TALK STANFORD (DEC 2011)

1 - SYM AS HOPF

2 - $K_0($ REP $S_n : n \geq 0) \longleftrightarrow$ SYM (WHAT DO WE GAIN)

3 - COMB. HOPF ALGEBRA

4 - Π : SYM. FUNCTIONS IN NON-COM. VAR

5 - $\Pi \longleftrightarrow K_0(?)$ (WHAT DO WE GAIN)

6 -
$$\begin{tikzcd} \text{HOPF MONOID} \arrow[r, leftrightarrow] \arrow[d] & K_0(SC_q[I]) \arrow[d] \\ \Pi \arrow[r, leftrightarrow] & K_0(SC_n(q)) \end{tikzcd}$$

(WHAT DO WE GAIN)

↳ SAME STRUCTURE SUPER CHAR DEPEND ON q.

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1. $\mathbb{Z}[x_1, \dots, x_n]^{S_n} = \mathbb{Z}[e_1, \dots, e_n]$

$$\sum_{k=0}^{n} t^k e_k = \prod_{i=1}^{n} (1 + t e_i)$$

$\boxed{\varprojlim_{n \to \infty}}$ $\Lambda = \mathbb{Z}[e_1, e_2, \dots]$

- $$m_\lambda = \sum_{x^\alpha \in \mathrm{ORBIT}(x^\lambda)} x^\alpha$$

Λ is ALGEBRA $[+, \cdot]$

is HOPF ALGEBRA

$\Delta : \Lambda \longrightarrow \Lambda \otimes \Lambda$

$f \in \Lambda \qquad f(x_1, x_2, \dots)$

$$f(Y+Z) = \sum f^{(1)}(Y) f^{(2)}(Z)$$

def $\Delta f = \sum f' \otimes f^2$

$\Rightarrow \Delta(fg) = \Delta(f)\Delta(g)$

$$\Delta(e_k) = \sum_{i=0}^{k} e_i \otimes e_{k-i}$$

+ ANTIPODE $\quad S : \Lambda \longrightarrow \Lambda$

- GRADING $\quad d(e_k) = k$

2. CAN WE LIFT THIS STRUCTURE IN A CATEGORY (OF REPRESENTATIONS)

$\mathcal{C}$: S_n ~~PROJ~~ PROJECTIVE MODULES, $n \geq 0$

$$K_0(\mathcal{C}) = \mathbb{Z}\left[\begin{array}{c}\text{iso CLASS OF} \\ S_n\text{-MODULE}\end{array}\right]$$

(irr CHARACTERS)

is A Hopf Alg. : Mult.

$$\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} \chi \otimes \psi$$

$$\Delta(\chi) = \sum_{k=0}^{n} \mathrm{Res}_{S_k \times S_{n-k}}^{S_n} \chi$$

FOR S_n

$$K_0(\mathcal{C}) \xleftrightarrow{\sim} \Lambda$$

$$\text{irr } \chi^\lambda \longleftrightarrow S_\lambda$$

NEW BASIS (SCHUR FUNCTIONS)

$\boxed{\text{WHAT DO WE GAIN}}$ BOTH SIDE.

3. COMB. HOPF. Alg.

SOMETHING LIKE THE ABOVE!

- GRADED HOPF ALGEBRA
- SINGLED OUT BASIS WITH POSITIVE STRUCTURE (COMB.)
- SINGLED OUT CHARACTER

$\qquad \zeta : H \longrightarrow k \quad$ Mult.

TRIVIAL CHARACTER

(1) ABS — BLL $\quad K_0(A) \xleftrightarrow{\sim} H$

VERY RESTRICTIVE

$\dim(A_n) = r^n n!$

4. Π : SYM. FUNCTIONS IN NON-COM VARIABLES

$$\mathbb{Z}\langle x_1, \dots, x_n \rangle^{S_n} \underset{\text{WOLF}}{\cong} \mathbb{Z}\langle f_1, f_2, \dots \rangle$$

$\varprojlim \quad \Pi \quad$ is HOPF

5. $\Pi \longleftrightarrow K_0(?)$

- ? = SUPER REP OU $UT_n(q)$
  "UNIPOTENT UPPERTRIANGULAR MATRICES OVER $\mathbb{F}_q$"
- REP THEORY IS WILD BUT...
- SUPERTHEORY:

  "GLUE TOGETHER ~~CONJ~~ CLASS"
  - SUPER CLASS

  "COMBINE irr CHARACTER"
  - SUPER CHARACTER

(A) $A \sim B \iff T(A-I)U+I = B$

$U,T \in UT_n$

G-E $\Rightarrow$ REPR. HAVE NO MORE THAN ONE NON-ZERO PER ROW AND COL. (OFF DIAGONAL)

<u>AIM</u> <u>SET PARTITIONS</u>

$\Pi \longleftrightarrow K_0(UT_n(2))$

(B) $A \sim B \iff T(A-I)U+I = B$

$U,T \in T_n(q) \times UT_n(q)$

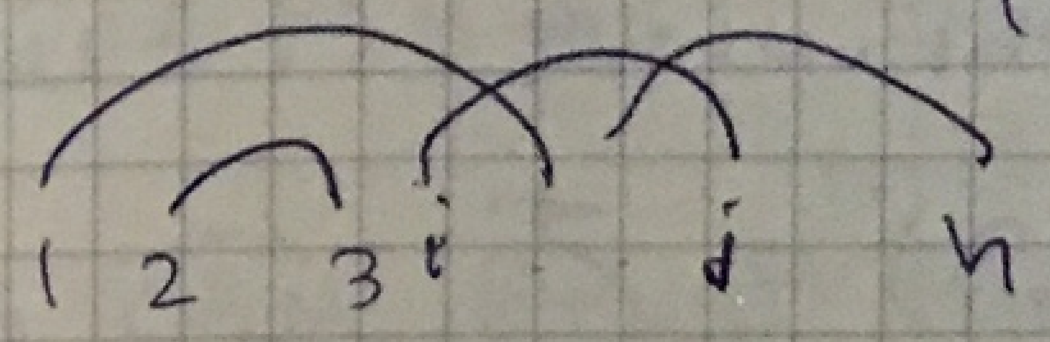
TORUS

G-E $\Rightarrow$ NO MORE THAN ONE "1" OFF DIAGONAL PER ROW & COL.

POSITION OF "1"s

$\lambda = \{(i,j) : i<j\}$

$\forall\ i<k<j \Rightarrow (i,k) \notin \lambda$
$(k,j) \in \lambda$



$\Pi \longleftrightarrow K_0(UT_n(q))$

WE HAVE ONE INTERESTING CHA (ONE FOR EACH q)

$\boxed{\text{~~WHAT DO WE GAIN~~}}$

MULT. $UT_{n+m} \Rightarrow UT_n \times UT_m$

- $\mathrm{Inf}^{UT_{m+n}}_{UT_n \times UT_m}\ \chi \otimes \psi$

COMULT. $UT_{|J|} \times UT_{|J^c|} \hookrightarrow UT_n$

- $\sum_{J \subseteq [n]} \mathrm{Res}^{UT_n}_{UT_{|J|} \times UT_{|J^c|}}\ \chi$

(IS REALLY CHA)

$\boxed{\text{WHAT DO WE GAIN}}$

- REP THEORI OF OP IN Π
- WITH THEM NEW BASIS & FACT

$$
\begin{array}{ccc}
\text{HOPF MONOID} & \longleftrightarrow & K_0(\cdots) \\
\downarrow & & \downarrow \\
\Pi & \longleftrightarrow & K_0(UT_n(q))
\end{array}
$$

<u>WHEN</u> DEFINING

$$\Delta(\lambda) = \sum_{J \subseteq [n]} \cdots$$

DEPEND ON <u>SUBSETS</u>

<u>HOPF MONOID</u> IS AN ALGEBRA STRUCTURE "GRADED" BY FINITE SET (AS OPOSITION TO "n") [SO IT IS BY NATURE MORE COMBINATORIA]

$\boxed{\text{FUNCTOR}}$

$$SC_q : \text{FINITE SET} \longrightarrow \text{VECT}$$

Mult

$$I \longmapsto SC_q[I]$$

SPAN OF LINEAR ORDER OF I WITH ARCS (ϕ, λ)

Mult.

$\boxed{K = I + J}$ $m_{I,J} : SC_q[I] \times SC_q[J] \to SC_q[K]$

$$(\phi, \lambda), (\psi, \mu) \longmapsto (\phi\tau, \lambda \cdot \mu)$$

$$\Delta_{I,J} : \cdots$$

- "COMPATIBLE
- WE CAN FIND NICE FORMULA IS VARIOUS BASIS
- USEFUL FOR Π

$$
\begin{array}{c}
SC_q \\
\downarrow \\
\Pi
\end{array}
$$