WATERLOO (TUTTE COLOQUIUM MAY 2016)

FOLLOWING PIPE DREAMS

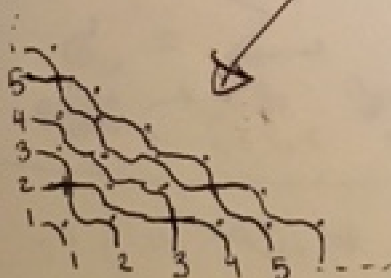
B-S BILLEY (1993) ..... B-CEBALLOS-PILAUD (2016)

- DEFINITION {(D, w(D))}
- WHY WE DEFINED THAT (BB) [RC DIAGRAMS]
- KNUTSON-MILLER [PIPE DREAMS)
  - TORIC DEGENERATION
  - SUBWORD COMPLEXES (BALLS OR SPHERE)
  - POLYTOPE CONJECTURE [BCL 2015]
- ALGEBRAIC STRUCTURE [BCP 2016]
- ENUMERATION

DEFINITION

$D \subseteq \mathbb{N} \times \mathbb{N}$ DIAGRAM $|D| < \infty$

w(D)



- REDUCED! (OR NOT)

PIPE DREAM

WHY SCHUBERT POLYNOMIALS

- $w \in S_n$ $\boxed{\mathfrak{S}_w \text{ POLYNOMIAL}}$
  - REPRESENTATIVE OF A SCHUBERT VARIETY IN FLAG MANIFOLD.

- $\boxed{\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_n^0}$

$$\partial_i = \frac{f(x_1 \cdots x_n)^{x_{i+1} x_i} - f}{x_{i+1} - x_i}$$

$\partial_i^2 = \partial_i$
$\partial_i \partial_j = \partial_j \partial_i$
$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i$

$\Downarrow$

WELL DEFINED RECURENCE

$w(i) < w(i+1)$

- $\boxed{\mathfrak{S}_w = \partial_i \mathfrak{S}_{ws_i}}$

- STABLE UNDER $S_n \hookrightarrow S_{n+1}$

"CONJECTURE STANLEY (1990)"

$$\boxed{\mathfrak{S}_w = \sum_{\substack{a_1 a_2 \cdots a_{\ell(w)} \\ i_1 \leq i_2 \leq \cdots \leq i_{\ell(w)}}} x_{i_1} x_{i_2} \cdots x_{i_{\ell(w)}}}$$

REDUCE WORD. $\rightarrow a_1 a_2 \cdots a_{\ell(w)}$

s.t $(a_j > a_{j+1} \Rightarrow i_j < i_{j+1})$

PROVED BY BILLEY - JOCKUSH - STANLEY

(with BB) WE WANTED COMBINATORIAL PROOF

$$\begin{matrix} a_1 & \cdots & a_\ell \\ i_1 & \cdots & i_\ell \end{matrix} \longrightarrow D = \{(i_j, a_j - i_j)\}$$

RC-DIAGRAM

(+ MONK-RULE INSERTION) ⊛

THM

$$\boxed{\mathfrak{S}_w = \sum_{\substack{D \text{ RC-DIAGRAM} \\ w(D) = w}} \prod_{(i,j) \in D} x_i}$$

- lots OF NICE PROPERTIES.

KNUTSON - MILLER

① RENAMED "PIPE DREAM"

② SHOW THAT

SCHUBERT VARIETIES $\xrightarrow{\text{TORIC DEGENERATION}}$ DECOMPOSE INTO ... GIVEN BY PIPE DREAMS.

MATRIX WITH RANK CONDITIONS

RANK(M) = RANK(w)

PRINC MINOR ↗ PRINC MINOR ↑

SUBWORD COMPLEX

$\Delta(Q,\pi)$ COMPLEX WITH FACES

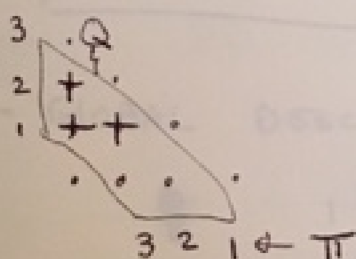
$Q = (q_1, q_2, \cdots, q_m)$

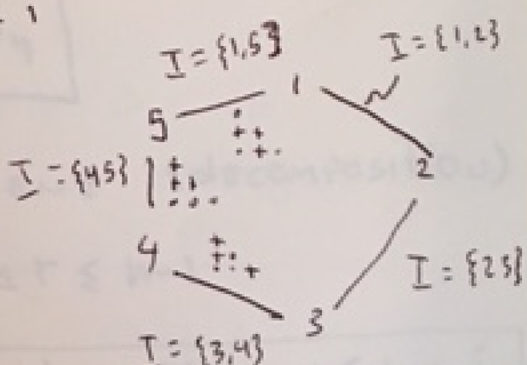
$I = (i_1, i_2 \cdots, i_n)$

$$\boxed{\pi = \overrightarrow{\prod_{j \notin I}} s_{q_j}}$$

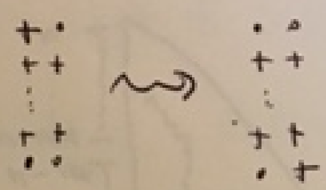
$$\text{Ex} \quad Q = (\overset{1}{1}, \overset{2}{2}, \overset{3}{1}, \overset{4}{2}, \overset{5}{1})$$

$$\pi = \begin{matrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{matrix}$$





"FLIP"



THM BALL OR SPHERE ($\Leftrightarrow \delta(Q) = \pi$)

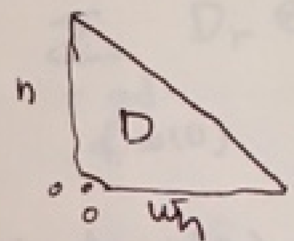
(*) CONJECTURE POLYTOPE?

BCL

ALGEBRAIC OPERATIONS

$(0,0) \notin D \subseteq \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$

Fix n

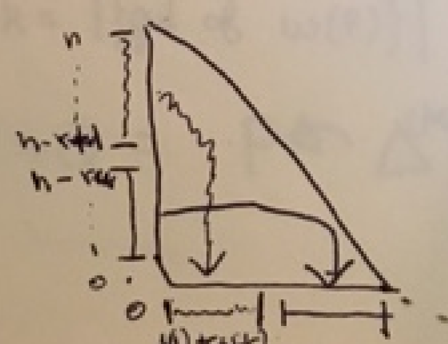


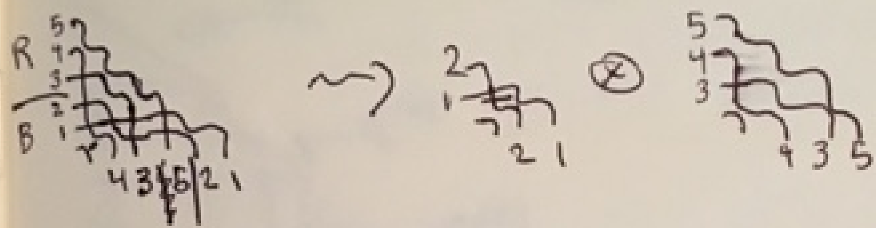
$$\boxed{w_n(D) \in S_n}$$

- GLOBAL DESCENT (decomposition)

$\exists \quad 1 \leq r \leq n-1$

$$\boxed{w(i) > w(j) \quad \forall \quad i \leq r < j}$$





$$\Delta(D) = \sum_{\substack{r \text{ gd} \\ \text{of } \omega(D)}} D_r \otimes D_r^c$$

$\{0, n\} \subseteq$ gd of $\omega(D)$

<u>THM</u> PD with Δ is a coalgebra

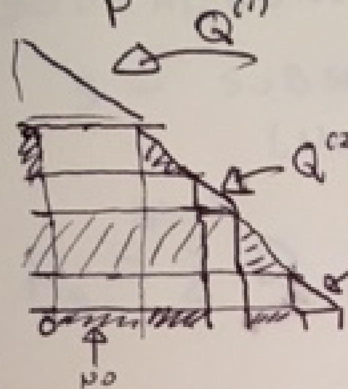
---

multiplication

$P * Q$

let $k = |\{\text{gd of } \omega(P)\}|$

$$P * Q = P \,\triangle\, \Delta^{(k+1)}(Q)$$



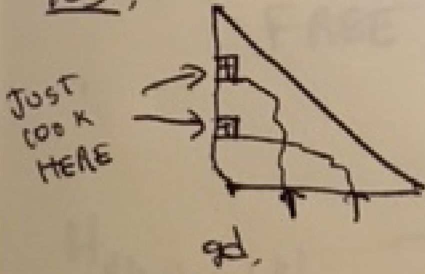
$Q^{(2)}$

<u>THM</u> PD with μ. is ALGEBRA
PD is Hopf
(Δ is HOM. FOR MULT.)

<u>THM</u> PD is "DENDRIFORM"
$\Rightarrow$ PD is FREE ALGEBRA.

CAN WE IDENTIFY <u>FREE GENERATOR</u>?

<u>YES</u>;



SUB-ALGEBRA

S – SUBSET OF IND. PERM.
(NO global DESCENT)

$$H_S = \overset{\text{SPAN}}{\{} D : \omega_n(D) = \sigma^{(1)} \# \sigma^{(2)} \# \cdots \; \sigma^{(i)} \in S \}$$

H_S is closed under product & co-product.

$H_{\{1\}}$ Hilbert series binary trees

SS

LODAY-RONCO "C_{n+1}"

FREE GENERATOR "C_{n}"

$H_{\{12\}}$ Hi

FREE GENERATOR "C_{2n+1}"

$2(n+1)-1$

⋮

$H_{\{(k-1)\cdots 1\, k\}} \longrightarrow$ "$C_{k(n+1)-1}$

$H_{\{1,\ 12,\ 123,\ \dots\}} \rightsquigarrow$ interesting paths.