COMBINATORIAL HOPF ALGEBRAS - ECCO'12
EXERCISES LECTURE 1

1. Antipode

Let H be a Hopf algebra and let S be its antipode.

(1). Show that

$$S(gh) = S(h)S(g)$$

for $g, h \in S$.

(2). Show that if H is commutative or cocommutative, then $S^2 = I_H$.

(3). Let $F = Hom_{Alg}(H, k)$ be the set of algebra morphisms from H to the ground field k. Show:

  (a) F is a group under the convolution product $\star$ where

$$g \star f := m_k(g \otimes f)\Delta_H$$

  (b) For $f \in F$ we have

$$f \circ S = f^{-1}$$

2. homogeneous and elementary symmetric functions

(1). Given that

$$h_n = S((-1)^n e_n) = -\sum_{i=1}^{n} (-1)^i h_{n-i} e_i$$

write h_i in terms of the e_i's for $i = 1, \ldots, n$.

(2). Use the identity

$$\sum_{i=0}^{m} e_i t^i = \prod_{i=1}^{m} (1 + t x_i)$$

to write an expression for $e_i(x_1, \ldots, x_m)$.

- Give an expression for $h_i(x_1, \ldots, x_m)$ using (1) and (2). [*Hint*: Guess and prove].
- Define the algebra map

$$\omega : Sym \to Sym$$

  such that $\omega(e_i) = h_i$.
  - Prove that ω is an involution.
  - Conclude that $Sym \cong \mathbb{Z}[h_1, h_2, \ldots]$.

- Compute $S(h_i)$.
- Compute $\Delta(h_i)$.

(5). Show the following:

(a) $h_k(x_1,\ldots,x_n) = h_k(x_2,\ldots,x_n) + x_1 h_{k-1}(x_1,\ldots,x_n)$.

(b) $h_k(x_k,\ldots,x_n) \in \langle Sym_n^+ \rangle$.

(c) Using the order $x_1 > \cdots > x_n$ show that $\mathrm{LM}(h_k(x_k,\ldots,x_n)) = x_k^k$ where $\mathrm{LM}(f)$ denotes the *leading monomial* of the polynomial f. [Note: Given two monomials $x_1^{a_1}x_2^{a_2}\cdots x_l^{a_l}$ and $x_1^{b_1}x_2^{b_2}\cdots x_k^{b_k}$, we say that $x_1^{a_1}x_2^{a_2}\cdots x_l^{a_l} \geq x_1^{b_1}x_2^{b_2}\cdots x_k^{b_k}$ whenever $(a_1,\ldots,a_l) \geq_{lex} (b_1,\ldots,b_k)$. The leading term $\mathrm{LM}(f)$ is the maximum of the monomials in f under the order $\geq$.]

($\star$) Show that the set $\{h_i(x_i,\ldots,x_n)\}_{i\in[n]}$ is a Groebner basis for $\langle Sym_n^+ \rangle$. Conclude that the dimension of the vector space $\mathbb{Z}[h_1,\ldots,h_k]/\langle Sym_n^+ \rangle$ is $n!$.