \sqcup S'$
$QQ' = (q_1, \ldots, q_r, q'_1, \ldots, q'_{r'})$
$\pi\pi'$= product of π and π' in $\overline{W}$
$II'_{\uparrow}$ denotes the "shifted" facet

Theorem (Bergeron–C., 2014+)

The graded vector space

$$k[Y_\infty] := \bigoplus_{n \geq 0} k[Y_n]$$

equipped with these multiplication and comultiplication is a graded Hopf algebra.

Antipode in $k[Y_\infty]$

Takeuchi's formula for bialgebra

$$S(\psi) = \sum_{\alpha \models n} (-1)^{\ell(\alpha)} m_\alpha \circ \Delta_\alpha(\psi)$$

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where for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$

$m_\alpha : \; k[Y_{\alpha_1}] \otimes \cdots \otimes k[Y_{\alpha_\ell}] \longrightarrow k[Y_n]$

$\Delta_\alpha : k[Y_n] \longrightarrow k[Y_{\alpha_1}] \otimes \cdots \otimes k[Y_{\alpha_\ell}]$

Antipode in $k[Y_\infty]$

Takeuchi's formula for $k[Y_\infty]$

$$S(W, Q, \pi, I) = \sum_{\mathcal{F} \in \mathcal{F}D} (-1)^{\ell(\mathcal{F})} (W_{\mathcal{F}}, Q_{\mathcal{F}}, \pi_{\mathcal{F}}, I_{\mathcal{F}})$$

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where $\mathcal{F}D$ is the set of all possible ℓ-flat decompositions $\mathcal{F} = (F_1, F_2, \ldots, F_\ell)$ for all $\ell \geq 1$.

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Many cancelation: resolve using sign reversing involution (similar to [Benedetti-Sagan 2014+])

Antipode in $k[Y_\infty]$

Cancelation free formula for $k[Y_\infty]$

$$S(W,Q,\pi,I) = \sum_{\mathcal{K}\in\Psi^0(\mathcal{F}D)} (-1)^{\ell(\mathcal{K})} a\big(G(\mathcal{K})\big)(W_{\mathcal{K}}, Q_{\mathcal{K}}, \pi_{\mathcal{K}}, I_{\mathcal{K}})$$

$(W, Q, \pi, I) = (A_3, Q, [4\ 3\ 2\ 1], \{1, 4, 5, 6, 7, 9, 11, 12, 13\})$

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$(W_{F_1}, Q_{F_1}, \pi_{F_1}, I_{F_1}) = (A_1, (s_1, s_1), [2\ 1], \{1\})$

$(W_{F_2}, Q_{F_2}, \pi_{F_2}, I_{F_2}) = (A_2, (s_1, s_2, s_1, s_2, s_1, s_2, s_1, s_2), [3\ 2\ 1], \{1, 2, 3, 6, 7\})$

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$(W_{F_1}, Q_{F_1}, \pi_{F_1}, I_{F_1}) = (A_1, (s_1, s_1), [2\ 1], \{1\})$
$(W_{F_2}, Q_{F_2}, \pi_{F_2}, I_{F_2}) = (A_2, (s_1, s_2, s_1, s_2, s_1, s_2, s_1, s_2), [3\ 2\ 1], \{1, 2, 3, 6, 7\})$

$(W_{\mathcal{F}}, Q_{\mathcal{F}}, \pi_{\mathcal{F}}, I_{\mathcal{F}}) =$

$(A_1 \times A_2, (s_1, s_1, s_3, s_4, s_3, s_4, s_3, s_4, s_3, s_4), [2\ 1\ 5\ 4\ 3], \{1, 3, 4, 5, 8, 9\})$

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$(W_{\mathcal{F}}, Q_{\mathcal{F}}, \pi_{\mathcal{F}}, I_{\mathcal{F}}) =$

$(A_1 \times A_2, (s_1, s_1, s_3, s_4, s_3, s_4, s_3, s_4, s_3, s_4), [2\ 1\ 5\ 4\ 3], \{1, 3, 4, 5, 8, 9\})$

Coefficient of $(W_{\mathcal{F}}, Q_{\mathcal{F}}, \pi_{\mathcal{F}}, I_{\mathcal{F}})$ in $S(W, Q, \pi, I))$

$(-1)^{\ell(\mathcal{F})} a\big(G(\mathcal{K})\big) = +a($  $) = 2$

Thanks, Gracias, Merci!