G-Basis in non-commutative vars and parallelism

YORK UNIVERSITÉ UNIVERSITY

Nantel Bergeron

www.math.yorku.ca/bergeron
E-Mail: **bergeron@yorku.ca**

Outline

- $\mathbb{Q}\langle a_1, a_2, ..., a_n\rangle$: Polynomials in non-commutatives variables
- Quotient: When is it finite?
- G-Basis for homogeneous ideal: very long computations.
- More questions: When is the G-basis finite?
- NC-Sym, NC-QSym, Fomin-Kirillov: Non-commutative Combinatorial Hopf Algebras.
- Algorithm: What can we parallelize?
- Co-algebra structure?: Can we use it??

Non-comm. Polynomials and Homogeneous ideals

$R = \mathbb{Q}\langle a_1, a_2, ..., a_n\rangle$ Polynomials in n non-commutatives variables

$w = x_1 x_2 \cdots x_k$ word (monomial) of degree k

$I = \langle f_1, f_2, \cdots, f_k\rangle$ two side ideal generated by the f_i's

I is homogeneous ideal if all f_i are homogeneous

QUESTION: Is the dim of quotient R/I finite or infinite?

Gröbner basis

Monomial order: $u < w$ lexicographic order on words.

$G(I) = \{g_1, g_2, \ldots\}$ is a G-basis of $I = \langle f_1, f_2, \cdots, f_k \rangle$ if

$G \subseteq I$ and

$$LT(I) \stackrel{\text{def}}{=} \langle LT(f) : f \in I \rangle \;=\; \langle LT(g_1), LT(g_2), \ldots \rangle$$

QUESTION: is G finite or infinite

LEMMA: For I homogeneous ideal we can find homogeneous G-basis. [proof: all S-poly are homogeneous]

Hilbert Series of R/I

THEOREM For G a G-basis of I (Homogeneous ideal)

$$H_{R/I}(t) = \sum_{d \geq 0} \dim(R/I^{(d)}) t^d$$

where

$$\dim(R/I^{(d)}) = \left| \{ w : deg(w) = d, \ \forall g \in G \ LT(g) \text{ is not factor of } w \} \right|$$

very long to compute!

Why do I care (1)? NCSym

$R_n = \mathbb{Q}\langle a_1, a_2, ..., a_n \rangle$

S_n acts $a_i \mapsto a_{\sigma(i)}$: $NCSym = \{P \in R_n : \forall \sigma \in S_n, \sigma.P = P\}$

$I_n = \langle P \in NCSym : P(0,0,...,0) = 0 \rangle$ homogeneous ideal

QUESTIONS: Is R_n/I_n finite dim? If not, is G-Basis finite?

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$n = 1$: $H_1(q) = 1$

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$n = 1$: $H_1(q) = 1$

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$n = 3$: $H_3(q) = 1 + 2q + 3q^2 + 3q^3$ [dim=9]

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$n = 1$: $H_1(q) = 1$

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$n = 3$: $H_3(q) = 1 + 2q + 3q^2 + 3q^3$ [dim=9]

$n = 4$: [dim=946]

$H_4(q) = 1 + 3q + 8q^2 + 20q^3 + 47q^4 + 102q^5 + 197q^6 + 308q^7 + 248q^8 + 12q^9$

months of computations

Why do I care (1)? NCSym

$R_n = \mathbb{Q}\langle a_1, a_2, ..., a_n \rangle$

S_n acts $a_i \mapsto a_{\sigma(i)}$: $NCSym = \{P \in R_n : \forall \sigma \in S_n, \sigma.P = P\}$

$I_n = \langle P \in NCSym : P(0,0,...,0) = 0 \rangle$ homogeneous ideal

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$n = 3$: $H_3(q) = 1 + 2q + 3q^2 + 3q^3$ [dim=9]

$n = 4$: [dim=946]

$H_4(q) = 1 + 3q + 8q^2 + 20q^3 + 47q^4 + 102q^5 + 197q^6 + 308q^7 + 248q^8 + 12q^9$

$n = 5$: [dim>12325]

$H_5(q) = 1 + 4q + 15q^2 + 55q^3 + 199q^4 + 712q^5 + 2520q^6 + 8819q^7 + \cdots$

$H_6(q) = 1 + 5q + 24q^2 + 114q^3 + 539q^4 + 2541q^5 + 11953q^6 + \cdots$

Why do I care (1)? NCSym

$R_n = \mathbb{Q}\langle a_1, a_2, ..., a_n \rangle$

S_n acts $a_i \mapsto a_{\sigma(i)}$: $NCSym = \{P \in R_n : \forall \sigma \in S_n, \sigma.P = P\}$

$I_n = \langle P \in NCSym : P(0, 0, ..., 0) = 0 \rangle$ homogeneous ideal

QUESTIONS: Is R_n/I_n finite dim? If not, is G-Basis finite?
QUESTIONS: What is the smallest n such that dim is infinite?

REMARK: This took months and months... and some computer crashes... etc... so I want to be faster, and save intermediate data.

Why do I care (2)? NCQSym

$R_n = \mathbb{Q}\langle a_1, a_2, ..., a_n\rangle$

$NCSym \subseteq NCQSym$

$J_n = \langle P \in NCQSym : P(0,0,...,0) = 0\rangle$ homogeneous ideal

QUESTIONS: Is R_n/J_n finite dim? If not, is G-Basis finite?

For what n it happen?

$H'_1(q) = 1$

$H'_2(q) = 1 + q$

$H'_3(q) = 1 + 2q + 2q^2$ [dim=5]

$H'_4(q) = 1 + 3q + 7q^2 + 9q^3$ [dim=20]

$H'_5(q) = 1 + 4q + 14q^2 + 42q^3 + 88q^4 + 28q^5$ [dim=177]

$H'_6(q) = 1 + 5q + 23q^2 + 99q^3 + 389q^4 + 1269q^5 + 2299q^6 + ?$ [dim≥4085]

Why do I care (3)? Fomin-Kirillov

$R_n = \mathbb{Q}\langle a_{ij} : 1 \leq i, j \leq n\rangle$

$K_n = \langle a_{ij} + a_{ji}, \quad a_{ij}a_{ij}, \quad a_{ij}a_{jk} + a_{jk}a_{ki} + a_{ki}a_{ij}, \quad a_{ij}a_{kl} - a_{kl}a_{ij}\rangle$
homogeneous

QUESTIONS: Is R_n/K_n finite dim? If not, is G-Basis finite?

For what n it happen?

$[m]_q = 1 + q + q^2 + \cdots + q^{m-1}$

$H_1^{FK} = 1$

$H_2^{FK} = [2]_q$

$H_3^{FK} = [2]_q^2[3]_q$

$H_4^{FK} = [2]_q^2[3]_q^2[4]_q^2$

$H_5^{FK} = [4]_q^4[5]_q^2[6]_q^4$

$H_6^{FK} = 1 + 15q + 125q^2 + 765q^3 + 3831q^4 + 16605q^5 + 64432q^6 + 228855q^7 + 755777q^8 + 2347365q^9 + 6916867q^{10} + 19468980q^{11} + \cdots ?$

Main Routines (Free Algebra)

```
# Free algebra of words over integer

FreeA =
AlgebrasWithBasis(QQ).example(('a','b','c','d','e','f'))

(a,b,c,d,e,f) = FreeA.algebra_generators()
```

Main Routines (What I want to do)

Given Homogeneous Ideal $I = \langle f_1, f_2, \ldots \rangle$

How to construct (iteratively) homogeneous components of G-Basis

Suppose we have constructed

$$\varnothing = GB_0 \subseteq GB_1 \subseteq \cdots \subseteq GB_{d-1}$$

How to get GB_d?

Main Routines (Reduce)

```
def Reduce(P,GB):          # Here GB = GB_{d-1}
  if P==0:  return 0
  LP=P.terms()
  result=0
  for T in LP:          # Parallelizable (?)
    (WT,CT)=T.leading_item()
    result=result+Reduce_term(CT,WT,GB)
  if P==result:
    return 1/(result.trailing_coefficient())*result
  return Reduce(result,GB)
```

Main Routines (Reduce)

```
def Reduce(P,GB):          # Here GB = GB_{d-1}
  if P==0:  return 0
  LP=P.terms()
  result=0
  for T in LP:         # Parallelizable (OPTIMIZE!)
    (WT,CT)=T.leading_item()
    result=result+Reduce_term(CT,WT,GB)
  if P==result:
    return 1/(result.trailing_coefficient())*result
  return Reduce(result,GB)
```

Main Routines (Reduce)

`Reduce_term(CT,WT,GB)`

$WT = abacbacba$ Find first $P \in GB$ where $WT = u \cdot LM(P) \cdot v$

$P = acba + babb - cccc \quad \Longrightarrow \quad acba \equiv -babb + cccc$

$WT = abacbacba \quad \mapsto \quad -abbabbcba + abcccccba$

REMARK: If GB is homogeneous, then Reduce preserve homogeneity

Main Routines (Homogeneous Syzygies)

$P = ababab + abcabc - bbaacc$

$Q = babc - bbab$

Sygyzies: $uQ - Pv$ such that $LM(uP) = LM(Qv)$

$$\begin{aligned} abaQ - Pc &= (abababc - ababbab) - (abababc + abcabcc + bbaaccc) \\ &= -ababbab - abcabcc - bbaaccc \end{aligned}$$

but also

$$ababaQ - Pabc = -abababbab - abcabcabc - bbaaccabc$$

REMARK: For homogeneous polynomials, sygygies preserve homogeneity, increase degree, but is unique for each degree and order of P and Q.

Main Routines (Homogeneous Syzygies)

```
def Syzygy(P,Q,n):
    (WP,CP)=P.trailing_item()
    (WQ,CQ)=Q.trailing_item()
    if n>= len(WP)+len(WQ): return 0
    overlap=len(WP)+len(WQ)-n
    for i in range(overlap):
        if WP[i-overlap]<>WQ[i]:  return 0
    result=CQ*P*FreeA.basis()[WQ[overlap:dQ]]
              -CP*FreeA.basis()[WP[0:-overlap]]*Q
    return result
```

Main Routines (Main loop)

```
def GdBasis(deg,n):       # Hom.  comp.  of degree "deg"
  G1B=GBasis(deg-1,n)     # Union of previous degrees
  GdB=[]
  GB=G1B
# LOOP A: Reduce Gen of degree “deg”. Parallelizable (tricky)
  for A in Gen(deg):
    G=Reduce(QM(A,n),GB)
    if G<>0:
      GdB=[Reduce(GG,[G]) for GG in GdB]+[G]
      GB=G1B+GdB
# LOOP B: Syzygies of degree “deg” Parallelizable (!!)
  return GdB
```

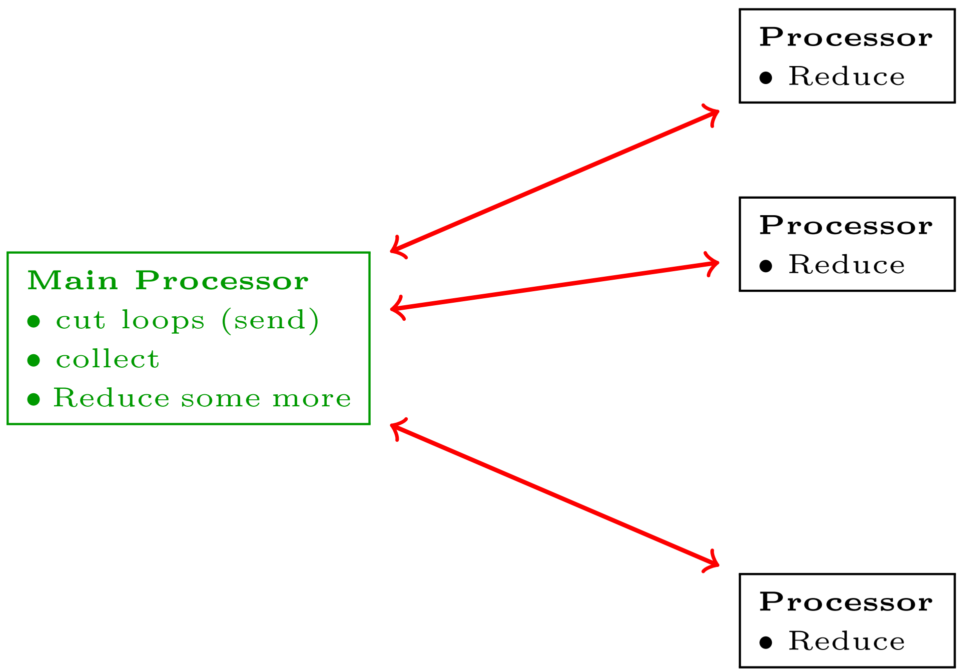
Main Routines (Main loop)

LOOP B: Syzygies of degree “deg” Parallelizable (!!)

```
for i in range(2,deg):
  for j in range(deg-i+1,deg):
    for P in GdBasis(i,n)[1]:
      for Q in GdBasis(j,n)[1]:
        G=Reduce(Syzygy(P,Q,deg),GB)
        if G<>0:
          GdB=Reduce(GG,[G]) for GG in GdB[1]]+[G]
          GB=G1B[1]+GdB[1]
return GdB
```

Parallelization (Reduce, reduce, reduce)

Need to perform a complete graph of reductions



We could cut in small pieces, collect answers and resend?

We could use bigger tree and resolve with available processors?

When do we stop?

(A) $GB_d = \varnothing$ for $d = k, k+1, \ldots, 2k-1$.

or

(B) Coefficient of Hilbert series is zero at degree d.

Open Questions

- Can we use the full Combinatorial Hopf structure?
- Is there relation for G-basis between n and n+1?
- We can do that for any realizable Combinatorial Hopf Algebra

Many CHA are realizable as series in (infinitely many) variables

Is there a nice result about quotient we get when we set trailing variable to zero?

  - Is quotient always finite?
  - Is G-basis always finite?

MERCI, THANKS