A Hopf Monoid of supercharacters

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(with **M. Aguiar and N. Thiem**)

(... ... ... **and many more**)

Outline

- Supercharacter theory of $U_n(q)$ $SC(q)$ a Hopf algebra
- Symmetric functions in noncommutative variables Π a Hopf algebra
- Isomorphism when q=2 (28 authors AIM paper, see ArXive)
- Hopf monoid (on species)
- Hopf monoid of supercharacters $\mathbf{SC}(q)$
- What can we do with this? conceptual and computational (antipode formulas)

Supercharacter theory of $U_n(q)$

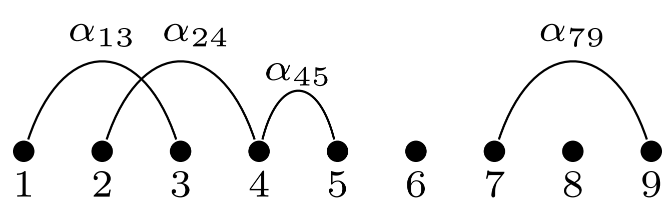
lumping conjugacy classes and characters together to get a more tame theory André, Diaconis-Isaac.

- Unipotent upper triangular matrices over finite Fields $\mathbf{F}_q$: $U_n(q)$.
- Superclasses in $U_n(q)$:

$$A \cong B \quad \leftrightarrow \quad (A - I) = M(B - I)N$$

superclass representative has at most one nonzero element in each rows and columns (off the diagonal).

$\lambda =$



Supercharacter theory of $U_n(q)$

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- Unipotent upper triangular matrices over finite Fields $\mathbf{F}_q$: $U_n(q)$.
- Superclasses in $U_n(q)$ λ
- Supercharacters χ^λ Hopf algebra structure see ArXive 28 author paper:

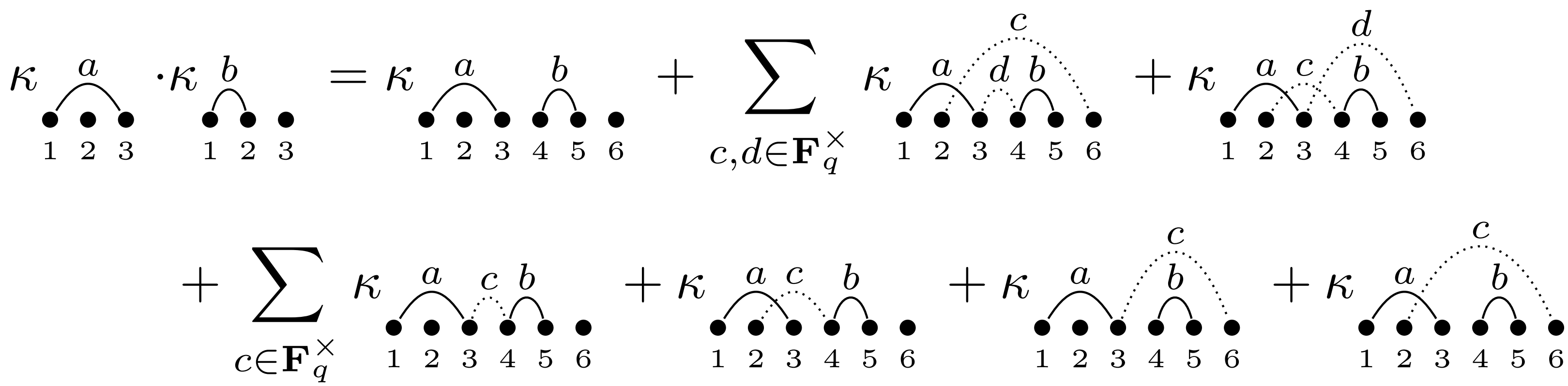
$$\Delta(\chi) = \sum_{A+B=[n]} \mathrm{Res}\,^{U_n(q)}_{U_{|A|}(q)\times U_{|B|}(q)}\, \chi$$

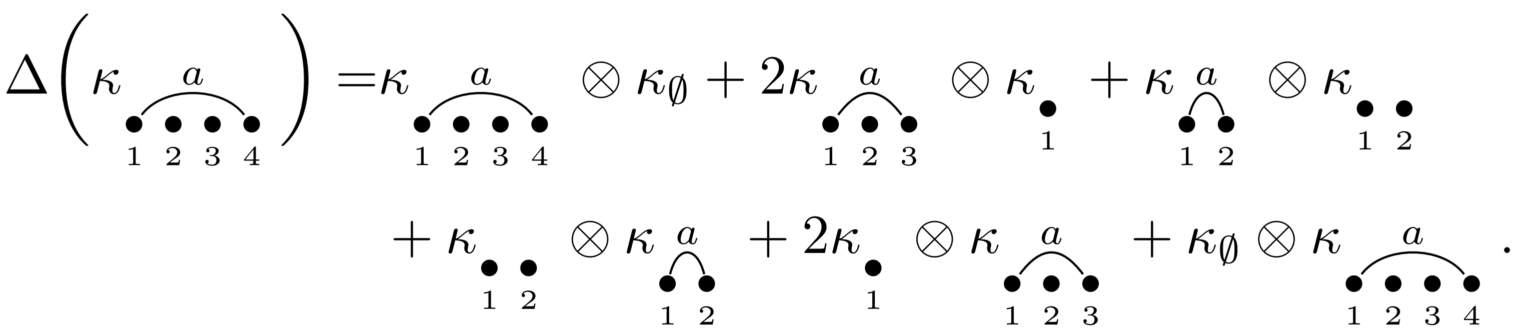
$$\chi \cdot \psi = \mathrm{Inf}\,^{U_{n+m}(q)}_{U_n(q)\times U_m(q)}\, \chi \otimes \psi = (\chi \otimes \psi) \circ \pi$$

where $\pi : U_{n+m}(q) \to U_n(q) \times U_m(q)$.

Supercharacter theory of $U_n(q)$

- Superclass functions κ_λ basis Hopf algebra structure is nice:

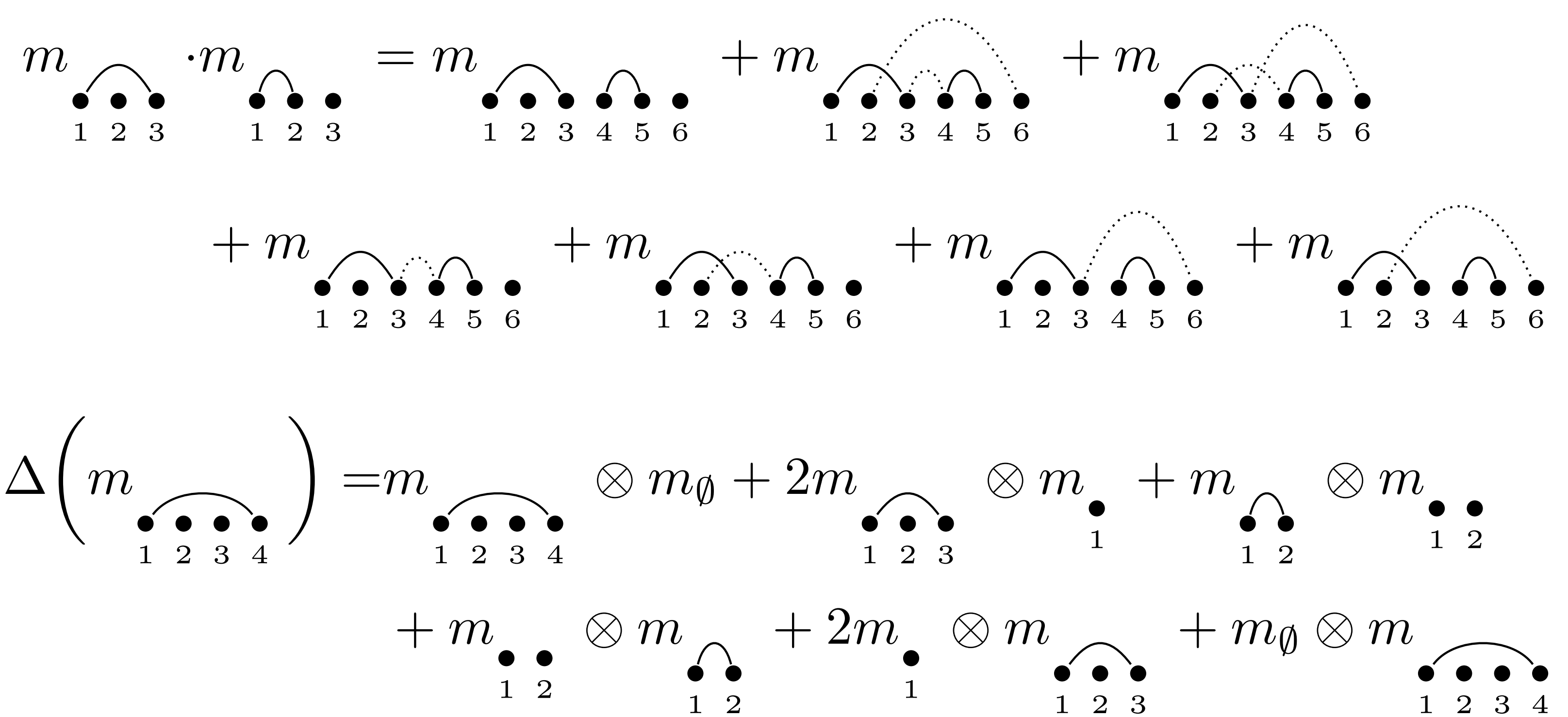




Symmetric functions in noncommutative variables

Wolf, Rosas-Sagan, Bergeron-Zabrocki, ...

- monomial symmetric functions m_λ basis (sum of orbit of a word)
- indexed by set partition Hopf algebra structure is nice:



Isomorphism when q=2

• the Hopf algebra of symmetric functions in noncommutative variables is isomorphic to the Hopf algebra of supperclass functions for $q = 2$.

• Now we want to better understand these structures.

Hopf Monoid

A nice Combinatorial Hopf Algebra is indexed by combinatorial object with a lot of structure. It should have a lift at the level of species (Hopf monoid).

This explains much of the structures of the Combinatorial Hopf algebra with more elegant and simplified formulas.

Aguiar-Mahajan

Hopf Monoid

- A Species $\mathbf{P}$ encode data: $\forall$ finite set $K \mapsto \mathbf{P}[K]$ vector space.

Typically, $\mathbf{P}[K]$ is the formal linear span of combinatorial objects of type "$\mathbf{P}$" (think Graphs, Set partitions, ...)

A bijection $K \xrightarrow{\sigma} T$ should induce a bijection $\mathbf{P}[K] \xrightarrow{\mathbf{P}[\sigma]} \mathbf{P}[T]$

and other natural conditions:

$\mathbf{P}[Id_K] = Id_{\mathbf{P}[K]}$,

$\mathbf{P}[\sigma \circ \tau] = \mathbf{P}[\sigma] \circ \mathbf{P}[\tau]$,

...

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- Monoidal structure '$\bullet$' on species:

$$(\mathbf{p} \bullet \mathbf{q})[K] = \bigoplus_{K = I \sqcup J} \mathbf{p}[I] \otimes \mathbf{q}[J] \,,$$

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• A Hopf Monoid $\mathbf{H}$ is a species such that $\mathbf{H}[\emptyset] = \mathbf{C}$ and structure maps m, Δ, u, ϵ, S.

(1) $m_{I,J}\colon \mathbf{H}[I] \otimes \mathbf{H}[J] \to \mathbf{H}[K], \qquad \forall I \sqcup J = K$

Associative multiplication $m\colon \mathbf{H} \bullet \mathbf{H} \to \mathbf{H}$ with unity $u\colon \mathbf{0} \to \mathbf{H}$

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Associative comultiplication $\Delta\colon \mathbf{H} \to \mathbf{H} \bullet \mathbf{H}$ with counity $\epsilon\colon \mathbf{H} \to \mathbf{0}$

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  (3)

$$\Delta_{I,J} \circ m_{I',J'} = (m_{A,C} \otimes m_{B,D}) \circ (\mathrm{id}_A \otimes \beta_{B,C} \otimes \mathrm{id}_D) \circ (\Delta_{A,B} \otimes \Delta_{C,D})$$

where for $K = I \sqcup J = I' \sqcup J'$ there is unique $A = I' \cap I$, $B = I' \cap J$, $C = J' \cap I$ and $D = J' \cap J$.

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The antipode $S \colon \mathbf{H} \to \mathbf{H}$ is constructed (for free) recursively.

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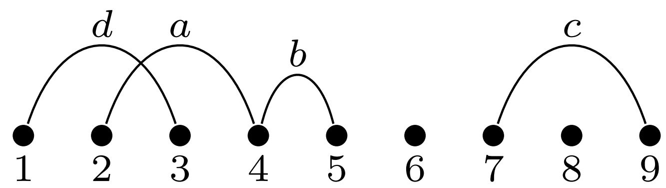
These structure are more natural that they *look*:

(1) given two combinatorial objects, give a rule to build larger one

(2) Given a decomposition of the ground set, give ways to decompose combinatorial object

(3) (typically) Find bijection between two constructions.

Hopf Monoid of Supercharacters



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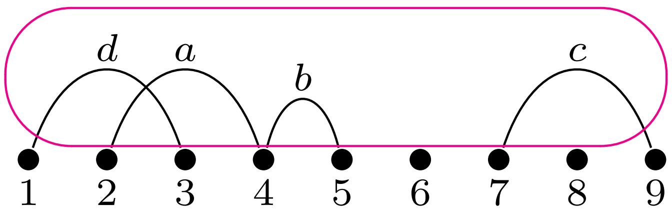
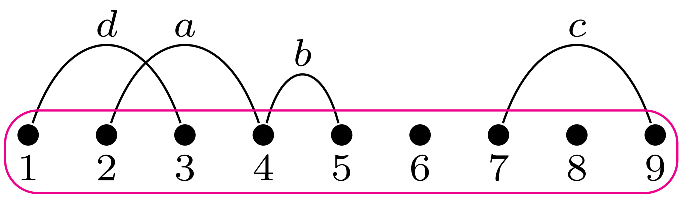


- Set of arcs λ (set partition when q=2)

Hopf Monoid of Supercharacters



- Set of arcs λ (set partition when q=2)
- Total order ϕ on the ground set.

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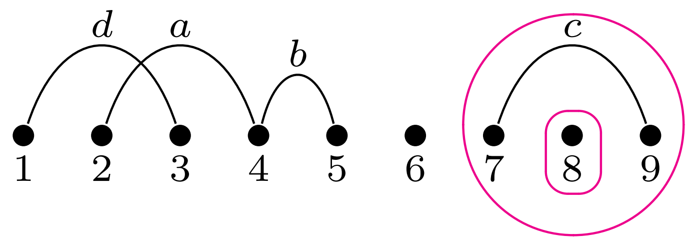
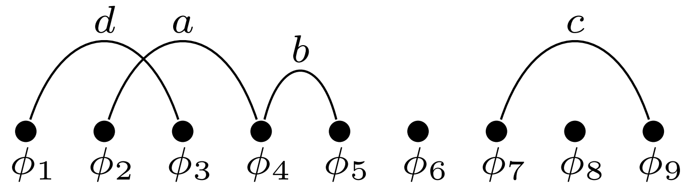


- Set of arcs λ (set partition when q=2)
- Total order ϕ on the ground set.

This is important as the dimension of supercharacter depends on the number of nodes under each arcs.

Hopf Monoid of Supercharacters

- **SC** is the species of Set of arcs with Total order: (ϕ, λ)

$(\phi, \lambda) =$ 

Hopf Monoid of Supercharacters

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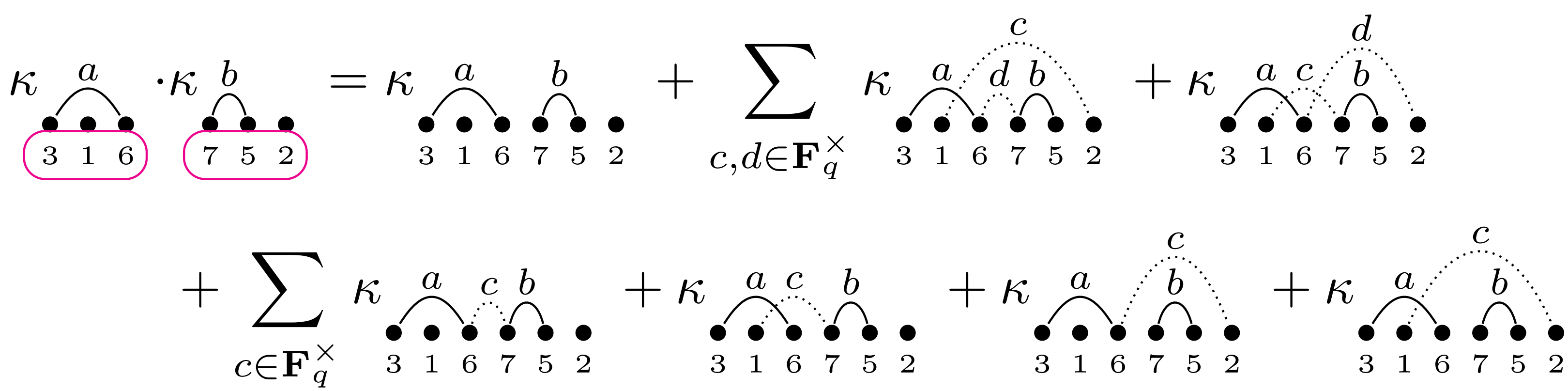
(1) $\kappa_{(\phi,\lambda)} \cdot \kappa_{(\tau,\nu)} = \mathrm{Inf}_{U_I^{\phi}(q) \times U_J^{\tau}(q)}^{U_K^{\phi\tau}(q)} \big(\kappa_{(\phi,\lambda)} \otimes \kappa_{(\tau,\nu)}\big)$

This is the same as before, with a twist:



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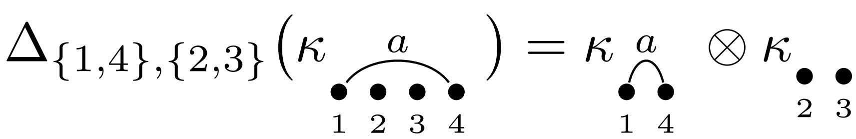
(2) $\Delta_{I,J}\big(\kappa_{(\phi,\lambda)}\big) = \mathrm{Res}_{U_I^{\phi|_I}(q) \times U_J^{\phi|_J}(q)}^{U_K^{\phi}(q)} \big(\kappa_{(\phi,\mu)}\big)$

$$= \begin{cases} \kappa_{(\phi|_I,\mu_I)} \otimes \kappa_{(\phi|_J,\mu_J)} & \text{if } \mu = \mu_I \cup \mu_J, \\ 0 & \text{otherwise.} \end{cases}$$

This is much simpler than before:



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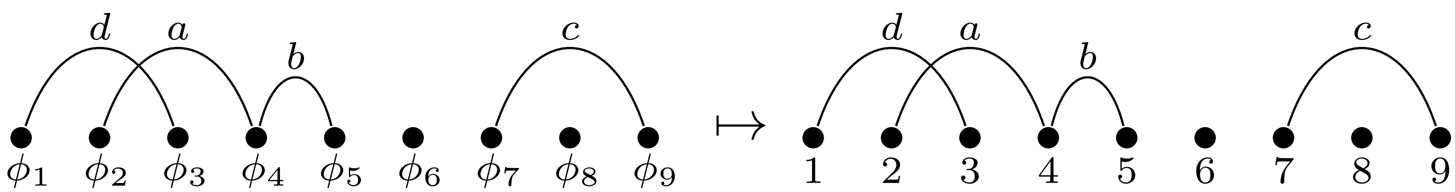
(3) "Check" that it works.

What can we do with this?

- Functor that send Hopf monoid to Hopf algebras

$$\mathbf{SC} \mapsto SC$$

$$(\phi, \lambda) \mapsto \phi^{-1} \circ \lambda$$



What can we do with this?

• Functor that send Hopf monoid to Hopf algebras

• The simplified structure for **SC** gives multiplicity free formula for the antipode of $\kappa_{(\phi,\lambda)}$ and other basis.

• Gives us some understanding of antipode for supercharacter χ^λ. In particular we show that given $\lambda = \lambda^{(1)}|\cdots|\lambda^{(k)}$ with its unique factorization into atomics.

$$S(\chi^\lambda) = (-1)^k S(\chi^{\lambda^{(k)}|\cdots|\lambda^{(1)}}) + \text{lower term in some order.}$$