Why find cancelation free (Positiv$_\epsilon$) formula for antipode?

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Outline

- Antipode and Generalized Chromatic Polynomials
- Hopf Algebra proof of Stanley's acyclic theorem
- Hopf monoid framework
- Toward cancelation free formula for antipode.

Combinatorial Hopf Algebra

$H = \bigoplus_{n\geq 0} H_n$ graded Hopf algebra with character $\zeta \colon H \to \mathbb{Q}$

$H_n = \mathbb{Q}\big[G : G \text{ iso class of graphs on } [n]\big]$

$G_1 \cdot G_2 = G_1 \cup G_2$

$\Delta(G) = \sum_{S\subseteq[n]} G|_S \otimes G|_{S^c}$

$$\zeta(G) = \begin{cases} 1 & \text{if } G \text{ has no edges} \\ o & \text{otherwise} \end{cases}$$

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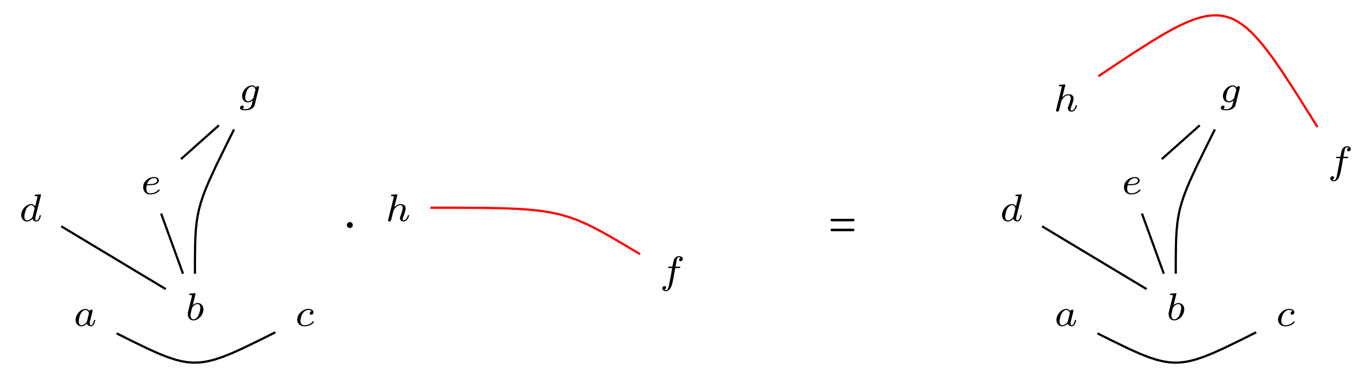
EXAMPLE



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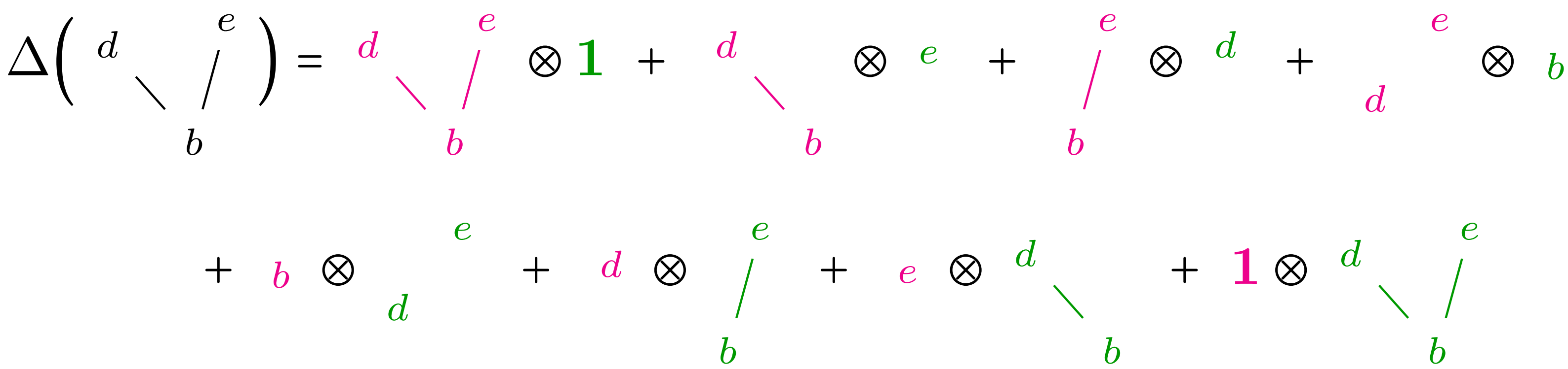

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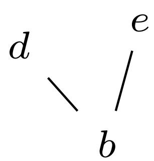
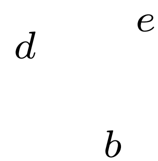
$\zeta\Big($ 

$\Big) = \mathbf{0}$ and $\zeta\Big($ 

$\Big) = \mathbf{1}$

Combinatorial Hopf Algebra

[Aguiar-Bergeron-Sottile]

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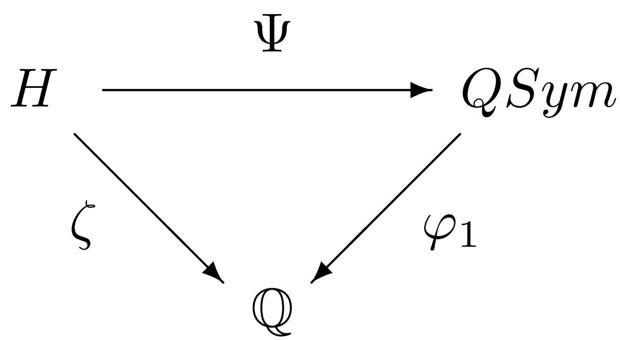

where $\varphi_1(f) = f(1, 0, 0, \cdots)$.

Combinatorial Hopf Algebra

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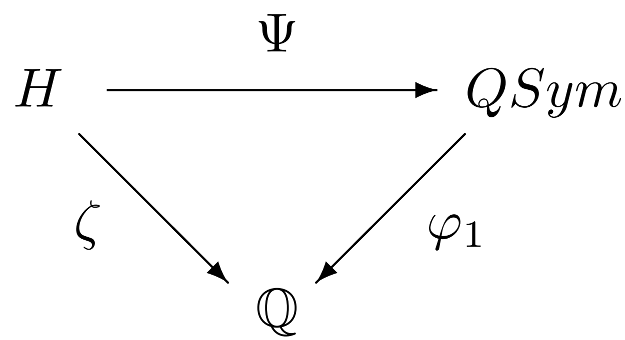
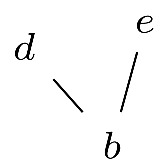


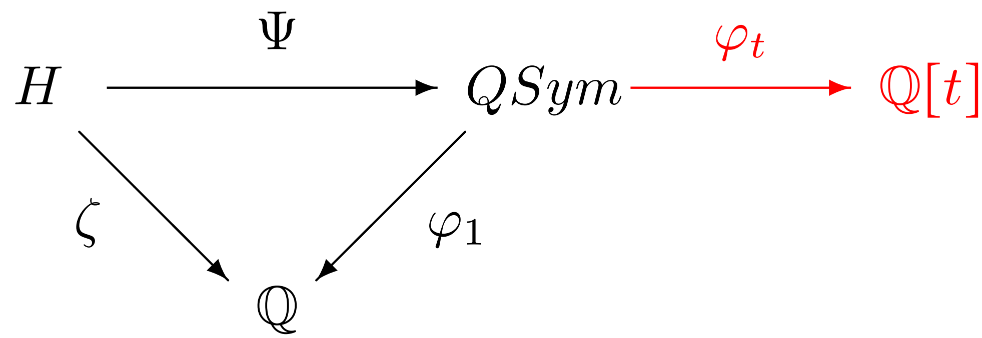
EXAMPLE

$\Psi\Big($  $\Big) = 0M_{(3)} + 1M_{(2,1)} + 1M_{(1,2)} + 6M_{(1,1,1)}$

$$= 6m_{(1,1,1)} + m_{(2,1)}$$

Combinatorial Hopf Algebra (ABS)

$H = \bigoplus_{n \geq 0} H_n$ graded Hopf algebra with character $\zeta\colon H \to \mathbb{Q}$



$\varphi_t(M_\alpha) = \binom{t}{\ell(\alpha)}$ is a Hopf morphism

$$\varphi_t\big(f(x_1, x_2, \ldots)\big)\Big|_{t=n} = f(\underbrace{1, \ldots, 1}_{n}, 0, 0, \ldots)\,.$$

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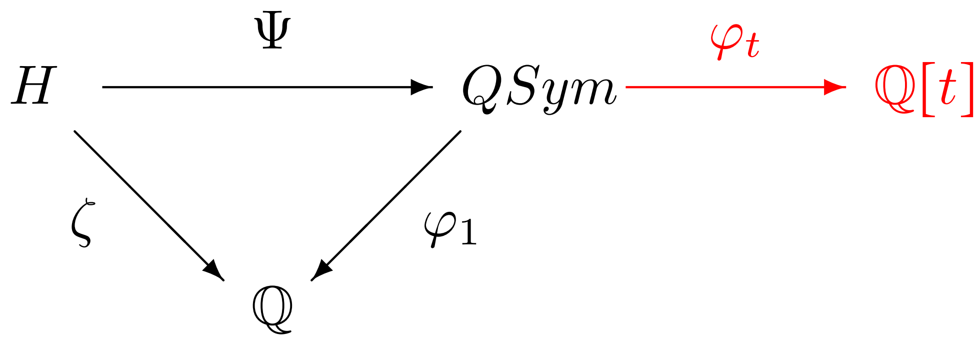

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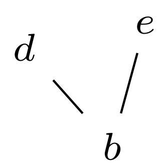
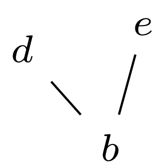
$\Psi\Big($  $\Big) = 0M_{(3)} + 1M_{(2,1)} + 1M_{(1,2)} + 6M_{(1,1,1)}$

$\varphi_t \circ \Psi\Big($  $\Big) = \binom{t}{2} + \binom{t}{2} + 6\binom{t}{3} = t(t-1)^2$

Generalized Chromatic Polynomial

[Grinberg-Reiner]

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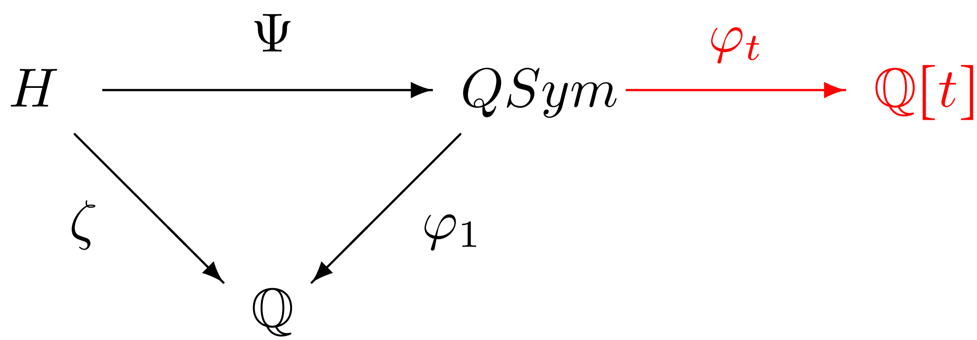

Generalized Chromatic Polynomial

$$\chi_x(t) = \varphi_t \circ \Psi(x)$$

$$\varphi_t \circ \Psi\Big|_{t=1} = \left(\varphi_t\Big|_{t=1}\right) \circ \Psi = \varphi_1 \circ \Psi = \zeta.$$

Generalized Chromatic Polynomial

[Grinberg-Reiner]

$H = \bigoplus_{n \geq 0} H_n$ graded Hopf algebra with character $\zeta : H \to \mathbb{Q}$

$$H \xrightarrow{\quad \Psi \quad} QSym \xrightarrow{\quad \varphi_t \quad} \mathbb{Q}[t]$$

When $H_n = \mathbb{Q}\big[G : G \text{ iso class of graphs on } [n]\big]$

$$\chi_G(t) = \varphi_t \circ \Psi(G)$$

is the usual Chromatic polynomial.

Theorem [Stanley]

$$\boxed{\chi_G(-1) = \pm a(G)}$$

For a graph G; $a(G)$ is number of acyclic orientation.

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Antipode: $S\colon \mathbb{Q}[t] \to \mathbb{Q}[t]$

$f(t) \mapsto f(-t)$

$$\chi_G(-1) = S \circ \varphi_t \circ \Psi(G)\Big|_{t=1} = \varphi_t \circ \Psi \circ S(G)\Big|_{t=1} = \zeta \circ S(G)$$

$$\boxed{\chi_G(-1) = \zeta\big(S(G)\big)}$$

Theorem [Humpert-Martin]

THEOREM

$$S(G) = \sum_{F \ flats} (-1)^{c(F)} a(G/_F) G|_F$$

Corollary $\chi_G(-1) = \zeta\big(S(G)\big) = (-1)^{c(G)} a(G)$

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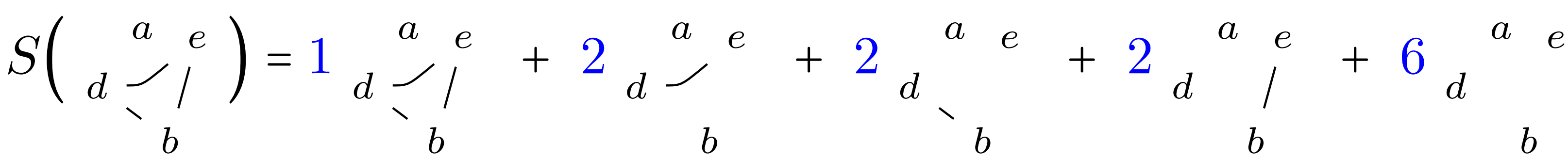

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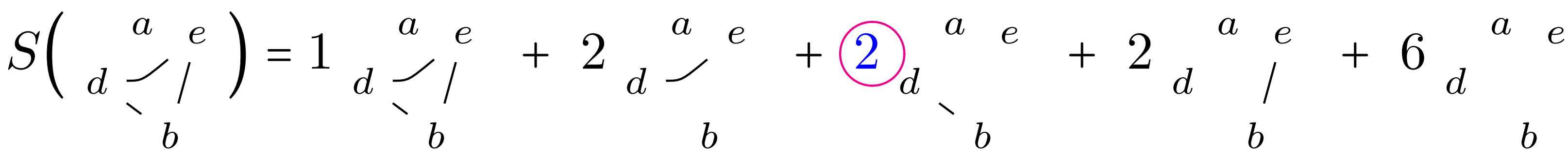

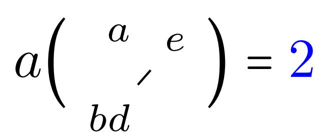

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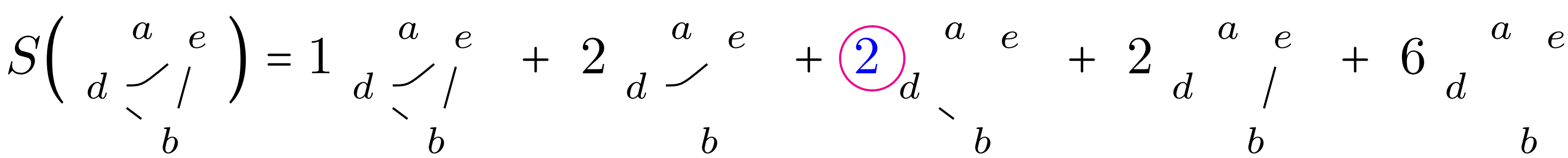

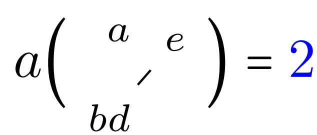

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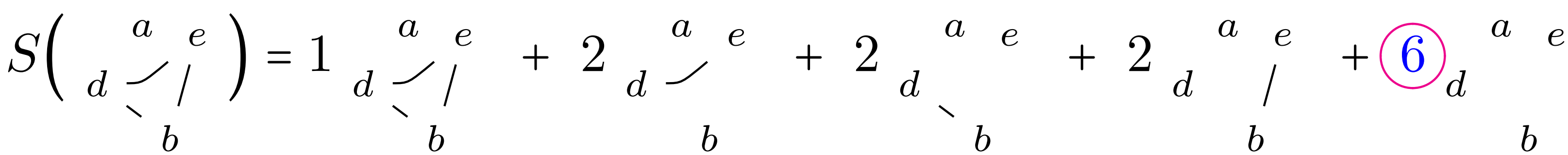

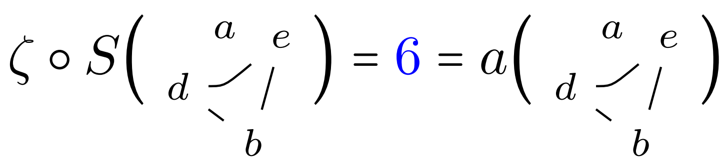

So now I want better formulas for antipode

Graded vector space: $H = \bigoplus_{n \geq 0} H_n$

multiplication: $m\colon H \otimes H \to H$

comultiplication: $\Delta\colon H \to H \otimes H$

ANTIPODE: $S\colon H \to H$ [Takeuchi] For $x \in H_n$

$$S(x) = \sum_{\alpha \vDash n} (-1)^{\ell(\alpha)} m_\alpha \Delta_\alpha(x)$$

where

$m_\alpha\colon H_{\alpha_1} \otimes \cdots \otimes H_{\alpha_\ell} \to H_n$ and $\Delta_\alpha\colon H_n \to H_{\alpha_1} \otimes \cdots \otimes H_{\alpha_\ell}$

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MANY CANCELATIONS

We WOULD LIKE a REFINED GRADING

So now I want better formulas for antipode

Graded vector space: $H = \bigoplus_{n \geq 0} H_n$

multiplication: $m\colon H \otimes H \to H$

graded: $m = \sum m_{a,b}$ where

$$m_{a,b}\colon H_a \otimes H_b \to H_{a+b}$$

comultiplication: $\Delta\colon H \to H \otimes H$

graded: $\Delta = \sum \Delta_{a,b}$ where

$$\Delta_{a,b}\colon H_{a+b} \to H_a \otimes H_b$$

Hopf Monoids

[Aguiar-Mahajan]

For

$$H = \bigoplus_{I \subseteq \mathbb{N}} H[I]$$

Hopf Monoids

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Hopf Monoids

[Aguiar-Mahajan]

(Vector Spaces) species

A Functor $H : (\text{Sets}, \text{Bijs}) \to (\text{Vects}, \text{Transfs})$

$H[I] = \text{Vector space}$

$m_{I,J}\colon H[I] \otimes H[J] \to H[I \uplus J]$ $\Delta_{I,J}\colon H[I \uplus J] \to H[I] \otimes H[J]$

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This gives us

$$m: H \bullet H \to H$$

$$(H \bullet H)[T] = \bigoplus_{(I,J) \vDash T} H[I] \otimes H[J]$$

Hopf Monoids

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$$m\colon H \bullet H \to H \quad \text{and} \quad \Delta\colon H \to H \bullet H$$

Hopf Monoids

[Aguiar-Mahajan]

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TAKEUCHI For $x \in H[I]$.

$$S(x) = \sum_{A \vDash I} (-1)^{\ell(A)} m_A \Delta_A(x)$$

Example of Hopf Monoids

- $G[I] = \text{Span}\{G : G \text{ is a simple graph on } I\}$

$m_{I,J}: G[I] \otimes G[J] \to G[I \uplus J]$ $\qquad$ $\Delta_{I,J}: G[I \uplus J] \to G[I] \otimes G[J]$

$g \otimes g' \mapsto g \cup g'$ $\qquad$ $g \mapsto g|_I \otimes g|_J$

- $L[I] = \text{Span}\{\alpha : \alpha \text{ is a total order on } I\}$

$m_{I,J}(\alpha \otimes \beta) = \alpha \cdot \beta$ $\qquad$ $\Delta_{I,J}(\alpha) = \alpha|_I \otimes \alpha|_J$

- $\Pi[I] = \text{Span}\{A : A \text{ is a set partition on } I\}$

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We can try resolve Takeuchi formula with ad hoc sign reversing involution and get cancelation free formula

Linearizable Hopf Monoids

[Marberg]

- a (Set) species $\mathbf{h}:(\text{Sets},\text{Bijs})\to\text{Sets}$, is a basis for H if

  $H[I]=\text{Span}\{x : x\in\mathbf{h}[I]\}$

- A hopf monoid H is linearizable if $m_{I,J}$ and $\Delta_{I,J}$ are linear extension of

  $m_{I,J}{:}\,\mathbf{h}[I]\otimes\mathbf{h}[J]\to\mathbf{h}[I\uplus J]$

  $\Delta_{I,J}{:}\,\mathbf{h}[I\uplus J]\to\mathbf{h}[I]\otimes\mathbf{h}[J]\cup\{0\}$

MANY HOPF MONOIDS ARE LINEARIZABLE

All our examples are... but not for all basis!!!

Functors from Hopf Monoids

[Aguiar-Mahajan]

$$\text{Species} \xrightarrow{K} \text{Gr-Vect}$$

$$\text{H} \longmapsto \bigoplus_{n \geq 0} H[n]$$

$$\text{Species} \xrightarrow{\overline{K}} \text{Gr-Vect}$$

$$\text{H} \longmapsto \bigoplus_{n \geq 0} H[n]/_{\langle x - \sigma(x) \rangle}$$

Both functors send Hopf monoids to graded Hopf algebras, but only $\overline{K}$ preserves antipode

$\overline{K}(L \times H) = K(H)$

Study antipode for $L \times H$ and H linearizable Hopf monoid.

For $(\alpha, x) \in (L \times \mathbf{h})[I]$, using Takeuchi

$$S_I(\alpha, x) = \sum_{(\beta, y) \in (\mathbf{l} \times \mathbf{h})[I]} \left(\sum_{A \in \mathcal{C}_{\alpha,x}^{\beta,y}} (-1)^{\ell(A)} \right) (\beta, y).$$

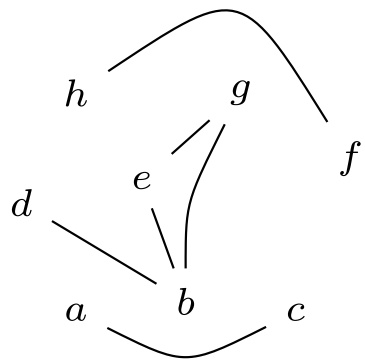
where

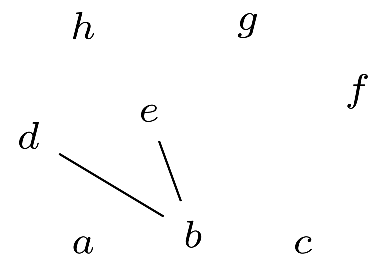
$$\mathcal{C}_{\alpha,x}^{\beta,y} = \left\{ A \vDash I : (\alpha_A, x_A) = (\beta, y) \right\}$$

$$\mathcal{C}_{\alpha,x}^{\beta,y} = \left\{ A \vDash I : (\alpha_A, x_A) = (\beta, y) \right\}$$

G: Hopf monoid of graph

$I = \{a, b, c, d, e, f, g, h\}$; $\alpha = abcdefgh$; $\beta = abdefghc$

$x =$ 

$y =$ 

$A = (a, b, d, e, f, g, h, c) \in \mathcal{C}_{\alpha,x}^{\beta,y}$ **?**

$$\mathcal{C}_{\alpha,x}^{\beta,y} = \{A \vDash I : (\alpha_A, x_A) = (\beta, y)\}$$

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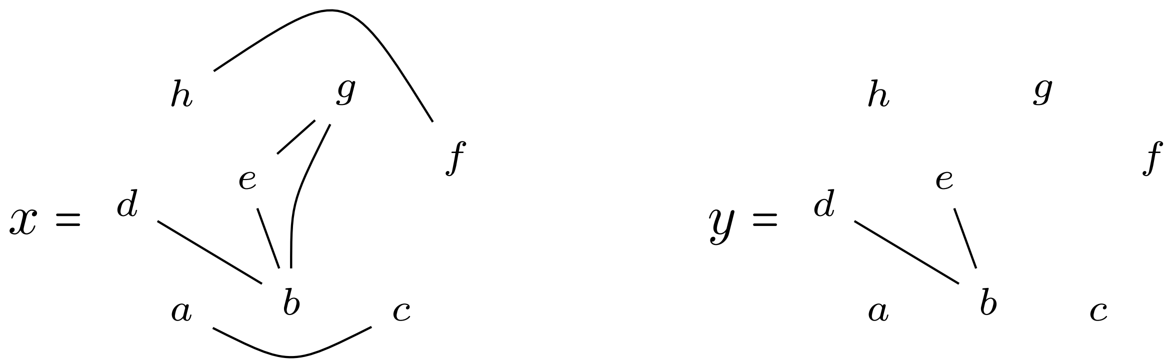

$A = (a,b,d,e,f,g,h,c) \in \mathcal{C}_{\alpha,x}^{\beta,y}$? NO

$$\alpha_A = \beta \quad \text{but} \quad x_A \neq y$$

$$\mathcal{C}_{\alpha,x}^{\beta,y} \;=\; \big\{ A \vDash I : (\alpha_A, x_A) = (\beta, y) \big\}$$

G: Hopf monoid of graph

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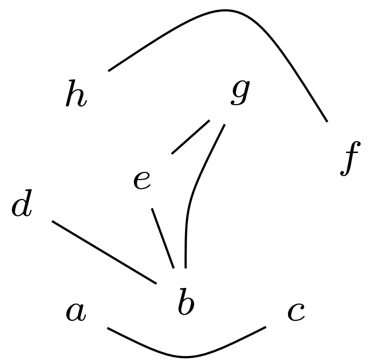

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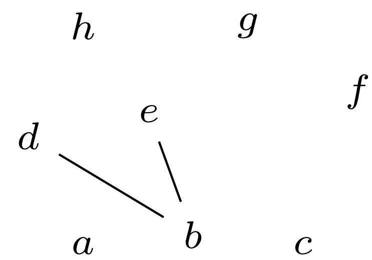
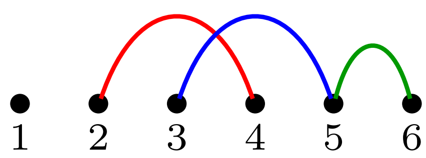


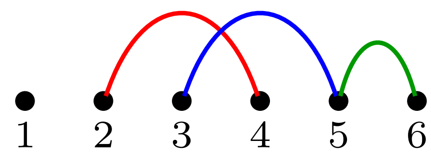
$\Lambda = (a, bde, f, g, h, c)$: unique minimum refinement in $\mathcal{C}_{\alpha,x}^{\beta,y}$

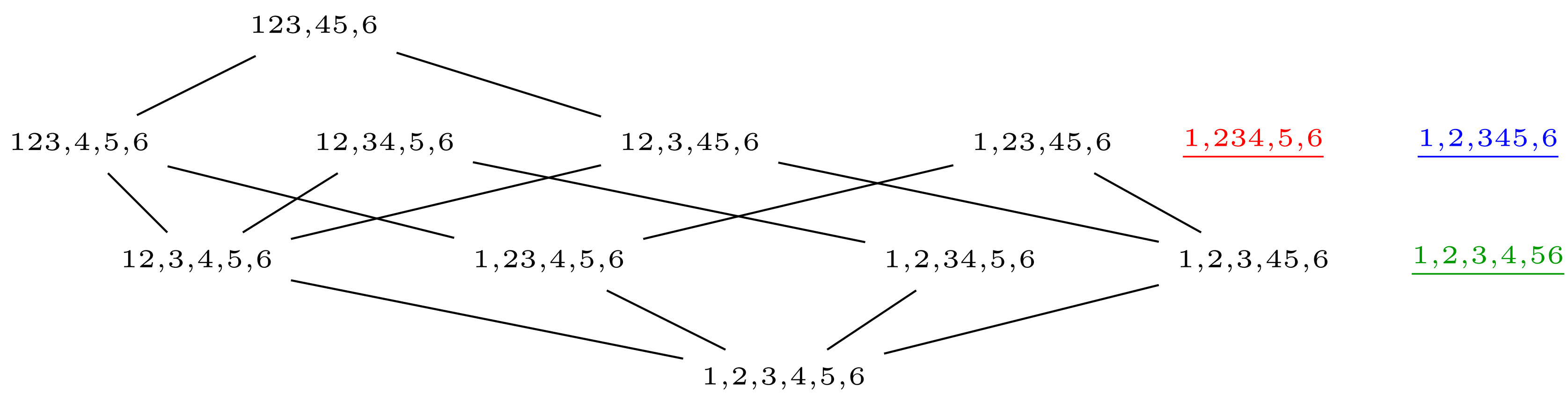
$G_{\alpha,x}^{\beta,y} =$



$$\mathcal{C}_{\alpha,x}^{\beta,y} = \left\{ A \vDash I : (\alpha_A, x_A) = (\beta, y) \right\}$$

$G_{\alpha,x}^{\beta,y} =$





$\longrightarrow$ coefficient of (β, y) in $S(\alpha, x)$ is $\sum_{A \in \mathcal{C}_{\alpha,x}^{\beta,y}} (-1)^{\ell(A)} = 0$

THEOREM $\mathcal{C}_{\alpha,x}^{\beta,y}$

H: Any linearizable Hopf monoid with basis $\mathbf{h}$

(α, x) and (β, y) in $(L \times \mathbf{h})[I]$

- $\mathcal{C}_{\alpha,x}^{\beta,y}$ has a unique minimal refinement Λ
- $\mathcal{C}_{\alpha,x}^{\beta,y}$ is a lower ideal in $[\Lambda, I]$
- $[\Lambda, I] \smallsetminus \mathcal{C}_{\alpha,x}^{\beta,y}$ has minimums of the form

$$(\Lambda_1, \ldots, \Lambda_{i-1}, \Lambda_i \cup \Lambda_{i+1} \cup \cdots \cup \Lambda_j, \Lambda_{j+1}, \ldots, \Lambda_m)$$

- This define a graph $G_{\alpha,x}^{\beta,y}$ with no NESTING

  [no $(a,d),(b,c)$ with $a \leq b < c \leq d$]

- We have $\varphi\colon \mathcal{C}_{\alpha,x}^{\beta,y} \to \mathcal{C}_{\alpha,x}^{\beta,y}$ sign-reversing involution.

$$\sum_{A \in \mathcal{C}_{\alpha,x}^{\beta,y}} (-1)^{\ell(A)} = 0 \text{ or } \pm 1$$

THEOREM $\mathcal{C}_{\alpha,x}^{\beta,y}$

$$\mathcal{C}_{\alpha,x}^{\beta,y} = \left\{ A \vDash I : (\alpha_A, x_A) = (\beta, y) \right\}$$

- $\mathcal{C}_{\alpha,x}^{\beta,y}$ has a unique minimal refinement Λ

Idea of proof Assume $A \neq B \in \mathcal{C}_{\alpha,x}^{\beta,y}$ two minimal refinement.

$$x_B = m_B \Delta_B(x) = m_B \Delta_B(x_B)$$

By assumption $x_A = x_B$ (I skip some details about α)

$$x_A = m_B \Delta_B(x_A) = m_B \Delta_B m_A \Delta_A(x) = m_C \Delta_C(x)$$

for a $C < A$ (contradiction)

THEOREM $\mathcal{C}_{\alpha,x}^{\beta,y}$

$$\mathcal{C}_{\alpha,x}^{\beta,y} = \left\{ A \vDash I : (\alpha_A, x_A) = (\beta, y) \right\}$$

- $\mathcal{C}_{\alpha,x}^{\beta,y}$ is a lower ideal in $[\Lambda, I]$

Idea of proof For any $A \neq B \in \mathcal{C}_{\alpha,x}^{\beta,y}$, we show $[\Lambda, A] \subseteq \mathcal{C}_{\alpha,x}^{\beta,y}$. For any $\Lambda \leq C \leq A$

$$y = m_\Lambda \Delta_\Lambda(x) = m_C \Delta_C m_\Lambda \Delta_\Lambda(x) = m_C \Delta_C m_A \Delta_A(x) = m_C \Delta_C(x)$$

and same with α

THEOREM $\mathcal{C}_{\alpha,x}^{\beta,y}$

$$\mathcal{C}_{\alpha,x}^{\beta,y} = \left\{ A \vDash I : (\alpha_A, x_A) = (\beta, y) \right\}$$

- $[\Lambda, I] \smallsetminus \mathcal{C}_{\alpha,x}^{\beta,y}$ has minimums of the form

$$(\Lambda_1, \dots, \Lambda_{i-1}, \Lambda_i \cup \Lambda_{i+1} \cup \cdots \cup \Lambda_j, \Lambda_{j+1}, \dots, \Lambda_m)$$

Idea of proof Let $B > \Lambda$ and $B \notin \mathcal{C}_{\alpha,x}^{\beta,y}$ minimal.

Suppose that B has more than one part that joint parts of Λ.

We refine each (one at a time) to get $\Lambda < C < B$: we get $y = x_C$.

This shows that $x_B = y$ since for each bloc we get the same thing as y (contradiction)

THEOREM $\mathcal{C}_{\alpha,x}^{\beta,y}$

$$\mathcal{C}_{\alpha,x}^{\beta,y} = \left\{ A \vDash I : (\alpha_A, x_A) = (\beta, y) \right\}$$

- We have $\varphi: \mathcal{C}_{\alpha,x}^{\beta,y} \to \mathcal{C}_{\alpha,x}^{\beta,y}$ sign-reversing involution.

$$\sum_{A \in \mathcal{C}_{\alpha,x}^{\beta,y}} (-1)^{\ell(A)} = 0 \text{ or } \pm 1$$

Idea of proof read paper!

THEOREM $\mathcal{C}_x^y$

H: Any lin. (co and coco)mutative Hopf monoid with basis $\mathbf{h}$

x and y in $\mathbf{h}[I]$

- $\mathcal{C}_x^y$ has a unique minimal refinement Λ (up to permutation)
- $\mathcal{C}_{\alpha,x}^{\beta,y}$ is a lower ideal in $\bigcup_\sigma[\sigma \cdot \Lambda, I]$
- $\Big(\bigcup_\sigma[\sigma \cdot \Lambda, I]\Big) \setminus \mathcal{C}_{\alpha,x}^{\beta,y}$ has minimums of the form

$$(\bigcup_{i \in U} \Lambda_i, \Lambda_{v_1}, \Lambda_{v_2}, \ldots, \Lambda_{v_r})$$

- This define a hypergraph G_x^y [hyperegdes $U \subset \Lambda$
- So far we do not have a complete cancelation free formula. The best is

$$\sum_{A \in \mathcal{C}_{\alpha,x}^{\beta,y}} (-1)^{\ell(A)} = a(G_x^y)$$

a $(0,1,-1)$-weighted sum of certain acyclic orientation of G_x^y

감사해요 K A M S A H A E Y O

M E R C I

T H A N K S

G R A C I A S